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Perturbation of $\Delta u + u^{(N+2)/(N-2)} = 0$, the Scalar Curvature Problem in $\mathbb{R}^N$, and Related Topics

A. Ambrosetti*

SISSA, via Beirut 2-4, Trieste 34014, Italy

and

J. Garcia Azorero and I. Peral†

UAM, Cantoblanco, Madrid, Spain

Some nonlinear elliptic equations on $\mathbb{R}^N$ which arise perturbing the problem with the critical Sobolev exponent are studied. In particular, some results dealing with the scalar curvature problem in $\mathbb{R}^N$ are given.

1. INTRODUCTION

This paper deals with some classes of elliptic equations which are perturbation of the problems with the critical Sobolev exponent $-\Delta u = u^{(N+2)/(N-2)}$ on $\mathbb{R}^N$, $N \geqslant 3$. Precisely, we will study the existence of positive solutions of problems like

$$\begin{cases} -\Delta u = u^p + \varepsilon \Psi(x, u), \\ u > 0, \qquad u \in \mathcal{D}^{1,2}(\mathbb{R}^N), \end{cases} \tag{1}$$

where, throughout the paper, we take

$$p = \frac{N+2}{N-2}, \qquad N \geqslant 3.$$

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In the first part of the paper we deal with the case that $\Psi = K(x)\, u^p$ when (1) becomes

$$\begin{cases} -\Delta u = [1 + \varepsilon K(x)]\, u^p, \\ u > 0, \qquad u \in \mathcal{D}^{1,2}(\mathbb{R}^N). \end{cases} \tag{2}$$

This is motivated by the scalar curvature problem in differential geometry, first posed by Nirenberg [26]. Let (M, g_0) be an N-dimensional Riemannian manifold and let S_0 be its scalar curvature. The problem is to find a metric g conformal to g_0 such that the corresponding scalar curvature is S. Letting $g = u^{4/(N-2)} g_0$, $N > 2$, one is led to solve the elliptic equation

$$-\frac{4(N-1)}{(N-2)}\, \Delta_{g_0} u + S_0 u = S u^p. \tag{3}$$

Here Δ_{g_0} denotes the Laplace–Beltrami operator. If $M = \mathbb{R}^N$, $g_0 = \sum dx_i^2$ is the standard metric and $S = 1 + \varepsilon K$, then ($S_0 = 0$ and) the problem becomes just (2), up to an uninfluent constant.

In general, there are several difficulties in facing this problem by means of variational methods. In addition to the lack of compactness, there are more intrinsic obstructions involving also the local behaviour of S and the nature of its critical points. For example, as a consequence of a Pohozaev-type identity, if u is any positive solution of (3) (with $M = \mathbb{R}^N$ and g_0 the standard metric), then $\int_{\mathbb{R}^N} \langle S'(x), x \rangle\, u^{p+1} = 0$ provided $u \to 0$, $|\nabla u| \to 0$ as $|x| \to \infty$. In particular, there are no positive solutions of (3) with such an asymptotic behaviour if $\langle S'(x), x \rangle$ does not change sign.

The prescribing scalar curvature problem has been widely investigated, see [4, 5, 10–13, 15, 18–21, 24, 28, 29, 31] and [6, 7, 14, 25] for the case of $M = \mathbb{R}^N$.

One group of existence results have been obtained under hypotheses involving the Laplacian ΔS at the critical points ξ of S, see [12] for $M = S^2$, [5] for $M = S^3$, and [20, 21] for $M = S^N$, $N \geqslant 3$. For example, in [5] it is assumed that S is a Morse function and

$$\Delta S(\xi) \neq 0, \qquad \forall \xi \colon S'(\xi) = 0.$$

Then, if $m(\xi)$ denotes the Morse index of the critical point ξ, problem (3) on the standard sphere S^3 has a solution provided

$$\sum_{\Delta S(\xi) < 0} (-1)^{m(\xi)} \neq -1.$$

The result has been extended to any S^N, $N \geqslant 3$ in [20, 21]. Roughly, it is assumed that there exists β, $N-2<\beta<N$, such that

$$S(y) = S(0) + \sum_{j=1}^{N} a_j \, |y_j|^{\beta} + \text{h.o.t.}, \tag{4}$$

where $a_j \neq 0$, $\sum a_j \neq 0$. Let $\Xi = \{\xi \colon S'(\xi) = 0, \ \sum_{j=1}^{N} a_j < 0\}$ and $i(\xi) = \sharp\{a_j \colon S'(\xi) = 0, \ a_j < 0\}$. Then (3) has a solution provided

$$\sum_{\xi \in \Xi} (-1)^{i(\xi)} \neq (-1)^N. \tag{5}$$

See Theorem 0.1 of [20]. The case that $\beta = N-2$ is handled in [21] under some further condition on the curvature S.

In the present paper we restrict our attention to curvatures close to a constant, $S = 1 + \varepsilon K$, and improve the preceding results because we require a condition like (4) allowing any $\beta \in \,]1, N[$ and assume merely (5). Actually, this is a particular case of more general results, see Theorems 3.7 and 3.11 in Section 3. Since it has been pointed out in [20, 21] that, in general, (3) could have no solution if $\beta < N-2$, the fact that we are dealing with curvatures close to constant is essential. We finally mention that perturbed problems like (2) are also discussed in Section 6 of [20] (see also [13] for a previous results) under some further algebraic condition on the top order terms in the Taylor expansion of S.

Our approach is completely different from the ones used in the above-mentioned papers and relies on a suitable use of an abstract perturbation method in critical point theory discussed in [1, 2] Solutions of (1) are the critical points on $\mathscr{D}^{1,2}(\mathbb{R}^N)$ of

$$f_\varepsilon(u) = \frac{1}{2} \int_{\mathbb{R}^N} |\nabla u|^2 - \frac{1}{p+1} \int_{\mathbb{R}^N} (1 + \varepsilon K) \, u_+^{p+1}.$$

For $\varepsilon = 0$ the unperturbed functional f_0 has a manifold of critical points Z of points of the form

$$\mu^{2-N/2} z_0((x-\xi)/\mu), \qquad \mu > 0, \quad \xi \in \mathbb{R}^N,$$

where

$$z_0(x) = C_N \cdot (1 + |x|^2)^{(2-N)/2}, \qquad C_N = [N(N-2)]^{(N-2)/4}$$

denote the "fundamental" solution of (2) with $\varepsilon = 0$. Consider the functional

$$\Gamma(\mu, \xi) = \frac{1}{p+1} \int_{\mathbb{R}^N} K(\mu y + \xi) \, z_0^{p+1}(y) \, dy.$$

After an appropriate finite dimensional reduction, variational in nature, one shows that "stable" critical points on $\mathbb{R}^+ \times \mathbb{R}^N$ of Γ correspond to points on Z from which "bifurcate" solutions of (1) whit $\varepsilon \neq 0$.

Unlike other applications, cfr. [1, 2], here the above abstract setting cannot be used in a straightforward manner. First of all, it is convenient to extend Γ to $\mu \leqslant 0$ by continuity and symmetry. This extended Γ has, for $\mu = 0$, the same critical points of K. But these critical points of the type $(0, \xi)$ do not give rise, in general, to solutions of (2). For example, let $K'(0) = 0$ and $\langle K'(x), x \rangle < 0$, $\forall x \neq 0$. Then

$$\langle \Gamma'(q), q \rangle = \frac{1}{p+1} \int_{\mathbb{R}^N} \langle K'(x), x \rangle \, z_0^{p+1}\left(\frac{x-\xi}{\mu}\right) dx < 0, \qquad q = (\mu, \xi).$$

Thus, the extended Γ has its unique critical point at $\mu = 0$, $\xi = 0$, while (2) has no positive solutions. It is just to prove that Γ has a critical point with $\mu > 0$ that a condition on the critical points of K comes out in a natural way. To give an idea of the arguments, let us consider the specific case that K is a Morse function and

$$\Delta K(\xi) \neq 0, \qquad \forall \xi\colon K'(\xi) = 0.$$

One shows that $D^2_{\mu, \xi} \Gamma(0, \xi) = 0$ while

$$D^2_{\mu, \mu} \Gamma(0, \xi) = c \cdot \Delta K(\xi), \qquad c > 0.$$

It follows that the Morse index $m(\Gamma, \xi)$ critical points $(0, \xi)$ of Γ is the same as the Morse index $m(K, \xi)$ of the critical point ξ of K if $\Delta K(\xi) > 0$, while $m(\Gamma, \xi) = m(K, \xi) + 1$ if $\Delta K(\xi) < 0$. Then a degree theoretical argument readily implies that Γ must possess at least a critical point (μ, ξ) with $\mu > 0$, giving rise to a solution of (2).

An advantage of our approach is that it gives rise to proofs that are rather simpler than the ones of, e.g., [20, 21]. It also highlights that, in general, the critical points we find are not mountain pass nor constrained minima of f_ε but they have a large Morse index. This can be useful to provide the correct insight in facing the nonperturbative case.

In Section 4 we consider the case that K is radial. In such a case our perturbational approach becomes very simple and yields a quite general and neat existence and multiplicity results that require only conditions on the qualitative behaviour of K at $r = 0$ and $r = \infty$. For example, we show that *a positive solution of* (2) *exists provided* K *is radial*, $K(0) = 0$, *and* $r^{-\alpha} K(r) \in L^1([1, \infty), r^{N-1}\, dr)$, *for some* $\alpha < N$. See Theorem 4.2. See also Theorems 4.4. and 4.5 for other existence statements. In addition we can also handle periodic K, see Theorem 4.9. The radial case has been studied in [6, 7]. A comparison with those results is made in Remark 4.6.(iii).

