Communication

A simple proof of the multiplicativity of directed cycles of prime power length

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Abstract


This paper gives a simple combinatorial proof of the multiplicativity of directed cycles of prime power length.

Let $G$ and $H$ be digraphs. We say $G$ is homomorphic to $H$, written as $G \rightarrow H$, if there is an edge preserving mapping (homomorphism) $f: V(G) \rightarrow V(H)$. If no such mapping exists, we say that $G$ is not homomorphic to $H$ and write $G \nrightarrow H$. A digraph $D$ is called multiplicative if the class of digraphs which are not homomorphic to $D$ is closed under products. Here the product is the categorial product, i.e., $(a, b)(c, d)$ is an edge of $G \times H$ if and only if $ac$ is an edge of $G$ and $bd$ an edge of $H$. Similar definitions apply to graphs.

Multiplicativity of graphs and digraphs is motivated by the conjecture, due to Hedetniemi [3], that the product of two graphs of chromatic number $n$ also has chromatic number $n$. (This is equivalent to asserting that the complete graph of order $n - 1$ is multiplicative.) For graphs the best result known is that all cycles are multiplicative [1,2]. For oriented cycles the situation is not as simple. However they have been investigated thoroughly and there is now a complete classification of multiplicative oriented cycles [2,5,8,9].

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For a positive integer \( n \), let \( C_n \) be the directed cycle with vertices \( 0, 1, 2, \ldots, n - 1 \) and edges \((i, i + 1)\) for \( 0 \leq i \leq n - 2 \) and \((n - 1, 0)\). If \( n \) is not a prime power, let \( n = km \) with \( k \) and \( m \) relatively prime. Then \( C_n \) is isomorphic to \( C_k \times C_m \), while \( C_k \not\cong C_n \) and \( C_m \not\cong C_n \). Therefore \( C_n \) is not multiplicative.

Directed cycles of prime power length were conjectured to be multiplicative by Nešetřil and Pultr [6]. This was first proved by Häggkvist, Hell, Miller and Neumann Lara [2]. However their proof is not combinatorial and uses a deep result from homotopy theory. In [8], Zhou gave a combinatorial, but quite complex proof based on the method of [1], which was later somewhat simplified in [4]. However even that proof was quite long. Here we give a simple combinatorial proof of this result.

**Theorem 1** [2]. The directed cycle \( C_n \) is multiplicative if and only if \( n \) is a prime power.

Let \( G \) be a digraph. An oriented walk \( P \) of \( G \) is a sequence of vertices and edges \( \langle u_0, e_1, v_1, e_2, \ldots, u_{n-1}, e_n, u_n \rangle \) such that \( e_i \) is either \((v_{i-1}, v_i)\) (a forward edge) or \((v_i, v_{i-1})\) (a backward edge). We will refer to such an oriented walk just by its sequence of vertices. If all the vertices are distinct, we have an oriented path. If \( u_0 = u_n \), we have a closed oriented walk. If \( u_0 = u_n \) and all other vertices are distinct, we have an oriented cycle. The inverse \( P^T \) of \( P \) has the same vertex set and edge set as \( P \) but is traversed in the opposite direction, i.e., \( P^T \) is the sequence \( \langle u_n, u_{n-1}, \ldots, u_1, u_0 \rangle \). The length \( l(P) \) of \( P \) is the number of forward edges of \( P \) minus the number of backward edges of \( P \). Hence \( l(P) = -l(P^T) \). For \( v \in P \), the level \( \lambda(v) \) of \( v \) in \( P \) is the length of the subpath of \( P \) from \( u_0 \) to \( v \). If \( P_1 = \langle u_0, v_1, \ldots, u_t \rangle \) and \( P_2 = \langle u_0, u_1, \ldots, u_m \rangle \) are two oriented walks of \( G \) and \( v_n = u_0 \), then \( P_1 \circ P_2 \) is the oriented walk of \( G \) obtained by adding \( P_2 \) to the end of \( P_1 \). It is easy to see that \( l(P_1 \circ P_2) = l(P_1) + l(P_2) \).

The following two facts were observed in [2]:

1. A digraph \( G \) is homomorphic to \( C_n \) if and only if every oriented cycle of \( G \) (or equivalently every closed oriented walk of \( G \)) has length 0 (mod \( n \)).

2. Suppose \( P_1 = \langle u_0, v_1, \ldots, u_t \rangle \) and \( P_2 = \langle u_0, u_1, \ldots, u_m \rangle \) are two oriented paths of length \( t \) such that for every \( v \in P_1 \) we have \( 0 \leq \lambda(v) \leq t \) and for every \( u \in P_2 \) we have \( 0 \leq \lambda(u) \leq t \). Then there is an oriented walk of length \( t \) in \( P_1 \times P_2 \) from \((u_0, u_0)\) to \((v_0, u_m)\).

