Similarity of Holomorphic Matrices

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ABSTRACT

Let Ω be a noncompact Riemann surface (e.g., the complex plane). The main result is that if A and B are two square holomorphic matrices on Ω such that for any z in Ω, there is a neighborhood of z with A and B holomorphically similar on the neighborhood, then A and B are holomorphically similar on Ω. This is then applied to extend results of Wasow on pointwise similarity. We actually prove these results for Bezout domains satisfying certain conditions, and then observe that these conditions are satisfied by the ring of holomorphic functions on Ω.

1. INTRODUCTION

Let R be a commutative ring (with 1) with Ω a subset of Spec R. Given a pair of square matrices A and B over R, there are various conditions we can consider:

(1.1) $A(P)$ and $B(P)$ are similar in R_P/PR_P for all P in Ω, or
(1.2) A and B are similar in R_P for all P in Ω.

Here R_P denotes the localization of R at P, and $A(P)$ denotes the image of A in R/P. Clearly, similarity over R implies (1.2), which implies (1.1). However, the reverse implications are not true in general (even if $\Omega = \operatorname{Spec} R$).

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A special case is when R is the ring of holomorphic functions on a complex manifold Ω. Here (1.2) can be replaced by

(1.2′) There exists a neighborhood Ω_z of z for each z in Ω such that A and B are holomorphically similar on Ω_z.

In fact (1.2) and (1.2′) are equivalent. We shall say A and B are pointwise similar on Ω if (1.1) holds, and A and B are locally similar on Ω if (1.2) or (1.2′) holds.

Recall that one is in the stable range of R if $aR + bR = R$ implies $a + bs$ is a unit in R for some s in R and that R is an LG ring if every polynomial over R which locally represents a unit does so globally (see [2]). The main result is the following:

THEOREM A. *Let R be a Bezout domain with quotient field K such that every proper homomorphic image of R is an LG ring. Assume that if K' is a finite dimensional field extension of K, and R' is the integral closure of R in K', then*

(i) *R' is Bezout,*

(ii) *one is in the stable range of R', and*

(iii) *there exists $0 \neq \delta$ in R with $\delta R'$ contained in a finitely generated R-module.*

If Ω is a subset of $\operatorname{Spec} R$ such that any nonunit of R is in some element of Ω, then two matrices which are locally similar on Ω are similar over R.

Note that condition (iii) is automatic if K is perfect (e.g., if R has characteristic zero.) Also, for a given pair of matrices, one only needs to verify that (i)–(iii) hold for $K' = K[\alpha_i]$, where the α_i are the eigenvalues of the matrices. Some R for which the hypotheses hold are:

(a) R semilocal ($\Omega = \operatorname{Spec} R$),

(b) the ring of all algebraic (or real) integers ($\Omega = \operatorname{Spec} R$), and

(c) the ring of holomorphic matrices on a noncompact Riemann surface Ω.

Note also that the examples in (a) and (b) are LG rings, as are rings with $R/\operatorname{rad} R$ von Neumann regular or of Krull dimension 0. However, the example in (c) is not an LG ring [8].

The proof involves looking more generally at representations of finitely generated R-algebras. We show that an analogous result holds for such representations.

As an application of the theorem, we derive a generalization of a result of Wasow [14] on pointwise similarity. This will be explained later. In particular one consequence is:

COROLLARY B. *Assume the hypotheses of Theorem* A. *Let A be a square matrix over R with rational canonical form C. Then A and C are similar over R if and only if they are pointwise similar on Ω.*

2. PRELIMINARY RESULTS

We present some results which will be needed later. Most of these are well known. R is a commutative ring with 1. The following result applies in particular to local rings.

LEMMA 2.1 ([2]; see also [7]). *Assume R is an LG ring. Let Λ be a module finite R-algebra and M a finitely generated Λ-module.*

(a) *One is in the stable range of $E = \mathrm{End}_{\Lambda}(M)$.*

(b) *If N and X are finitely generated Λ-modules, then $M \oplus X \cong N \oplus X$ implies $M \cong N$.*

(c) *Let tM denote t copies of M. Then $tM \cong tN$ implies $M \cong N$.*

(d) *If M and N are finitely presented, then $M_P \cong N_P$ for all P in* $\mathrm{Spec}\, R$ *implies $M \cong N$.*

The next result is standard.