In the second part of the paper, see Sections 5 and 6 below, we take

$$\Psi = hu^q + Ku^p$$

and either $q = 1$ and $N > 4$, or $K \equiv 0$ and $1 < q < p$. The same abstract setting applies to these cases, too, yielding several existence theorems for the problem

$$-\Delta u = \varepsilon h(x)\, u^q + [1 + \varepsilon K(x)]\, u^p, \qquad x \in \mathbb{R}^N. \tag{6}$$

Roughly, the presence of hu^q, with $1 \leqslant q < p$, permits us to find rather general results. For example, we show that *if* $q = 1$ *and* $K \equiv 0$, *then a positive solution of* (6) *exists for* $|\varepsilon|$ *small provided* $N > 4$, *h has compact support and is not identically zero*. It is worth pointing out that here we do not need to require $h > 0$. The results of Sections 5 and 6 are new, to the best of our knowledge. Indeed, usual variational techniques would, in general, require more restrictive assumptions on h to overcome the lack of compactness.

Finally, let us remark that, according to the local L^∞ estimate by Trudinger [30] and a standard bootstrap argument, the solutions we find are classical C^2 solutions. See also [8].

Notation

If $x, y \in \mathbb{R}^N$, $\langle x, y \rangle$ and $|x|^2$ denote, respectively, the euclidean scalar product and the euclidean norm.

If $\Omega \subset \mathbb{R}^N$ and $h \in C(\Omega, \mathbb{R}^N)$ the topological degree of h with respect to Ω and 0 (when it is well defined) is denotes by $\deg(h, \Omega, 0)$. The local degree (i.e., the index of) an isolated solution p to $h = 0$ is denoted by $\deg_{loc}(h, p)$.

We will work mainly in

$$\mathcal{D}^{1,2}(\mathbb{R}^N) = \left\{ u \in L^{2N/(N-2)}(\mathbb{R}^N) \;\middle|\; \int_{\mathbb{R}^N} |\nabla u|^2\, dx < \infty \right\}$$

that coincides with the completion of $C_0^\infty(\mathbb{R}^N)$ with respect to the L^2-norm of the gradient.

If E is a Hilbert space and $f \in C^2(E, \mathbb{R})$ is a functional, we denote by f' its gradient.

A critical point of f is a $u \in E$ such that $f'(u) = 0$. We set $\mathrm{Crit}(f) = \{u \in E: f'(u) = 0\}$ and $\mathrm{Crit}(f, a) = \{u \in \mathrm{Crit}(f): f(u) = a\}$.

If $u \in \mathrm{Crit}(f)$ we denote by $m(f, u)$ the Morse index of u, namely the dimension of the subspace where $D^2 f(u)$ is negatively defined.

2. THE ABSTRACT PERTURBATION METHOD

In this section we state the abstract results we will use in the rest of the paper. They are closely related, but not equal to those discussed in [1, 2] (for other perturbation results see also [3]) and are reported below for the reader's convenience.

Let E be a Hilbert space and let f_0, $G \in C^2(E, \mathbb{R})$ be given. Consider the perturbed functional

$$f_\varepsilon(u) = f_0(u) - \varepsilon G(u).$$

Suppose that f_0 satisfies

(1) f_0 has a finite dimensional manifold of critical points Z; let $b = f_0(z)$, for all $z \in Z$;

(2) for all $z \in Z$, $D^2 f_0(z)$ is a Fredholm operator with index zero;

(3) for all $z \in Z$ there results $T_z Z = \mathrm{Ker}\, D^2 f_0(z)$.

Hereafter we denote by Γ the functional $G_{|Z}$.

THEOREM 2.1. *Let f_0 satisfy* 1–3 *above and suppose that there exists a critical point $\bar{z} \in Z$ of Γ such that one of the following conditions hold*:

(i) *$\bar{z}$ is nondegenerated*;

(ii) *$\bar{z}$ is a proper local minimum or maximum*;

(iii) *$\bar{z}$ is isolated and the local topological degree of Γ' at $\bar{z}$, $\deg_{loc}(\Gamma', 0)$ is different from zero.*

Then for $|\varepsilon|$ small enough, the functional f_ε has a critical point u_ε such that $u_\varepsilon \to \bar{z}$ as $\varepsilon \to 0$.

The proof lies, roughly, in three steps.

Step 1. Using assumptions 1 and 2 and the implicit function theorem, one finds for $|\varepsilon|$ small a manifold Z_ε, locally diffeomorphic to Z, whose points have the form $z + w(\varepsilon, z)$, where $w \perp T_z Z$ and verifies $\|w\| = O(\varepsilon)$ as $\varepsilon \to 0$. Furthermore, the critical points of f_ε constrained on Z_ε give rise to critical point of f_ε, namely to solutions of $f'_\varepsilon = 0$. This fact will be expressed by saying that Z_ε is a *natural constraint* for f_ε.

Step 2. One shows that, for $u \in Z_\varepsilon$, there results

$$f_\varepsilon(u) = b - \varepsilon \Gamma(z) + o(\varepsilon), \qquad (\varepsilon \to 0).$$

Step 3. Step 1 and the preceding formula allow us to show that after perturbation $\bar{z}$ gives rise to critical points of f_ε which are close to $\bar{z}$. Let us point out that case (iii) is not explicitly considered in [1, 2] but can be readily handled with the arguments therein.

Remark 2.2. If $Z_0 := \{z \in Z: \Gamma(z) = \min_Z \Gamma\}$ is compact then one can still prove that f_ε has a critical point near Z_0. The set Z_0 could also consist of local minima and the same for maxima. Likewise in statement (iii) we could allow that Γ has an isolated set of critical points $\mathscr{C}$ such that $\deg(\Gamma', \Omega, 0) \neq 0$, where Ω is an open bounded neighbourhood of $\mathscr{C}$.

Remark 2.3. According to the results of Section 3 of [2] we can evaluate the Morse index of u_ε. Precisely, if Γ is nondegenerated, then for $\varepsilon > 0$ small one has $m(f_\varepsilon, u_\varepsilon) = m(f_0, \bar{z}) + m(-\Gamma, \bar{z})$. Let us recall that the functional Γ in [2] corresponds to $-\Gamma$ here. Actually, f_ε had in [2] the form $f_\varepsilon = b + \varepsilon\Gamma + o(\varepsilon)$.

Let us explicitly point out that in the above arguments we do not need to assume that Z is complete.

3. THE SCALAR CURVATURE PROBLEM

In this section we deal with problem (2), namely

$$-\Delta u = [1 + \varepsilon K(x)]\, u^p, \qquad u > 0.$$

3.1. *The Unperturbed Problem*

Let K be bounded, let $E = \mathscr{D}^{1,2}(\mathbb{R}^N)$, and consider the functional $f_\varepsilon: E \to \mathbb{R}$ by setting

$$f_\varepsilon(u) = \frac{1}{2}\int_{\mathbb{R}^N} |\nabla u|^2 - \frac{1}{p+1}\int_{\mathbb{R}^N} u_+^{p+1} - \frac{\varepsilon}{p+1}\int_{\mathbb{R}^N} K(x)\, u_+^{p+1}, \tag{7}$$

where $u_+ = \max\{u, 0\}$. Plainly, $f_\varepsilon \in C^2(E, \mathbb{R})$ and any critical point $u \in E$, $u \neq 0$, of f_ε is a solution of (2). Moreover, according to [8, 30], u is smooth. It is also easy to check that $u > 0$. Actually, using as a test function $u_- = \min\{u, 0\}$ from $(f'_\varepsilon(u), u_-) = 0$ it follows that $u \geqslant 0$. Thus the strong maximum principle yields $u > 0$.

As pointed out in the Introduction, if $\varepsilon = 0$ the positive solutions of (2) are given by

$$z_{\mu,\xi}(x) = \mu^{-(N-2)/2} z_0\left(\frac{x-\xi}{\mu}\right), \tag{8}$$

where $\mu > 0$, $\xi \in \mathbb{R}^N$, and

$$z_0(x) = C_N \cdot (1 + |x|^2)^{(2-N)/2}, \qquad C_N = [N(N-2)]^{(N-2)/4}.$$

Letting

$$Z = \{z_{\mu,\xi} \mid \mu > 0, \xi \in \mathbb{R}^N\}, \tag{9}$$

Z is a $N+1$ dimensional manifold of the critical points of f_0. It is worth pointing out that $Z \subset W^{1,2}(\mathbb{R}^N)$ when $N > 4$.

In order to apply the abstract setting we will check the assumptions on f_0 introduced in the preceding section. The following lemma is essentially known. In particular, for the nondegeneracy condition (statement 3 below) we refer to [27]. However, we report a sketch of the proof for the reader's convenience and to make the paper as self-contained as possible.

LEMMA 3.1. *f_0 satisfies the following properties*:

1. $\dim Z = N+1$
2. $D^2 f_0(z) = I - C$ *where* C *is compact.*
3. $T_{\mu,\xi} Z = \ker\{D^2 f_0(z_{\mu,\xi})\} \subset \mathscr{D}^{1,2}(\mathbb{R}^N)$.