As noted in [2], the proof of (1) is straightforward and (2) can be proved by double induction on \( t \), the length of \( P_1 \) and \( P_2 \), and \( q \), the total number of vertices of \( P_1 \) and \( P_2 \) of level 0. Now we are ready to prove Theorem 1.

**Proof of Theorem 1.** We may assume that \( n \) is a prime power, since we have already observed that \( C_n \) is not multiplicative otherwise. Suppose \( G \) and \( H \) are digraphs such that \( G \not\cong C_n \) and \( H \not\cong C_n \). We need to show that \( G \times H \not\cong C_n \). By (1), \( G \) has an oriented cycle \( C = \langle u_0, v_1, \ldots, v_k, u_0 \rangle \) of length \( m_1 \neq 0 \) (mod \( n \)) and \( H \) has an oriented
cycle $C' = \langle u_0, u_1, \ldots, u_m, u_0 \rangle$ of length $m_2 \neq 0 \pmod{n}$. Without loss of generality, assume that $m_1 > 0$ and $m_2 > 0$ and let $m^* = \text{lcm}(m_1, m_2) = k_1 m_1 - k_2 m_2$. Since $n$ is a prime power, we have $m^* \neq 0 \pmod{n}$. To prove $G \times H \cong C_n$, it is enough to show that $C \times C' \subset G \times H$ has a closed walk $W$ of length $m^*$ from $(v_0, u_0)$ to $(u_0, u_0)$.

First we show that there is a point $v_i \in C$ such that each of the walks $\langle v_0, \ldots, v_{i+j} \rangle$ (addition modulo $k + 1$) has nonnegative length. To see this, let $\alpha_i = l((v_0, v_1, \ldots, v_i)) (1 \leq i \leq k)$ and let $\alpha_i = \min\{\alpha_i : 0 \leq i < k\}$. It is easy to see that $v_i$ has the claimed property (observe that the length of $C$ is $m_1 > 0$).

Without loss of generality, we assume that $u_0$ has this property in $C$ and $u_0$ has the corresponding property in $C'$.

Let $P_1 = \langle u_0, u_1, \ldots, u_k, u_0, u_1, \ldots, u_k, u_0, \ldots, u_0 \rangle$ be the closed walk of $C$ which goes around $C$ exactly $k_1$ times. For convenience let $v_i^j$ be the $j$th occurrence of $u_i$ in $P_1$ (i.e., write $P_1$ as $\langle v_0^1, v_1^1, \ldots, v_0^1, v_1^1, \ldots, v_k^1 \rangle$). Similarly, let $P_2 = \langle u_0^1, u_1^1, \ldots, u_0^2, u_1^2, \ldots, u_k^2, \ldots, u_0^2 \rangle$ be the closed walk of $C'$ which goes around $C'$ exactly $k_2$ times, where $u_i^j$ denotes the $j$th occurrence of $u_i$ in $P_2$.

Let $M_1 = \max\{\lambda_{P_1}(x) : x \in P_1\}$ and let $M_2 = \max\{\lambda_{P_2}(y) : y \in P_2\}$ and let $M = \max\{M_1, M_2\}$.

Let $P_1' = \langle u_0^{k_1}, u_1^{k_1+1}, \ldots, v_i^{k_1+1}, \ldots, u_0^1 \rangle$ be the walk of $C'$ which is the extension of $P_1$ by continuing around $C$ until the length first reaches $M + 1$. Similarly let $P_2' = \langle u_0^1, \ldots, u_k^{k_2+1}, u_1^{k_2+1}, \ldots, u_0^1 \rangle$ be the walk of $C'$ which is the extension of $P_2$ by continuing around $C'$ until the length first reaches $M + 1$. Let $P_1''$ be the subpath of $P_1'$ from $u_i^{k_1}$ to $v_i^j$ and let $P_2''$ be the subpath of $P_2'$ from $u_i^{k_1+1}$ to $u_j^k$. It is easy to check that

- $P_1'$ and $P_2'$ are oriented paths of length $M + 1$ such that for every $x \in P_1'$ we have $0 \leq \lambda_{P_1'}(x) \leq M + 1$ and for every $y \in P_2'$ we have $0 \leq \lambda_{P_2'}(y) \leq M + 1$.
- $P_1''$ and $P_2''$ are oriented paths of length $M + 1 - m^*$ such that for every $x \in P_1''$ we have $0 \leq \lambda_{P_1''}(x) \leq M + 1 - m^*$ and for every $y \in P_2''$ we have $0 \leq \lambda_{P_2''}(y) \leq M + 1 - m^*$.

By (2), $P_1' \times P_2'$ has a walk $W'$ from $(v_0, u_0)$ to $(u_j, u_j)$ of length $M + 1$ and $P_1'' \times P_2''$ has a walk $W''$ from $(v_0, u_0)$ to $(u_j, u_j)$ of length $M + 1 - m^*$. Now $W' \circ (W'')^{-1}$ is a closed walk of length $m^*$ in $C \times C'$. This finishes the proof of the theorem.