LEMMA 2.2. *Let Λ be an R-algebra and M a finitely presented Λ-module.*

(a) *If R' is a flat commutative extension of R, then*

$$\mathrm{Hom}_{\Lambda'}(M', N') \cong \mathrm{Hom}_{\Lambda}(M, N) \otimes_R R'.$$

Here $T' = T \otimes_R R'$.

(b) *The map $\theta: N \to \mathrm{Hom}_{\Lambda}(M, N)$ is an isomorphism between the categories* $\mathrm{Div}\, M$ *and finitely generated projective E-modules. Here* $\mathrm{Div}\, M$ *are the Λ-modules which are summands of tM for some t, and $E = \mathrm{End}_{\Lambda}(M)$. Moreover θ induces a bijection between the genus of M, $G(M) = \{N \mid N_P \cong M_P$ for all P in* $\mathrm{Spec}\, R\}$, *and $G(E)$.*

LEMMA 2.3 (cf. [13]). *If one is in the stable range of Λ, and M is a finitely generated projective Λ-module, then one is in the stable range of* $\mathrm{End}_{\Lambda}(M)$.

LEMMA 2.4. *Assume Λ is a subring of Γ, and I is a common two sided ideal. If one is in the stable range of Γ and Λ/I, then one is in the stable range of Λ.*

Proof. The proof is essentially identical to that given in [6, 4.4]. ■

The next result is a version of the Noether-Deuring theorem. See [3] for different proofs and references.

PROPOSITION 2.5 (Grothendieck). *Let R be a local ring with Λ a module finite R-algebra. Suppose M and N are finitely presented Λ-modules. If R' is a faithfully flat commutative extension of R, then $M' \cong N'$ (as Λ'-modules) implies $M \cong N$.*

Proof. Let σ be the isomorphism between M' and N'. By Lemma 2.2(a), $\sigma = \sum_{i=1}^{t} s_i \otimes \sigma_i$, with $\sigma_i \in \mathrm{Hom}_\Lambda(M, N)$ and $s_i \in R'$. Let P be the maximal ideal of R. Each σ_i induces a map $\bar{\sigma}_i$ from $\bar{M} = M/PM$ to $\bar{N}$. Note $M' \cong N'$ implies $\bar{M}$ and $\bar{N}$ have the same dimension over $\bar{R} = R/P$. So consider $f = \det \sum x_i \bar{\sigma}_i$, a polynomial over $\bar{R}$. By hypothesis, f is nonzero. Thus if $\bar{R}$ is infinite (or sufficiently large), $f(\bar{r}_1, \ldots, \bar{r}_t) \neq 0$ for some $r_1, \ldots, r_t \in R$.

By Nakayama's lemma, this implies $\sigma' = \sum r_i \sigma_i \in \mathrm{Hom}_\Lambda(M, N)$ is surjective. Similarly, there exists a surjection τ' in $\mathrm{Hom}_\Lambda(N, M)$. Thus $\tau'\sigma' \in \mathrm{End}_\Lambda(M)$ is a surjection. Since M is a finitely generated R-module, this implies $\tau'\sigma'$ is an automorphism, whence σ' is an isomorphism.

If $\bar{R}$ is to small, let R'' be a free commutative local extension of R with sufficiently large residue field (e.g. take $R'' = R[x]/g(x)$), where $g(x)$ is monic of large degree and $g(x)$ is irreducible over $\bar{R}$). The argument above shows $M'' \cong N''$ as Λ''-modules. Thus $tM \cong tN$ as Λ-modules, where t is the rank of R'' over R. By Lemma 2.1, $M \cong N$. ■

Note that by Lemma 2.1, the above result also applies to R semilocal or more generally when R is an LG ring.

3. LATTICES OVER R-ORDERS

In this section, assume R is a Prüfer domain R (i.e. all finitely generated ideals are projective). Let Λ be a finitely generated R-algebra. Let Lat Λ be the category of Λ-modules which are finitely generated projective R-modules.

LEMMA 3.1. *If $M, N \in$ Lat Λ, then* $\mathrm{Hom}_\Lambda(M, N)$ *is a finitely generated projective R-module.*

Proof. Let $\lambda_1, \ldots, \lambda_s$ be generators for Λ. Consider the exact sequence of R-modules

$$0 \to \operatorname{Hom}_\Lambda(M, N) \to \operatorname{Hom}_R(M, N) \xrightarrow{\tau} \bigoplus_{i=1}^{s} \operatorname{Hom}_R(M, N),$$

where $\tau(\alpha) = (\sigma\lambda_1 - \lambda_1\sigma, \ldots, \sigma\lambda_s - \lambda_s\sigma)$. Since $\operatorname{Hom}_R(M, N)$ is a finitely generated projective R-module and R is Prüfer, the image of τ is projective. Thus $\operatorname{Hom}_\Lambda(M, N)$ is a direct summand of $\operatorname{Hom}_R(M, N)$.