Proof (Sketch of Proof of 3). It is always true that $T_{\mu,\xi} Z \subset \ker\{f_0''(z_{\mu,\xi})\}$. We will show the converse, i.e., that if $u \in \mathscr{D}^{1,2}(\mathbb{R}^N)$ is a solution of

$$-\Delta u = p z_{\mu,\xi}^{p-1} u \tag{10}$$

then $u \in T_{\mu,\xi} Z$, namely

$$u = \alpha D_\mu z_{\mu,\xi} + \langle \nabla_x z_{\mu,\xi}, b \rangle, \qquad \alpha \in \mathbb{R}, \quad b = (\beta_1, ..., \beta_N) \in \mathbb{R}^N.$$

Let us denote by D_μ, D_ξ the derivatives with respect to the parameters μ and ξ, respectively. In particular, in our case $D_\xi \equiv \nabla_x$. Up to a translation, we can assume that $\xi \equiv 0$ and, for simplicity of notation, we consider $\mu = 1$. We look for solutions $u \in \mathscr{D}^{1,2}(\mathbb{R}^N)$ of (10) of the form

$$u = \sum_{k=0}^{\infty} \psi_k(r)\, Y_k(\theta), \qquad \text{where} \quad \psi_k(r) = \int_{S^{N-1}} u(r\theta)\, Y_k(\theta)\, d\theta,$$

and Y_k denotes the kth spherical harmonic satisfying ($\Delta_{S^{N-1}}$ stands for the Laplace–Beltrami operator)

$$k(N+k-2)\, Y_k(\theta) + (\Delta_{S^{N-1}} Y_k(\theta)) = 0. \tag{11}$$

For $k \geqslant 0$ one finds

$$\left(-\psi_k'' - \frac{N-1}{r}\psi_k'\right) Y_k - \psi_k \Delta_{S^{N-1}} Y_k = p z_\mu^{p-1} \psi_k Y_k,$$

hence by (11),

$$-\psi_k'' - \frac{N-1}{r}\psi_k' + \frac{k(N+k-2)}{r^2}\psi_k = pz_\mu^{p-1}\psi_k \tag{12}$$

and by standard regularity theory, $\psi_k(0)=0$ if $k \geqslant 1$. Notice that if $k=0$ one has

$$-\psi_0'' - \frac{N-1}{r}\psi_0' = pz_\mu^{p-1}\psi_0,$$

and then $w = D_\mu z_\mu$ is a solution of (12) which belongs to $\mathscr{D}^{1,2}(\mathbb{R}^N)$. If we look for a second linearly independent solution of the form $u(r) = c(r)\, w(r)$, we will check that u is not in $\mathscr{D}^{1,2}(\mathbb{R}^N)$. A direct computation gives

$$-(c''w + 2c'w' + cw'') - \frac{(N-1)}{r}(c'w + cw') = pz^{p-1}cw$$

and, because w is a solution,

$$-c''w - c'\left(2w' - \frac{(N-1)}{r}w\right) = 0.$$

Setting $v = c'$ we obtain

$$-\frac{v'}{v} = 2\frac{w'}{w} + \frac{(N-1)}{r},$$

namely

$$v(r) = \frac{1}{r^{(N-1)}w^2(r)} \approx C\frac{(r^2+1)^{(N-2)}}{r^{(N-1)}},$$

where C is a constant. This implies $c'(r) \approx r^{N-3}$ as well as

$$u(r) \approx C\frac{r^{N-2}}{(1+r^2)^{(N-2)/2}}$$

as $r \to \infty$. Hence $u \notin L^{p+1}$ and *a fortiori*, $u \notin \mathscr{D}^{1,2}(\mathbb{R}^N)$. Next, by derivation into the equation, we get that $z_\mu'(r)$ (i.e., the radial derivative) is a solution to (12) for $k=1$. Moreover, the same argument as in the case $k=0$ shows that a second linearly independent solution has the form

$$u(r) \approx Cr^N \frac{r}{(1+r^2)^{N/2}} \approx r \qquad \text{as} \quad r \to \infty.$$

Thus, $u \notin L^{p+1}$.

For $k \geqslant 1$ we set

$$A_k \psi \equiv -\psi'' - \frac{N-1}{r}\psi' + \frac{k(N+k-2)}{r^2}\psi - p z_\mu^{p-1}\psi.$$

Since A_1 has a solution z'_μ with constant sign in $r \in (0, \infty)$, then it is a *ground state* corresponding to the principal eigenvalue $\lambda = 0$. Then A_1 is a nonnegative operator.

Finally, if $k \geqslant 2$, we can write

$$A_k \psi = A_1 \psi + \frac{\delta_k}{r^2}\psi,$$

where $\delta_k = k(N+k-2) - (N-1) > 0$ if $k \geqslant 2$ and therefore A_k is a positive operator for $k \geqslant 2$. In particular, it follows that the problem corresponding to $k \geqslant 2$ has no solution. ∎

Remark 3.2. For future reference, let us point out that if we look for radial solutions we will work on $E_r = \{u \in \mathcal{D}^{1,2}(\mathbb{R}^N): u = u(|x|)\}$ and hence the critical manifold becomes $Z_r = \{\mu^{-(N-2)/2} z_0(x/\mu): \mu > 0\}$. In such a case $\dim(Z) = 1$ and Lemma 3.1 still holds. Indeed, the kernel of the second derivative is the space of solutions of the case $k = 0$.

Applying the abstract method we find the perturbed manifold

$$Z_\varepsilon = \{z_{\mu,\xi} + w(\varepsilon, \mu, \xi)\}, \qquad \|w(\varepsilon)\| = O(\varepsilon)$$

which is a natural constraint for f_ε. Moreover, according to the discussion carried out in Section 2, there results

$$f_\varepsilon(z_{\mu,\xi} + w(\varepsilon, z_{\mu,\xi})) = b - \varepsilon \Gamma(\mu, \xi) + o(\varepsilon), \qquad (\varepsilon \to 0),$$

where

$$\begin{aligned}(p+1)\,\Gamma(\mu, \xi) &= \int_{\mathbb{R}^N} K(x)\, z_{\mu,\xi}^{p+1}(x)\, dx \\ &= \mu^{-N} \int_{\mathbb{R}^N} K(x)\, z_0^{p+1}\left(\frac{x-\xi}{\mu}\right) dx \\ &= \int_{\mathbb{R}^N} K(\mu y + \xi)\, z_0^{p+1}(y)\, dy.\end{aligned}$$

Hereafter we will write $\Gamma(\mu, \xi)$ instead of $\Gamma(z_{\mu,\xi})$. Similarly, we will speak about critical points $q = (\mu, \xi)$ of Γ instead of $z_{\mu,\xi}$.

3.2. *Study of Γ*

We begin by proving some general properties of Γ. First of all, it is convenient to extend Γ by continuity to $\mu = 0$ for all fixed $\xi \in \mathbb{R}^N$ by setting,

$$\Gamma(0, \xi) = \frac{1}{p+1} \int_{\mathbb{R}^N} K(\xi)\, z_0(y)^{p+1}\, dy \equiv c_0 K(\xi),$$

where $c_0 = (1+p)^{-1} \int_{\mathbb{R}^N} z_0^{p+1}(y)\, dy$. Moreover, one has

$$D_\mu \Gamma(\mu, \xi) = \frac{1}{p+1} \int_{\mathbb{R}^N} \langle K'(\mu y + \xi), y \rangle\, z_0^{p+1}(y)\, dy.$$

Since $\int_{\mathbb{R}^N} y_i z_0^{p+1}(y)\, dy = 0$ for all $i = 1, ..., N$, then for the extended Γ one has

$$D_\mu \Gamma(0, \xi) = 0. \tag{13}$$

As a consequence we can further extend Γ by symmetry to $\mathbb{R}^{N+1}$ as a C^1 function. We will use the same symbol Γ for such a function.

Let us also explicitly remark that from (13) it follows

$$\xi \in \mathrm{Crit}(K) \Leftrightarrow (0, \xi) \in \mathrm{Crit}(\Gamma). \tag{14}$$

In Theorem 3.7 we will need the following lemma:

LEMMA 3.3. *Suppose that* $K \in L^\infty(\mathbb{R}^N) \cap C^1(\mathbb{R}^N)$ *satisfies*

$$\begin{cases} \text{(a)} \quad \exists \rho > 0 \colon \langle K'(x), x \rangle < 0 \qquad \forall\, |x| \geqslant \rho, \\ \text{(b)} \quad \langle K'(x), x \rangle \in L^1(\mathbb{R}^N), \qquad \int_{\mathbb{R}^N} \langle K'(x), x \rangle\, dx < 0. \end{cases} \tag{15}$$

Then there exists $R > 0$ *such that*

$$\langle \Gamma'(\mu, \xi), (\mu, \xi) \rangle < 0 \qquad \text{for all} \quad |\mu| + |\xi| \geqslant R.$$

Proof. Letting $q = (\mu, \xi)$, one has

$$\begin{aligned} (p+1)\langle \Gamma'(q), q \rangle &= \int_{\mathbb{R}^N} \langle K'(\mu y + \xi), \mu y + \xi \rangle\, z_0^{p+1}(y)\, dy \\ &= \mu^{-N} \int_{\mathbb{R}^N} \langle K'(x), x \rangle\, z_0^{p+1}\left(\frac{x - \xi}{\mu}\right) dx \\ &= \mu^{-N} [J_{1,R} + J_{2,R}], \end{aligned}$$

where

$$J_{1,R} = \int_{B_R} \langle K'(x), x \rangle \, z_0^{p+1}\left(\frac{x-\xi}{\mu}\right) dx$$

$$J_{2,R} = \int_{\mathbb{R}^N - B_R} \langle K'(x), x \rangle \, z_0^{p+1}\left(\frac{x-\xi}{\mu}\right) dx$$

and B_R denotes the ball of radius R in $\mathbb{R}^N$. From (15b), there exists R_0 such that if $R \geqslant R_0$ then

$$I(R) := \int_{B_R} \langle K'(x), x \rangle \, dx < 0.$$

Set $g(x) = \langle K'(x), x \rangle$, $g_+(x) = \max\{g(x), 0\}$, $g_-(x) = \max\{-g(x), 0\}$ and

$$\mathrm{Max}(\mu, \xi) := \max_{x \in B_R} z_0^{p+1}\left(\frac{x-\xi}{\mu}\right),$$

$$\mathrm{Min}(\mu, \xi) := \min_{x \in B_R} z_0^{p+1}\left(\frac{x-\xi}{\mu}\right).$$

One has

$$\begin{aligned} J_{1,R} &= \int_{B_R} g_+(x) \, z_0^{p+1}\left(\frac{x-\xi}{\mu}\right) dx - \int_{B_R} g_-(x) \, z_0^{p+1}\left(\frac{x-\xi}{\mu}\right) dx \\ &\leqslant \mathrm{Max}(\mu, \xi) \cdot \int_{B_R} g_+(x) \, dx - \mathrm{Min}(\mu, \xi) \cdot \int_{B_R} g_-(x) \, dx. \end{aligned}$$