LEMMA 3.2. *Suppose Ω is a subset of* Spec R *such that if I is a finitely generated proper ideal of R, then I is contained in some element of Ω. If $M, N \in$* Lat Λ *with $M_P \cong N_P$ for all P in Ω, then $M_P \cong N_P$ for all P in* Spec R.

Proof. First assume that M and N are free over R. Since $M_P \cong N_P$ for at least one P, the Λ-modules M and N have the same rank. Thus there is a well-defined determinant on $\operatorname{Hom}_R(M, N)$. Choose generators $\sigma_1, \ldots, \sigma_s$ for $\operatorname{Hom}_\Lambda(M, N)$. Define $f(x_1, \ldots, x_s) = \det \sum x_i\sigma_i \in R[x_1, \ldots, x_s]$. Since $M_P \cong N_P$ for P in Ω, f takes on values outside of P for each P in Ω. This implies that the coefficients of f are relatively prime.

If Q is in Spec R and R/Q is infinite, then by the above remarks f takes on values outside of Q, whence $M_Q \cong N_Q$. In general, set $S = R_Q$. Let T be a faithfully flat local extension of S with infinite residue field (e.g. take $T = S[x]_Q$). Then $M \otimes_S T \cong N \otimes_S T$ as $\Lambda \otimes_S T$-modules. Hence by the Noether-Deuring theorem (see Section 2), $M_Q \cong N_Q$.

If M and N are not free, choose projective R-modules M' and N' so that $M \oplus M' \cong N \oplus N'$ are free R-modules. Without loss of generality, Λ can be assumed to be a free R-algebra. Extend the action of Λ by making the generators act trivially on M' and N'. By the previous paragraph $M \oplus M'$ and $N \oplus N'$ are locally isomorphic for all Q in Spec R. Since M' and N' have the same rank, they are locally isomorphic. By local cancellation (Lemma 2.1), $M_Q \cong N_Q$ for all Q in Spec R. ■

4. A COMPARISON RESULT FOR K_0

We consider the following situation. R is a commutative ring with 1. Let $\Lambda \leqslant \Gamma$ be two module finite R-algebras. Set $C = \{r \in R \mid r\Gamma \leqslant \Lambda\}$ and $\bar{\Lambda} = \Lambda/C\Gamma$. Recall that $K_0(T)$ is the abelian group generated by isomorphism classes $[M]$ of finitely generated projective T-modules with relations given by short exact sequences. By a result of Milnor (cf. [1]), there is an exact sequence

$$K_1(\Gamma)\oplus K_1(\overline{\Lambda})\to K_1(\Gamma/C\Gamma)\to K_0(\Lambda)\xrightarrow{\lambda} K_0(\Gamma)\oplus K_0(\overline{\Lambda}),\quad (4.1)$$

where λ is given by extension [i.e. $\lambda(M)=(M\otimes_\Lambda \Gamma, M\otimes_\Lambda \overline{\Lambda})$].

In particular, if one is in the stable range of Γ, then $K_1(\Gamma)$ maps onto $K_1(\Gamma/C\Gamma)$. Thus (4.1) yields

$$0\to K_0(\Lambda)\xrightarrow{\lambda} K_0(\Gamma)\oplus K_0(\Lambda/C\Gamma).\quad (4.2)$$

Now assume that R/C is an LG ring. Suppose M and N are finitely generated projective Λ-modules with $M_P\cong N_P$ for all P in Spec R. By [2], $\overline{M}=M\otimes_\Lambda \overline{\Lambda}\cong \overline{N}$. If also $M\otimes_\Lambda \Gamma\cong N\otimes_\Lambda \Gamma$, then by (4.2) $[M]=[N]$ in $K_0(\Lambda)$. However, by Lemma 2.4, one is in the stable range of Λ. Thus $M\oplus d\Lambda\cong N\oplus d\Lambda$ implies $M\cong N$. This yields:

Theorem 4.1. *Let R be a commutative ring with 1. Assume $\Lambda\leqslant\Gamma$ are two module finite R-algebras satisfying*:

(i) *one is in the stable range of Γ, and*
(ii) *R/C is an LG ring, where $C=\{r\in R\mid r\Gamma\leqslant\Lambda\}$.*

Then if M and N are finitely generated projective Λ-modules, $M\cong N$ if and only if $M\otimes_\Lambda\Gamma\cong N\otimes_\Lambda\Gamma$ and $M_P\cong N_P$ for all P in Spec R.

One can verify Theorem 4.1 directly without using Milnor's result. See [10].

5. THE MAIN RESULT

In this section, we shall fix a Prüfer domain R with quotient field K with R/C an LG ring for all $0\neq C$. Let Λ be a finitely generated R-algebra with M a Λ-lattice. Set $E=\mathrm{End}_\Lambda(M)\leqslant \mathrm{End}_\Lambda(KM)=B$, where $KM=K\otimes_R M$. Let J be the Jacobson radical of B. Thus J is nilpotent and $\overline{B}=B/J\cong \oplus M(n_i,D_i)$ for some division rings D_i.

THEOREM 5.1. *Suppose the following hold:*

(i) *each D_i is commutative,*
(ii) *the integral closure R_i of R in D_i is Bezout,*
(iii) *one is in the stable range of R_i, and*
(iv) *there exists $0 \neq \delta$ in R with δR_i contained in a finitely generated R-module.*

Let Ω be a subset of Spec R *such that every proper finitely generated ideal of R is contained in some element of Ω. Then $N_P \cong M_P$ for all P in Ω implies $N \cong M$.*

Proof. By Lemma 3.2, $M_P \cong N_P$ for all P in Spec R. Thus by Lemma 2.2(b), we can replace Λ by E and assume M and N are finitely generated projective E-modules (by Lemma 3.1, E is a finitely generated projective R-module). Note $J \cap E$ is nilpotent. Thus if $\overline{M}$ denotes going mod $J \cap E$, $\overline{M} \cong \overline{N}$ if and only if $M \cong N$. So we can replace E by $\overline{E}$, which is an R-subalgebra of $\overline{B}$. Let F be a maximal R-order of $\overline{B}$ containing $\overline{E}$. Since each R_i is Bezout, $F \cong \oplus M(n_i, R_i)$, and $\overline{M} \otimes_E F \cong \overline{N} \otimes_E F$. In particular, there exist finitely generated R-subalgebras R_i' of R_i with $\overline{E} \leqslant F_0 \cong \oplus M(n_i, R_i')$ and $\overline{M} \otimes_E F_0 \cong \overline{N} \otimes_E F_0$. By assumption (iv), there exists $0 \neq \gamma$ in R with $\gamma R_i \leqslant R_i'$. By (iii), and Lemmas 2.3 and 2.4, this implies one is in the stable range of F_0. Since F_0 and E have the same quotient ring, there exists $0 \neq \lambda \in R$ with $\lambda F_0 \leqslant \overline{E}$. So $R/\lambda R$ is an LG ring. Thus Theorem 4.1 applies, and $\overline{M} \cong \overline{N}$, as desired. ∎

To apply this result to matrices, we need the following observation.

LEMMA 5.2. *Let α be an $n \times n$ matrix over K. Let $B = C(\alpha) = \{\beta \in M_n(K) \mid \beta\alpha = \alpha\beta\}$. Then $B/\text{rad}\, B \cong \oplus M(n_i, K_i)$, where $K_i = K[y_i]$ and the y_i are eigenvalues of α.*

Proof. This follows directly by considering α in rational canonical form. ∎

COROLLARY 5.3. *Let A and B be square matrices over R. Let Ω be a subset of* Spec R *such that every proper finitely generated ideal of R is contained in some element of Ω.*

For each eigenvalue y_i of A, assume

(i) *the integral closure R_i of R in $K[y_i]$ is Bezout,*
(ii) *R_i has one in the stable range, and*
(iii) *there exists $0 \neq \delta$ in R with δR_i contained in a finitely generated R-module.*

If A and B are similar over R_P for each P in Ω, then A and B are similar over R.

Proof. Let $\Lambda = R[x]$. Let M and N be free R-modules of the rank as the size of A. Make M and N into Λ-modules by declaring $xv = Av$ or Bv, respectively. Note that $M \cong N$ if and only if A and B are similar. The result is now an immediate consequence of Theorem 5.1 and Lemma 5.2 ■

Note that Theorem A is a special case of the corollary.