For $|\mu| + |\xi|$ large there results

$$\mathrm{Max}(\mu, \xi) \simeq \frac{C_N \mu^{2N}}{(\mu^2 + (R - |\xi|)^2)^N}$$

and

$$\mathrm{Min}(\mu, \xi) \simeq \frac{C_N \mu^{2N}}{(\mu^2 + (R + |\xi|)^2)^N}.$$

Hence

$$\lim_{\mu + |\xi| \to \infty} \frac{\mathrm{Max}(\mu, \xi)}{\mathrm{Min}(\mu, \xi)} = 1,$$

and $I(R)<0$ implies that $J_{1,R}<0$ provided $\mu+|\xi|$ is large enough. Moreover, by (15a) $|x|\geqslant\rho\Rightarrow\langle K'(x), x\rangle<0$. In conclusion, if $|q|\geqslant R=\max\{R_0,\rho\}$, one has that $\langle\Gamma'(q), q\rangle<0$ and the proof is completed. ∎

We conclude this subsection by showing some lemmas concerning the behaviour of Γ near the critical points of the type $(0,\xi)$. Let

$$c_1=\frac{1}{N(p+1)}\int_{\mathbb{R}^N}|y|^2\,z_0^{p+1}(y)\,dy. \tag{16}$$

Lemma 3.4. *Suppose that $K\in L^\infty(\mathbb{R}^N)\cap C^2(\mathbb{R}^N)$. There results*

$$D^2_{\mu,\xi_i}\Gamma(0,\xi)=0, \qquad \forall i=1,\ldots,N, \tag{17}$$

$$D^2_{\mu,\mu}\Gamma(0,\xi)=c_1\Delta K(\xi). \tag{18}$$

Proof. Formula (17) follows immediately from (13). For the latter, one has by a straight computation

$$D^2_{\mu,\mu}\Gamma(\mu,\xi)=\frac{1}{p+1}\int_{\mathbb{R}^N}\sum D^2_{ij}K(\mu y+\xi)\;y_i y_j z_0^{p+1}(y)\,dy.$$

Since

$$\int_{\mathbb{R}^N}y_i y_j z_0^{p+1}(y)\,dy=0\Leftrightarrow i\neq j,$$

the lemma follows. ∎

More in general, one has

Lemma 3.5. *Given $\xi\in\mathbb{R}^N$, suppose there exists $\beta=\beta_\xi$, $1<\beta<N$, and $Q_\xi\colon\mathbb{R}^N\to\mathbb{R}$ such that*

$$\begin{cases}\text{(a)} & Q_\xi(\lambda x)=\lambda^\beta Q_\xi, & \forall\lambda\geqslant 0,\\ \text{(b)} & K(x)=K(\xi)+Q_\xi(x-\xi)+o(|x-\xi|^\beta), & \textit{as}\quad x\to\xi\end{cases} \tag{19}$$

and let

$$A_\xi=\frac{1}{p+1}\int_{\mathbb{R}^N}Q_\xi(y)\,z_0^{p+1}(y)\,dy.$$

Then

$$\lim_{\mu\to 0^+}\frac{\Gamma(\mu,\xi)-\Gamma(0,\xi)}{\mu^\beta}=A_\xi. \tag{20}$$

Proof. Remark that $Q_\xi z_0^{p+1} \in L^1(\mathbb{R}^N)$ provided $\beta < N$. By (19) we immediately find that the above limit equals

$$\frac{1}{p+1}\int_{\mathbb{R}^N} \frac{Q_\xi(\mu y)}{\mu^\beta}\, z_0^{p+1}(y)\, dy = \frac{1}{p+1}\int_{\mathbb{R}^N} Q_\xi(y)\, z_0^{p+1}(y)\, dy$$

and the result follows. ∎

From Lemma 3.5 we infer:

LEMMA 3.6. *Let $\xi \in \mathrm{Crit}(K)$ be isolated and suppose*

(*) *there exist $\beta = \beta_\xi \in]1, N[$ and $Q_y \colon \mathbb{R}^N \to \mathbb{R}$, depending continuously on y locally near ξ, such that $A_\xi \neq 0$ and there results*

$$Q_y(\lambda x) = \lambda^\beta Q_y(x), \qquad \forall \lambda \geqslant 0,$$

$$K(x) = K(y) + Q_\xi(x - y) + o(|x - y|^\beta), \qquad \text{as} \quad x \to y.$$

Then $q = (0, \xi)$ is an isolated critical point of Γ and there results

$$A_\xi > 0 \Rightarrow \deg_{loc}(\Gamma', q) = \deg_{loc}(K', \xi)$$

$$A_\xi < 0 \Rightarrow \deg_{loc}(\Gamma', q) = -\deg_{loc}(K', \xi).$$

Proof. That $q \in \mathrm{Crit}(\Gamma)$ has been pointed out in (14). From the assumptions it follows that $\exists \delta > 0$ such that $A_y \neq 0$ for all $y \in B(\xi, \delta)$. From (20) one infers that $\Gamma(\mu, y) \sim \Gamma(0, y) + A_y \mu^\beta$ for $y \in B(\xi, \delta)$. This and the fact that ξ is isolated implies that q does. Let $T_\delta = [-\delta, \delta] \times B(\xi, \delta)$. For $\delta > 0$ small the degree $\deg(\Gamma', T_\delta, 0)$ is well defined and the multiplicative property yields

$$\deg(\Gamma', T_\delta, 0) = \deg_{loc}(K', \xi) \cdot \deg_{loc}(D_\mu \Gamma, 0),$$

where $D_\mu \Gamma$ denotes the map $\mu \mapsto D_\mu \Gamma(\mu, \xi)$. Using again (20) we infer that $\deg_{loc}(D_\mu \Gamma, 0) = 1$, resp. -1, if $A_\xi > 0$, resp. $A_\xi < 0$, and the lemma follows. ∎

3.3. *Main Existence Results*

Let K satisfy the following conditions:

Assumption (K1). (K1.a) $K \in L^\infty(\mathbb{R}^N) \cap C^1(\mathbb{R}^N)$ satisfies (15);

(K1.b) K has finitely many critical points;

(K1.c) for all $\xi \in \mathrm{Crit}(K)$, (*) holds;

(K1.d) $\sum_{A_\xi < 0} \deg_{loc}(K', \xi) \neq (-1)^N$.

THEOREM 3.7. *Let* $(K1)$ *hold. Then for* ε *small* (2) *has a (positive) solution* $u_\varepsilon \in \mathcal{D}^{1,2}(\mathbb{R}^N)$.

Proof. Let $R \geqslant \rho$, where ρ is defined in (15). We set $B_R^N = \{x \in \mathbb{R}^N : |x| < R\}$. Since $\langle K'(x), x \rangle < 0$ for all $|x| = R (R \geqslant \rho)$, then one immediately has

$$\deg(K', B_R^N, 0) = (-1)^N.$$

This and the properties of the topological degree yield

$$\sum_{\xi \in \mathrm{Crit}(K)} \deg_{loc}(K', \xi) = (-1)^N.$$

Since $A_\xi \neq 0$ for all $\xi \in \mathrm{Crit}(K)$, we can also write

$$(-1)^N = \sum_{A_\xi > 0} \deg_{loc}(K', \xi) + \sum_{A_\xi < 0} \deg_{loc}(K', \xi). \tag{21}$$

Let $\mathscr{C}^+$ denote the set of $(\mu, \xi) \in \mathrm{Crit}(\Gamma)$ such that $\mu > 0$. According to Lemmas 3.3 and 3.5, $\mathscr{C}^+$ is a (possibly empty) compact set. Since the extended Γ is even in μ, then also $\mathscr{C}^- = \{(-\mu, \xi) : (\mu, \xi) \in \mathscr{C}^+\}$ consists of critical points of Γ. We claim:

LEMMA 3.8. *There is a bounded open set* $\Omega \subset]0, \infty) \times \mathbb{R}^N$ *with* $\mathscr{C}^+ \subset \Omega$ *such that*

$$\deg(\Gamma', \Omega, 0) \neq 0.$$

Proof. Using Lemma 3.3 we infer

$$\deg(\Gamma', B_R^{N+1}, 0) = (-1)^{N+1}.$$

By contradiction, take an open bounded set Ω with $\mathscr{C}^+ \subset \Omega \subset]0, \infty) \times \mathbb{R}^N$ and such that $\deg(\Gamma', \Omega, 0) = 0$. Let $\Omega^- = \{(-\mu, \xi) : (\mu, \xi) \in \Omega\}$ and set $\Omega' = \Omega \cup \Omega^-$. Of course one has that $\deg(\Gamma', \Omega^-, 0) = 0$ and hence

$$(-1)^{N+1} = \deg(\Gamma', B_R^{N+1} \backslash \bar{\Omega}', 0). \tag{22}$$

According to (14) any $q \in \mathrm{Crit}(\Gamma) \backslash \mathscr{C}$ has the form $q = (0, \xi)$, with $\xi \in \mathrm{Crit}(K)$. Using Lemma 3.6 we infer that

$$\begin{aligned}\deg(\Gamma', B_B^{N+1} \backslash \bar{\Omega}, 0) &= \sum \deg_{loc}(\Gamma', q) \\ &= \sum_{A_\xi > 0} \deg_{loc}(K', \xi) - \sum_{A_\xi < 0} \deg_{loc}(K', \xi).\end{aligned}$$

Then (22) becomes

$$(-1)^{N+1} = \sum_{A_\xi > 0} \deg_{loc}(K', \xi) - \sum_{A_\xi < 0} \deg_{loc}(K', \xi). \tag{23}$$

Putting together (21) and (23) we find

$$\sum_{A_\xi < 0} \deg_{loc}(K', \xi) = (-1)^N,$$

a contradiction with (K1.d). ▮

Proof of Theorem 3.7 *completed.* Lemma 3.8 allows us to apply Theorem 2.1(iii) jointly with Remark 2.2 and this completes the proof. ▮

The following Corollary shows that Theorem 3.7 covers the cases discussed in [5, 20, 21].