6. RIEMANN SURFACES

Let Ω be a noncompact Riemann surface with R the ring of holomorphic functions on Ω. We first record same properties of R. These are generally known (at least for Ω the complex plane). One can derive them fairly easily from results in [4] and [11]. If $z \in \Omega$, let $P_z = \{ f \in R \mid f(z) = 0\}$. Let K be the quotient field of R.

LEMMA 6.1. *Let X be a discrete subset of Ω.*

(a) *Given positive integers n_x, $x \in X$, there exists $f \in R$ such that $\{z \in \Omega \mid f(z) = 0\} = X$ and the multiplicity of the zero of f at x is n_x.*

(b) *Given positive integers n_x and f_x holomorphic in a neighborhood of x, there exists $f \in R$ such that $f \equiv f_x \bmod (P_x)^n$. Moreover, if $f_x(x) \neq 0$ for all x, then f can be chosen to be an exponential.*

Proof. This follows easily from [4, Theorems 25.5 and 26.3].

THEOREM 6.2.

(a) *One is in the stable range of R.*
(b) *R is Bezout.*
(c) *R/C is an LG ring for all $0 \neq C$.*

Proof. (a): Suppose $f, g \in R$ with no common zero. We can assume $g \neq 0$. Let X be the set of zeros of g. By Lemma 6.1(c), we can choose a unit u in R such that $f - u$ has a zero of order $n_x =$ (order of the zero of g at x) for all x in X. Hence $h = (f - u)/g \in R$ and $u = f - gh$. Thus (a) holds.

(b): Let $0 \neq f, g \in R$. Let X be the set of common zeros of f and g. By Lemma 6.1, there exists $h \in R$ such that h vanishes only at X, and the order of each zero is the minimum of the orders of the zeros of f and g. Clearly h divides both f and g in R. By (a), $h = fr + gs$. Hence $hR = fR + gR$, and (b) holds.

Let $0 \neq f \in R$. Lemma 6.1(b) implies that

$$R/fR \cong \prod_{x \in X} R/P^{n_x},$$

where X is the set of zeros of f and n_x is the multiplicity of the zero. Hence R/fR is Von Neumann regular modulo its Jacobson radical, whence (c) holds by [7]. ■

Note that if K' is a finite dimensional field extension of K, the meromorphic functions on Ω, then K' is the field of meromorphic functions on some branched covering Ω' of Ω, and the integral closure R' of R in K' is the ring of holomorphic functions on Ω'. Thus Proposition 6.1 applies to R' as well.

Another fact that is useful is:

THEOREM 6.3. *If D is a division ring finite dimensional over K, then D is a field.*

The main idea of the proof of the above result is a certain cohomology calculation suggested by M. Artin See [10] for details.

THEOREM 6.4. *Let Λ be a finitely generated R-algebra with M and N Λ-lattices. Suppose for each z in Ω, there exists a neighborhood Ω_z with M and N holomorphically isomorphic on Ω_z (to be precise, $M \otimes_R S \cong N \otimes_R S$ as $\Lambda \otimes_R S$-lattices, where S is the ring of holomorphic functions on Ω_z). Then $M \cong N$.*

Proof. Let R_z denote the localization of R at the maximal ideal P_z. Let $\tilde{R}_z$ denote the ring of germs of holomorphic functions at z. The hypothesis certainly implies that $\tilde{M}_z = M \otimes \tilde{R}_z \cong \tilde{N}_z$ for each z. Since $\tilde{R}_z$ is a faithfully flat extension of the local pid R_z, the Noether-Deuring theorem implies that $M_z \cong N_z$. The hypotheses of Theorem 5.1 now are all satisfied [note that (iv) holds because R has characteristic zero].

COROLLARY 6.5. *If A and B are matrices over R which are locally similar on Ω, then A and B are similar over R.*

7. POINTWISE SIMILARITY

In general, pointwise similarity is much weaker than local similarity. For example,

$$A = \begin{bmatrix} z^2 & z \\ 0 & 0 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} z^2 & 0 \\ 0 & 0 \end{bmatrix}$$

are pointwise similar but are not similar in any neighborhood of 0. Another such example is obtained by taking

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & z \\ 0 & 0 & 0 \end{bmatrix} \quad \text{and} \quad B = A^T.$$

However, under certain circumstances, pointwise similarity does imply local similarity at the point. The first result of this type was obtained by Wasow [14] and has been generalized in [12] and [8]. The result can be stated as follows:

PROPOSITION 7.1 [8, Theorem 3.2]. *Let R be a commutative ring with 1. Let A and B be $n \times n$ matrices over R. Define $T(X) = AX - XB$, If $P \in \operatorname{Spec} R$ and $\operatorname{rank} T = \operatorname{rank} T(P)$, then A and B pointwise similar at P implies that A and B are similar over R_P.*

Here we are using rank as meaning the largest minor with nonvanishing determinant. By using Corollary 5.3, we obtain a global version of this result.