COROLLARY 3.9. *Problem* (2) *has a* (*positive*) *solution* $u_\varepsilon \in \mathcal{D}^{1,2}(\mathbb{R}^N)$ *for* $|\varepsilon|$ *small provided K satisfies* (K1.a), (K1.b), *and one of the following conditions*:

(K2) $\Delta K(\xi) \neq 0$ *for each* $\xi \in \mathrm{Crit}(K)$ *and*

$$\sum_{\Delta K(\xi) < 0} \deg_{loc}(K', \xi) \neq (-1)^N.$$

(K2′) $\forall \xi \in \mathrm{Crit}(K)\ \exists \beta \in]1, N[$ *and* $a_j \in C(\mathbb{R}^N)$, *with* $\tilde{A}_\xi := \sum a_j(\xi) \neq 0$ *and such that* $K(x) = K(\eta) + \sum a_j |x-\eta|^\beta + o(|x-\eta|^\beta)$ *as* $x \to \eta$, *for all* η *locally near* ξ. *Moreover there results*

$$\sum_{\tilde{A}_\xi < 0} \deg_{loc}(K', \xi) \neq (-1)^N.$$

Proof. (K2) This is essentially the case handled in [5] (where it is taken $N=3$ and it is also assumed that K is Morse). It suffices to take $\beta = 2$ and $Q_\xi = D^2_{jk} K(\xi)(x-\xi)^2$. As in Lemma 3.4 one finds $A_\xi = c_1 \Delta K(\xi)$. Hence $A_\xi \neq 0$ and Theorem 3.7 applies.

(K2′) This is the case discussed in [20, 21]. However, it is worth mentioning that in those papers it is also assumed that $a_j \neq 0$ for all j. Here one finds

$$A_\xi = \sum a_j(\xi) \cdot \int_{\mathbb{R}^N} |y_1|^\beta z_0^{p+1}(y)\, dy$$

and Theorem 3.7 applies. ▮

Another example in which Theorem 3.7 applies is when

$$K(x) = K(\xi) + \sum a_b\,|x-\eta|^b + o(|x-\eta|^\beta) \qquad \text{as} \quad x \to \eta,$$

where $b=(b_1, ..., b_N)$ is a multi-index and $\beta = |b| \in]1, N[$. In such a case one finds, as in Lemma 3.4, that

$$\frac{\partial^\beta}{\partial\mu^\beta}\,\Gamma(0,\xi) = \frac{\beta!}{p+1}\sum a_b(\xi)\cdot\int_{\mathbb{R}^N} y^b z_0^{p+1}(y)\,dy.$$

Hence, setting

$$C_b = \frac{\beta!}{p+1}\int_{\mathbb{R}^N} y^b z_0^{p+1}(y)\,dy,$$

and letting $\mathscr{B}$ denote the set of b whose components are all even integers, there results

$$\frac{\partial^\beta}{\partial\mu^\beta}\,\Gamma(0,\xi) = \sum_{b\in\mathscr{B}} a_b(\xi)\,C_b.$$

Then, assuming that $A_\xi^* := \sum_{b\in\mathscr{B}} a_b(\xi)\,C_b \neq 0$, condition (K1.d) becomes

$$\sum_{A_\xi^*<0} \deg_{loc}(K', \xi) \neq (-1)^N.$$

Remarks 3.10. (i) Obviously there exists a natural counterpart of the first statement of Corollary 3.9, when we take the reverse inequalities in (15) and the corresponding condition $\sum_{\Delta K(\xi)>0} (-1)^{m(K,\xi)} \neq 1$.

(ii) A condition like (15) is somewhat needed. Actually, if $K=K(r)$ is radial, $\int_0^r s^N K'(s)\,ds \geqslant 0$ for all $r\geqslant 0$ and $\exists R>0$ such that $\int_0^r s^N K'(s)\,ds \geqslant \int_0^R s^N K'(s)\,ds > 0$ for all $r>R$, then (2) does not have any positive radial solution on $\mathbb{R}^N$, see Theorem 5.13 of [14].

(iii) Suppose that Γ has a nondegenerated critical point $\bar{z}=(\bar{\mu},\bar{\xi})$ with $\bar{\mu}>0$ and let u_ε be the critical point of f_ε obtained by using Theorem 2.1. Then we can use Remark 2.3 to evaluate the Morse index of u_ε. Actually, one has that $m(f_0, z_0)=1$ (z_0 can be found by means of the m-p procedure); if, for example, $\bar{z}$ is also a mountain pass critical point then one finds:

$$\varepsilon<0 \Rightarrow m(f_\varepsilon, u_\varepsilon) = 1 + m(\Gamma, \bar{z}) = 2$$

$$\varepsilon>0 \Rightarrow m(f_\varepsilon, u_\varepsilon) = 1 + m(-\Gamma, \bar{z}) = 1 + N.$$

(iv) The solutions u_ε satisfy $u_\varepsilon(x) = O(|x|^{2-N})$ as $|x| \to \infty$. This implies that the corresponding solutions of (3), through the stereographic projection, are smooth.

We conclude this section by showing how condition (K1.b) can be dropped. The arguments are quite similar to those discussed so far and thus we will be sketchy.

Suppose that for all $\xi \in \mathrm{Crit}(K)$ condition (*) holds and set

$$\mathscr{K}^+ = \{\xi \in \mathrm{Crit}(K) : A_\xi > 0\}, \qquad \mathscr{K}^- = \{\xi \in \mathrm{Crit}(K) : A_\xi < 0\}$$

Recall that, by (15) and (*), $\mathrm{Crit}(K) = \mathscr{K}^+ \cap \mathscr{K}^-$ is compact and let $U^\pm$ be neighbourhoods of $\mathscr{K}^\pm$ such that $\deg(K', U^\pm, 0)$ is well defined.

THEOREM 3.11. *Let K satisfy* (K1.a) *and suppose that for all $\xi \in \mathrm{Crit}(K)$ condition* (*) *holds and that*

$$\deg(K', U^-, 0) \neq (-1)^N.$$

Then for $|\varepsilon|$ small problem (25) *has a positive solution $u_\varepsilon \in \mathscr{D}^{1,2}(\mathbb{R}^N)$.*

Proof. Let $\mathscr{C}_0^\pm = \{(0, \xi) \in \mathbb{R}^N : \xi \in \mathscr{K}^\pm\}$. As in Lemma 3.6 one shows that there exist neighbourhoods $U^\pm$ of $\mathscr{C}_0^\pm$ such that

$$\deg(\Gamma', V^+, 0) = \deg(K', U^+, 0), \qquad \deg(\Gamma', V^+, 0) = -\deg(K', U^-, 0).$$

Repeating the arguments used to prove Theorem 3.7 the result follows. ∎

4. THE SCALAR CURVATURE PROBLEM: THE RADIAL CASE

In this section we deal with the scalar curvature problem in the case in which K is radial. We will see that in such a case assumption (K1) can be greatly relaxed. Precisely, we will assume

(K3) $K \in L^\infty(\mathbb{R}^N) \cap C^1(\mathbb{R}^N)$, $K(x) = K(r)$, $r = |x|$, and $r^{-\alpha} K(r) \in L^1([1, \infty), r^{N-1}\, dr)$, for some $\alpha < N$.

In this section, we shall work in the space $\mathscr{D}_r^{1,2}$ of radial functions of $\mathscr{D}^{1,2}(\mathbb{R}^N)$ and hence we shall consider the critical manifold associated to the unperturbed problem

$$Z_r = \left\{ z_\mu \equiv \mu^{-(N-2)/2} z_0\left(\frac{r}{\mu}\right) \,\middle|\, \mu > 0 \right\} \approx \mathbb{R}^+.$$

It is easy to check that the arguments discussed in the previous section can be repeated here to show that the abstract setting applies; see Remark 3.2. In particular, one has

$$f_\varepsilon(z_\mu + w(\varepsilon, \mu)) = b - \varepsilon \omega_N \Gamma_r(\mu) + o(\varepsilon),$$

where $b = f_0(z_\mu)$, ω_N is the measure of the unit $(N-1)$-sphere and

$$(p+1)\,\Gamma_r(\mu) = \int_0^\infty K(r)\, z_\mu^{p+1}(r)\, r^{N-1}\, dr = \int_0^\infty K(\mu r)\, z_0^{p+1}(r)\, r^{N-1}\, dr.$$

Next we show:

LEMMA 4.1. *If* (K3) *holds then* $\Gamma_r(\mu) \to 0$ *as* $\mu \to +\infty$.

Proof. We have (below a_1, a_2, a_3 denote positive constants depending on N and α only)

$$\begin{aligned}(p+1)\,\Gamma_r(\mu) &= a_1 \int_0^\infty K(r)\, \frac{\mu^N}{(\mu^2+r^2)^N}\, r^{N-1}\, dr \\ &\leqslant a_2 \mu^{-N} \int_0^1 K(r)\, r^{N-1}\, dr + a_3 \mu^{\alpha-N} \int_1^\infty \frac{K(r)}{r^\alpha}\, r^{N-1}\, dr.\end{aligned}$$

Since $r^{-\alpha} K(r) \in L^1([1,\infty), r^{N-1} dr)$ and $\alpha < N$, the result follows. ∎

Furthermore, as before we have:

(i) Γ_r can be extended by continuity to $\mu = 0$ by setting $(p+1)\,\Gamma_r(0) = K(0) \cdot \int_0^\infty z_0^{p+1}(r)\, r^{N-1}\, dr$;

(ii) $\Gamma_r'(0) = 0$ and hence Γ_r can be further extended to $\mathbb{R}$ by symmetry.

We first deal with the case that $K(0) = 0$, when the following general result holds true.

THEOREM 4.2. *Let* (K3) *hold and suppose that* $K(0) = 0$ *and that* $K \not\equiv 0$. *Then for* $|\varepsilon|$ *small* (2) *has a positive radial solution* $u_\varepsilon \in \mathscr{D}_r^{1,2}$.

Proof. We claim that $\Gamma(\mu)$ is not identically equal to 0. To prove this fact we will use Fourier analysis. Let us introduce some notation. If $g \in L^1([0,\infty), dr/r)$, we define

$$\mathscr{M}[g](s) = \int_0^\infty r^{-is} g(r)\, \frac{dr}{r}.$$

$\mathscr{M}$ is nothing but the Mellin transform, see [16]. The associated convolution is defined by

$$(g \times h)(s) = \int_0^\infty g(r)\, h\left(\frac{s}{r}\right) \frac{dr}{r},$$

There results $\mathscr{M}[g \times h] = \mathscr{M}[g] \cdot \mathscr{M}[h]$. With this notation we can write our Γ in the form

$$\Gamma(\mu) = \int_0^\infty K(r)\, z_0^{p+1}\left(\frac{r}{\mu}\right)\left(\frac{r}{\mu}\right)^N \frac{dr}{r}.$$

We set $m = N - \alpha$ and

$$g(r) = K(r)\, r^m, \qquad h(r) = z_0^{p+1}\left(\frac{1}{r}\right)\left(\frac{1}{r}\right)^{N-m}.$$

Remark that $g, h \in L^1([0, \infty), dr/r)$. There results $\Gamma(\mu) = \mu^{-m}(g \times h)(\mu)$ and hence if, by contradiction, $\Gamma \equiv 0$ then $g \times h \equiv 0$ and one has

$$\mathscr{M}[g] \cdot M[h] = \mathscr{M}[g \times h] \equiv 0.$$

On the other hand, $\mathscr{M}[h]$ is real analytic and so has a discrete number of zeros. By continuity it follows that $\mathscr{M}[g] \equiv 0$. Then g and hence K are identically equal to 0, a contradiction proving the claim. Since $\Gamma(0) = 0$, $\lim_{\mu \to \infty} \Gamma(\mu) = 0$ and $\Gamma \not\equiv 0$, it follows that Γ has a maximum or a minimum at some $\bar{\mu} \neq 0$. By a straight application of Theorem 2.1 jointly with Remark 2.2 the result follows. ∎

We now consider the case that $K(0) \neq 0$. First, letting $K \in C^2$ we find as in the previous section,

$$\text{(iii)} \quad \Gamma_r''(0) = \frac{1}{p+1} K''(0) \cdot \int_0^\infty z_0^{p+1}(r)\, r^{N+1}\, dr.$$

In addition, using arguments similar to those carried out in the proof of Lemmas 4.1 and 3.3 one can show:

Lemma 4.3. *Suppose that*

$$K'(r)\, r \in L^1((0, \infty), r^{N-1}\, dr) \tag{24}$$

and let

$$\kappa := \int_0^\infty K'(r)\, r^N\, dr.$$

Then, if $\kappa < 0$, *resp.* > 0, *one has that* $\Gamma_r'(\mu) \to 0^-$, *resp.* 0^+, *as* $\mu \to +\infty$.

Theorem 4.4. *Let* (K3) *hold. Then for* $|\varepsilon|$ *small* (2) *has a positive radial solution* $u_\varepsilon \in \mathscr{D}_r^{1,2}$ *provided one of the following conditions is satisfied*:

(a) $K \in C^2(\mathbb{R}^N)$ and $K(0) \cdot K''(0) > 0$;

(b) (24) *holds and* $K(0) \cdot \kappa > 0$.

Proof. (a) Let us suppose that, for example, $K(0) > 0$. Then (i) yields $\Gamma_r(0) > 0$. If $K''(0) > 0$ then (iii) implies that $\Gamma_r''(0) > 0$. Using also Lemma 4.1 it follows that Γ_r has a (global) maximum at some $\bar{\mu} > 0$ and (a) follows.

(b) On the other hand, if (24) holds and, say $K(0)<0$ as well as $\kappa<0$, we use Lemma 4.3 to infer that $\Gamma_r'(\mu)\to 0^-$ as $\mu\to+\infty$. Then Γ has still a maximum at some $\bar{\mu}>0$ and (b) follows. ∎

In the following result we make an assumption on

$$\gamma := C_N \int_{\mathbb{R}^N} K(|x|)(1+|x|^2)^{-N}\, dx \qquad \text{and/or}$$

$$\gamma' := C_N \int_0^\infty K'(r)(1+r^2)^{-N} r^N\, dr.$$

THEOREM 4.5. *Let* (K3) *hold. Then for* $|\varepsilon|$ *small* (2) *has a positive radial solution* $u_\varepsilon \in \mathscr{D}_r^{1,2}$ *provided one of the following conditions is satisfied*:

(a) $\gamma\neq 0$ *and* $\gamma\cdot K(0)\leqslant 0$;

(b) $\gamma=0$ *and* $\gamma'\neq 0$.

Furthermore, in case (b) *there exists a second positive radial solution* $v_\varepsilon \in \mathscr{D}_r^{1,2}$ *provided* $\gamma'\cdot K(0)\geqslant 0$.

Proof. (a) It suffices to observe that there results $\Gamma_r(1)=\gamma/(p+1)$ and hence by (i), Γ_r has a maximum or a minimum at some $\bar{\mu}>0$.

(b) Since $\Gamma_r'(1)=\gamma'$ then Γ_r has a maximum or a minimum in $(1,\infty)$. If, in addition, $\gamma'\cdot K(0)\geqslant 0$ then Γ_r has another maximum or minimum in $(0,1)$ yielding a second solution. ∎

Remarks 4.6. (i) Likewise in the preceding section we can substitute the assumption that $K''(0)>0$ with weaker ones.

(ii) It is clear that one could state other possible existence results, in the same spirit of the preceding theorems. For example, if K is bounded, $\gamma=0$, $\gamma'\neq 0$, and $\gamma'\cdot K(0)\geqslant 0$ then (2) has a positive radial solution (namely, $r^{-\alpha}K\in L^1$ is unnecessary in such a case). Similarly, if in Theorem 4.5 one has that $K\in C^2$, $\gamma'=0$ and $\Gamma_r''(1)=\int_0^\infty K''(r)(1+r^2)^{-N} r^{N+1}\, dr\neq 0$ then $\mu=1$ is a local minimum or maximum for Γ_r which gives rise to a solution of (2).

(iii) Let us make a comparison with [6, 7]. The former deals with the radial case only, but possibly not perturbative. In the perturbative case our Theorem 4.2 improves Theorem 0.1 of [6], because we do not need to assume here that $K(\infty):=\lim_{r\to\infty} K(r)$ exists. Likewise Theorem 0.2 of [6] is essentially covered by our Theorem 4.5(a). The remaining results of [6] require various kind of conditions involving integrals such as γ. Unlike Theorem 4.5, these conditions are made in [6] jointly with further assumptions on the behaviour and decay of $K(r)$ at $r=0$ and $r=\infty$.

As anticipated in the Introduction, the solutions are found in [6] as constrained critical points, either minima or m-p. The former ones correspond to the case that our Γ_r has a minimum, the latter to a maximum

($\varepsilon > 0$). In particular, these critical points have Morse index $\leqslant 2$. According to Remark 3.10(iii), this highlights that such a procedure cannot be extended to handle the non radial case with $\varepsilon > 0$ where the critical points that give rise to solutions of (2) have a larger Morse index.

As for [7], it deals with the perturbative, radial case. Roughly, the results proved therein have the following form: if $\mu_0 > 0$ is such that $\Gamma_r'(\mu_0) = 0$ and $\Gamma_r''(\mu_0) \neq 0$ then (2) has a solution (of course, we are using our notation). The condition $\Gamma_r'(\mu_0) = 0$ is deduced from the Pohozaev identity, thanks to the homogeneity of the problem. However, this kind of result is nothing but a particular case of the abstract Theorem 2.1. Indeed, as we have shown before, the condition $\Gamma_r''(\mu_0) \neq 0$ is, in general, not necessary. Let us also mention Theorem 3.2 of [7] dealing with perturbative problems with nonradial K. This result is also a particular case of Theorem 2.1. However, as we have stressed before, critical points of Γ with $\mu = 0$ do not give rise, in general, to solutions of (2), and this crucial point is not discussed in [7].

In the next result we consider the case in which, instead of (K3), we suppose that $K(r)$ is periodic, namely

(K3′) $K \in C^2(\mathbb{R}^N)$, $K(x) = K(r)$, $K(r)$ is T-periodic and $\int_0^T K(r)\, dr = 0$.

Hypothesis (K3′) allows us to use the following Riemann–Lebesgue convergence result.

LEMMA 4.7. *Let $Q = [0, T]^N$ be a cube in $\mathbb{R}^N$, and $f \in L^q(Q)$ be a T-periodic function. Consider $f_\mu(x) = f(\mu x)$, then*

$$f_\mu \rightharpoonup \bar{f} = \frac{1}{|Q|} \int_Q f\, dx, \qquad \text{weakly in } L^q_{loc}(\mathbb{R}^N), \quad \text{as } \mu \to +\infty$$

LEMMA 4.8. *If* (K3′) *holds, then*

$$\Gamma_r(\mu) \to 0 \qquad \mu \to +\infty$$

Proof. Given $\varepsilon > 0$, there exists $R > 0$ large enough such that

$$\left| \frac{1}{p+1} \int_R^\infty K(r)\, z_\mu^{p+1}(r)\, r^{N-1}\, dr \right| \leqslant \frac{1}{p+1} \|K(r)\|_\infty \int_R^\infty z_\mu^{p+1}(r)\, r^{N-1}\, dr < \varepsilon.$$

On the other hand, the remainder integral over the interval $0 \leqslant r < R$ tends to 0 as $\mu \to \infty$ because of hypothesis (K3′) and the Riemann–Lebesgue lemma. ∎

THEOREM 4.9. *Let K satisfy* (K3′) *and condition* (a) *of Theorem* 4.4. *Then the same conclusion holds true.*

Proof. It suffices to repeat the arguments used to prove (a) of Theorem 4.4 using Lemma 4.8 instead of Lemma 4.3. ∎

Remark 4.10. Functions $K(x)$ which are periodic in one variable have been considered in [20, 21]. However, those results require an additional nondegeneracy condition that is not needed here.