THEOREM 7.2. *Let R be Bezout domain with every proper homomorphic image an LG ring. Assume that A and B are squares matrices over R and that A satisfies conditions* (i)–(iii) *of Corollary* 5.3.

Let $\Omega = \{P \in \operatorname{Spec} R \mid A(P)$ and $B(P)$ are similar and $\operatorname{rank} T = \operatorname{rank} T(P)\}$. Then there exists an $n \times n$ matrix U over R that $UA = BU$ and $u = \det U$ is not in any element of Ω.

Proof. Let R' be the ring obtained from R by inverting all elements $0 \neq r$ in R with r not in P for all P in Ω. Then R' is Prüfer, and if I is a proper finitely generated ideal if R', then $I \leqslant PR'$ for some P in Ω (since in fact R' is Bezout).

Now Corollary 5.3 applies to R'. Hence A and B are similar over R'. Let U be the given similarity. Without loss of generality, the entries of U are in R. Thus $u = \det U$ is a unit in R', as desired. ■

In particular, the result applies to Riemann surfaces.

COROLLARY 7.3. *Let* Ω *be a noncompact Riemann surface. Suppose* A *and* B *are square holomorphic matrices on* Ω. *Set* $T(X) = AX - XB$. *Let* $\Omega' = \{z \text{ in } \Omega \mid A(z) \text{ and } B(z) \text{ are similar and } \operatorname{rank} T(z) = \operatorname{rank} T\}$. *If* Ω' *is nonempty, then* Ω' *is a codiscrete subset of* Ω *and* A *and* B *are holomorphically similar on* Ω'.

Proof. By the previous result, there exists U holomorphic on Ω with $UA = BU$ and $u = \det U$ nonvanishing on Ω'. Let Ω'' be the complement of the zeros of u. It is easy to see that $\Omega''' = \{z \text{ in } \Omega \mid \operatorname{rank} T(z) = \operatorname{rank} T\}$ is codiscrete. Thus $\Omega' = \Omega'' \cap \Omega'''$ is codiscrete. Now U is the desired holomorphic similarity on Ω'. ∎

COROLLARY 7.4. *Let* R *be a Bezout domain, and suppose* A *is a square matrix over* R. *Let* C *be the rational canonical form of* A. *Assume the hypotheses of Corollary* 5.3 *are satisfied. Let* $\Omega = \{P \in \operatorname{Spec} R \mid A(P) \text{ and } C(P) \text{ are similar}\}$. *Then there exists* U *with* $AU = UC$ *with* $\det U = u$ *not in any element of* Ω.

Proof. By [8, Theorems 3.7 and 5.2], A and C are similar over R_P if and only if $A(P)$ and $C(P)$ are similar. Let R' be the ring obtained from R by inverting all elements not in any element of Ω. Exactly as in the proof of Theorem 7.2, A and C are similar over R', as desired. ∎

As a final result, we use the Noether-Deuring theorem to give a new proof of a result on pointwise similarity. This result also follows from [12]. See [9].

PROPOSITION 7.5. *Let* R *be a commutative ring with* 1 *with classical ring of quotients* K *a finite direct product of fields. Let* A *and* B *be square matrices over* R. *Let* $\Omega = \{P \in \operatorname{Spec} R \mid A(P) \text{ and } B(P) \text{ are similar}\}$. *If*

$$\bigcap_{P \in \Omega} P = 0,$$

then A *and* B *are similar over* K.

Proof. Let

$$R' = \prod_{P \in \Omega} R_P / PR_P.$$

The R embeds in R' via the diagonal map. Hence K embeds in $K' = K \otimes_R R'$. Since A and B are similar over R', they are clearly similar over K'. By the Noether-Deuring theorem (or one can use canonical forms), A and B are similar over K. ∎

Note that most of the matrix results have obvious analogs for Λ-lattices with almost identical proofs. Some of these results (with some extra hypotheses) are still valid for Λ-modules as well. See [10] for such results.

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