5. EXISTENCE RESULTS FOR PROBLEM (4), I

In these last two sections we will study problem (6). First we deal with the case that $q=1$ that requires a restriction on the dimension. Precisely, let us consider the equation

$$-\Delta u=\varepsilon h(x)\,u+[1+\varepsilon K(x)]\,u^p, \qquad x\in\mathbb{R}^N, \quad N>4. \tag{25}$$

We will suppose that (K1a and b) holds and that h satisfies:

(h1.a) $h\in C^2(\mathbb{R}^N)$ and $\Sigma:=\operatorname{supp}\{h\}$ is compact;

(h1.b) $\langle h'(x), x\rangle\in L^1(\mathbb{R}^N)$ and $\langle h'(x), x\rangle\leqslant 0$.

We can repeat the general argument with

$$\hat{f}_\varepsilon(u)=f_0(u)-\varepsilon\hat{G}(u), \qquad u\in\mathscr{D}1, 2(\mathbb{R}^N),$$

where the perturbation is given by

$$\hat{G}(u)=\frac{1}{p+1}\int_{\mathbb{R}^N}K(x)\,u_+^{p+1}+\frac{1}{2}\int_\Sigma h(x)\,u^2.$$

Let us point out that $\hat{f}_\varepsilon$ is of class C^2. By the same kind of arguments as above we find

$$\hat{f}_\varepsilon|_{Z_\varepsilon}=b-\varepsilon\hat{\Gamma}(\mu,\xi)+o(\varepsilon), \tag{26}$$

where

$$\hat{\Gamma}=\frac{\mu^{-N}}{p+1}\int_{\mathbb{R}^N}K(x)\,z_0^{p+1}\left(\frac{x-\xi}{\mu}\right)dx+\frac{1}{2}\mu^{2-N}\int_\Sigma h(x)\,z_0^2\left(\frac{x-\xi}{\mu}\right)dx,$$

or, changing variables,

$$\hat{\Gamma}(\mu,\xi)=\frac{1}{p+1}\int_{\mathbb{R}^N}K(\mu y+\xi)\,z_0^{p+1}(y)\,dy+\frac{1}{2}\mu^2\int_{\mathbb{R}^N}h(\mu y+\xi)\,z_0^2(y)\,dy.$$

Let

$$c_2 = \int_{\mathbb{R}^N} z_0^2(y)\, dy.$$

Notice that $c_2 < \infty$ if and only if $N > 4$.

LEMMA 5.1. *Let* (K1.a *and* b) *hold and suppose* $h \in L^1(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$. *Then there results*

$$\lim_{\mu \to 0^+} \hat{\Gamma}(\mu, \xi) = \Gamma(0, \xi) = \frac{1}{p+1} \int_{\mathbb{R}^N} K(\xi)\, z_0^{p+1}(y)\, dy.$$

If, in addition, $h \in C^2(\mathbb{R}^N)$ *one has*:

$$\lim_{\mu \to 0^+} D_\mu \hat{\Gamma}(\mu, \xi) = 0, \qquad \lim_{\mu \to 0^+} D^2_{\mu, \xi_i} \hat{\Gamma}(\mu, \xi) = 0,$$

$$\lim_{\mu \to 0^+} D^2_{\mu, \mu} \hat{\Gamma}(\mu, \xi) = c_1 \Delta K(\xi) + c_2 h(\xi). \tag{27}$$

Moreover, if (h1.b) *holds then there exists* $R > 0$ *such that for all* $q = (\mu, \xi)$, $|q| \geq R$ *there results*

$$\langle \hat{\Gamma}'(q), q \rangle < 0. \tag{28}$$

Proof. Let

$$\Phi(\mu, \xi) = \tfrac{1}{2} \mu^2 \int_{\mathbb{R}^N} h(\mu y + \xi)\, z_0^2(y)\, dy$$

so that $\hat{\Gamma}(\mu, \xi) = \Gamma(\mu, \xi) + \Phi(\mu, \xi)$. One has

$$|\Phi(\mu, \xi)| \leq \tfrac{1}{2} \mu^2 c_2\, \|h\|_\infty \to 0,$$

proving the first statement. One also has

$$D_\mu \Phi(\mu, \xi) = \mu \int_{\mathbb{R}^N} h(\mu y + \xi)\, z_0^2(y)\, dy + \tfrac{1}{2} \mu^2 \int_{\mathbb{R}^N} \langle h'(\mu y + \xi), y \rangle\, z_0^2(y)\, dy$$

and, as before, we find

$$\lim_{\mu \to 0^+} D_\mu \Phi(\mu, \xi) = 0.$$

Similarly one shows that $\lim_{\mu \to 0^+} D^2_{\mu, \xi_i} \Phi(\mu, \xi) = 0$, proving the second statement. As for (27) one has

$$D^2_{\mu, \mu} \Phi(\mu, \xi) = \int_{\mathbb{R}^N} h(\mu y + \xi)\, z_0^2(y)\, dy + 2\mu \int_{\mathbb{R}^N} \langle h(\mu y + \xi), y \rangle\, z_0^2(y)\, dy$$

$$+ \tfrac{1}{2} \mu^2 \int_{\mathbb{R}^N} \langle h''(\mu y + \xi)\, y, y \rangle\, z_0^2(y)\, dy.$$

The last two integrals tend to 0 as $\mu \to 0^+$ and since $z_0 \in L^1$ one finds

$$\lim_{\mu \to 0^+} D^2_{\mu,\mu} \Phi(\mu, \xi) = c_2 h(\xi),$$

and (27) follows.

Finally, as for (28), there results

$$\langle \hat{\Gamma}'(q), q \rangle = \langle \hat{\Gamma}'(q), q \rangle + \tfrac{1}{2} \mu^2 \int_{\mathbb{R}^N} \langle h'(\mu y + \xi), \mu y + \xi \rangle \, z_0^2(y) \, dy.$$

Using (h1.b) one has that

$$\langle \hat{\Gamma}'(q), q \rangle \leqslant \langle \Gamma'(q), q \rangle,$$

and the result follows from Lemma 3.3. ∎

The previous lemma allows us to extend, as in the previous section, by continuity and symmetry, $\hat{\Gamma}$ to $\mu \leqslant 0$.

In the following, we deal, for simplicity, with K satisfying conditions which are the counterpart of assumption (K2) in Corollary 3.9. It is easy to check that the more general case related to assumption (K1.c and d) could also be handled. In view of (27) we set

$$X^+ = \{\xi \in \mathrm{Crit}(K) \colon c_1 \Delta K(\xi) + c_2 h(\xi) > 0\},$$
$$X^- = \{\xi \in \mathrm{Crit}(K) \colon c_1 \Delta K(\xi) + c_2 h(\xi) < 0, \}$$

where c_1 is defined in (16).

Theorem 5.2. *Let* (K1.a *and* b) *and* (h1) *hold and suppose that* $c_1 \Delta K(\xi) + c_2 h(\xi) \neq 0$ *for all* $\xi \in \mathrm{Crit}(K)$. *Furthermore, assume*

$$\sum_{\xi \in X^-} \deg_{loc}(K', \xi) \neq (-1)^N. \tag{29}$$

Then for $|\varepsilon|$ *small problem* (25) *has a positive solution* $u_\varepsilon \in \mathcal{D}^{1,2}(\mathbb{R}^N)$.

Proof. As in Lemma 3.6 one shows that $q = (0, \xi)$ is an isolated critical point of Γ and there results

$$\xi \in X^+ \Rightarrow \deg_{loc}(\hat{\Gamma}', q) = \deg_{loc}(K', \xi)$$
$$\xi \in X^- \Rightarrow \deg_{loc}(\hat{\Gamma}', q) = -\deg_{loc}(K', \xi).$$

Taking also into account the preceding discussion one can repeat the argument used in Section 3 to show that $\hat{\Gamma}$ has a critical point with $\mu > 0$, corresponding to a solution u_ε of (25). It remains to prove that $u_\varepsilon > 0$.

We know that $u_\varepsilon = z_{\theta_\varepsilon, \mu_\varepsilon} + w(\theta_\varepsilon, \mu_\varepsilon, \varepsilon)$ for suitable $(\mu_\varepsilon, \theta_\varepsilon)$ near a critical point of $\hat{\Gamma}$ on $]0, \infty) \times \mathbb{R}^N$. In particular the proof of Theorem 3.7 shows that $(\mu_\varepsilon, \theta_\varepsilon)$ remain bounded for $\varepsilon \leqslant \varepsilon_0$. Then, up to a subsequence, $(\theta_\varepsilon, \mu_\varepsilon) \to (\theta, \mu)$ and

$$\|w_\varepsilon(\theta_\varepsilon, \mu_\varepsilon, \varepsilon)\|_{\mathscr{D}^{1,2}(\mathbb{R}^N)} \to 0, \qquad \text{as} \quad \varepsilon \to 0.$$

By using the L^∞ estimate in [30] on a ball $B(0, R_0)$ containing Σ and the previous decay of the energy, we obtain

$$\sup_{x \in B_{R_0}} |w_\varepsilon(x)| \leqslant C(R_0) \|w_\varepsilon\|_{L^{2N/(N-2)}} \leqslant C_1(R_0) \|w_\varepsilon\|_{D^{1,2}}.$$

Hence u_ε converges in L^∞ on B_{R_0} and in particular in Σ, the support of h. Let $\eta = \inf_{x \in B_{R_0}} z_{\theta, \mu}(x)$ and let $\varepsilon_0 > 0$ be such that

$$\|w_\varepsilon\|_{L^\infty(B_{R_0})} \leqslant \frac{\eta}{2}, \qquad \text{if} \quad |\varepsilon| \leqslant \varepsilon_0.$$

For such ε the support of $(u_\varepsilon)_-$ is disjoint with B_{R_0} and *a fortiori* with the support of h. Consider the corresponding Euler equation

$$-\Delta u_\varepsilon = \varepsilon h(x)\, u_\varepsilon + (1 + \varepsilon K(x))(u_\varepsilon)_+^p,$$

multiplying by $(u_\varepsilon)_-$ and integrating by parts we obtain $(u_\varepsilon)_- \equiv 0$. Then $y_\varepsilon \geqslant 0$ and by the strong maximum principle we get $u_\varepsilon > 0$ for $|\varepsilon|$ small enough. ∎

As anticipated in he Introduction, in some cases we can take advantage of the presence of h to greatly weaken assumption (K1). First, we need a lemma.

LEMMA 5.3. *Suppose that* $K \in L^1(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$, $K(x) \to 0$ *as* $|x| \to \infty$ *and let* $h \in L^1(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$. *Then*

$$\lim_{\mu + |\xi| \to \infty} \hat{\Gamma}(\mu, \xi) = 0.$$

Proof. Let us proof separately that

(i) $\lim_{\mu + |\xi| \to \infty} \Gamma(\mu, \xi) = 0$,

(ii) $\lim_{\mu + |\xi| \to \infty} \Phi(\mu, \xi) = 0$.

Proof of (i). If $\mu \to 0$ (and $|\xi| \to \infty$) then the result follows by the dominated convergence theorem, because

$$|K(\mu y + \xi)\, z_0^{p+1}(y)| \leqslant \|K\|_\infty\, z_0^{p+1} \in L^1(\mathbb{R}^N)$$

and

$$\lim_{(\mu, \xi) \to (0, \infty)} K(\mu y + \xi)\, z_0^{p+1}(y) = 0 \qquad a.e.$$

So we can assume that $\mu \to \bar{\mu} \in (0, \infty]$, and $|\mu| + |\xi| \to \infty$. Then

$$(p+1)\, \Gamma(\mu, \xi) = \mu^{-N} \int_{\mathbb{R}^N} K(x)\, z_0^{p+1}\left(\frac{x-\xi}{\mu}\right) dx = I_{1,R} + I_{2,R},$$

where

$$I_{1,R} = \mu^{-N} \int_{|x| \leqslant R} K(x)\, z_0^{p+1}\left(\frac{x-\xi}{\mu}\right) dx,$$

$$I_{2,R} = \mu^{-N} \int_{|x| > R} K(x)\, z_0^{p+1}\left(\frac{x-\xi}{\mu}\right) dx.$$

Since μ is bounded away from 0 then there exists $C > 0$ such that

$$\mu^{-N} z_0^{p+1}(y) \leqslant C$$

and hence, given $\varepsilon > 0$, by the integrability of K, we can choose R large enough such that

$$I_{2,R} \leqslant C \int_{|x| > R} K(x)\, dx \leqslant \varepsilon.$$

On the other hand, changing variables,

$$I_{1,R} = \int_{|y - \xi/\mu| < R/\mu} K(\mu y + \xi)\, z_0^{p+1}(y)\, dy$$

$$\leqslant \|K\|_\infty \int_{|y - \xi/\mu| < R/\mu} z_0^{p+1}(y)\, dy.$$

It is easy to see that the last integral tends to zero. Actually, $z_0^{p+1} \in L^1(\mathbb{R}^N)$ and either $\mu \to \infty$ or $\mu \to \bar{\mu} > 0$. In the former case R/μ, the radius of the domain of integration, tends to zero, in the latter the center ξ/μ tends to infinity and the radius is bounded.

Proof of (ii). If $\mu \to 0$ the result follows as in Lemma 5.1. If $\mu \to +\infty$ we have that

$$|\Phi(\mu, \xi)| \leqslant \tfrac{1}{2} \mu^{2-N} \|h\|_{L^1} \cdot \|z_0\|_\infty^2 \to 0.$$

Finally, when $\mu \to \bar{u} \in (0, \infty)$ and $|\mu| + |\xi| \to \infty$ the result follows from the preceding formula and the fact that $\sup_\Sigma z_0^2((x-\xi)/\mu) \to 0$ as $\mu \to \bar{\mu} \in (0, \infty)$, and $|\mu| + |\xi| \to \infty$. ∎

THEOREM 5.4. *Suppose that* $K \in L^1(\mathbb{R}^N) \cap C^2(\mathbb{R}^N)$, $K(x) \to 0$ *as* $|x| \to \infty$, $K(x) \geqslant 0$, *resp.* $K(x) \leqslant 0$ *and that there exists* $\xi_0 \in \mathbb{R}^N$ *such that* $K(\xi_0) = 0$. *In addition, let* h *satisfy* $(h1.\text{a})$ *and be such that*

$$h(\xi_0) < -\frac{c_1}{c_2} \Delta K(\xi_0), \qquad \text{resp.} \quad h(\xi_0) > -\frac{c_1}{c_2} \Delta K(\xi_0).$$

Then for $|\varepsilon|$ *small problem* (25) *has a positive solution* $u_\varepsilon \in \mathcal{D}^{1,2}(\mathbb{R}^N)$.

Proof. Let $K \geqslant 0$ (the other case is similar). One has that $\hat{\Gamma}(0, \xi_0) = 0$ and

$$D^2_{\mu,\mu} \hat{\Gamma}(0, \xi_0) = c_1 \Delta K(\xi_0) + c_2 h(\xi_0) < 0.$$

Using Lemma 5.3, we deduce that $\hat{\Gamma}$ has a global negative minimum at some $(\bar{\mu}, \bar{\xi})$. Since $\hat{\Gamma}(0, \xi) = c_0 K(\xi) \geqslant 0$, we infer that $\bar{\mu} > 0$ and the conclusion follows. ∎

It is worth completing the preceding result with the case when $K \equiv 0$. Actually, in such a case one has that $\hat{\Gamma}(0, \xi) \equiv 0$ and

$$D^2_{\mu,\mu} \hat{\Gamma}(0, \xi) = c_2 h(\xi).$$

Then, if h is somewhere positive (negative) then $\hat{\Gamma}$ has a positive global maximum (negative global minimum), with $\mu > 0$. This implies:

THEOREM 5.5. *Let* h *satisfy* $(h1.\text{a})$ *and be not identically zero. Then for* $|\varepsilon|$ *small*

$$-\Delta u = \varepsilon h(x)\, u + u^p, \qquad x \in \mathbb{R}^N, \quad N > 4 \tag{30}$$

has a positive solution $u_\varepsilon \in \mathcal{D}^{1,2}(\mathbb{R}^N)$. *Furthermore, if there exist* $\xi_1, \xi_2 \in \mathbb{R}^N$ *such that*

$$h(\xi_1) > 0, h(\xi_2) < 0$$

then for $|\varepsilon|$ *small* (30) *has at least two distinct positive solutions in* $\mathcal{D}^{1,2}(\mathbb{R}^N)$.

In the next result we do not need $N > 4$.

THEOREM 5.6. *Let* $h \in L^1(\mathbb{R}^N)$ *have compact support and suppose that* $\int_{\mathbb{R}^N} h(x)\, dx \neq 0$. *Then for* $|\varepsilon|$ *small*

$$-\Delta u = \varepsilon h(x) u + u^p, \qquad x \in \mathbb{R}^N, \quad N \geqslant 3 \tag{31}$$

has a positive solution $u_\varepsilon \in \mathcal{D}^{1,2}(\mathbb{R}^N)$.

Proof. By the Dominated Convergence Theorem, we find

$$\int_{\mathbb{R}^N} h(x)\, z_0^2\left(\frac{x}{\mu}\right) dx \to C_N^2 \int_{\mathbb{R}^N} h(x)\, dx \neq 0, \qquad \text{as} \quad \mu \to \infty.$$

Since $\hat{\Gamma}(0, \xi) \equiv 0$ and $\hat{\Gamma}(0, \xi) \to 0$ as $|\mu| + |\xi| \to +\infty$ (remark that the proof of Lemma 5.3 does not make use of $N > 4$) $\hat{\Gamma}$ has a global maximum or minimum and the result follows. ▌

Remark 5.7. The assumption that h has compact support has been used only to prove the positivity of the solutions. Let us also emphasize that we obtain positive solutions independent of the sign of h (and ε).

6. EXISTENCE RESULTS FOR PROBLEM (4), II

In this final section we deal with

$$-\Delta u = \varepsilon h(x)\, u^q + u^p, \qquad x \in \mathbb{R}^N, \tag{31}$$

where $1 < q < p = (N+2)/(N-2)$. In contrast with the preceding section, here we neither assume $N > 4$ nor that h has compact support, but merely that $h \in L^1(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$.

Let

$$\tilde{f}_\varepsilon = f_0(u) - \varepsilon \tilde{G}(u),$$

where

$$\tilde{G}(u) = \frac{1}{q+1} \int_{\mathbb{R}^N} h(x)\, u_+^{q+1}.$$

Let us point out that $\tilde{G}$ is of class C^2 because $q > 1$. As before, one finds

$$\tilde{f}_\varepsilon|_{Z_\varepsilon} = b - \varepsilon \tilde{\Gamma}(\mu, \xi) + o(\varepsilon), \tag{32}$$

where

$$\tilde{\Gamma}(\mu, \xi) = \frac{\mu^{N-\theta}}{q+1} \int_{\mathbb{R}^N} h(\mu y + \xi)\, z_0^{q+1}(y)\, dy$$

and $\theta = (N-2)(q+1)/2$. Since $N > \theta$ we can repeat the arguments carried out in Lemmas 5.1 and 5.3 to show that $\tilde{\Gamma}$ can be extended to all of $\mathbb{R} \times \mathbb{R}^N$ and there results

LEMMA 6.1. *Let* $h \in L^1(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$. *Then there results*:

(i) $\tilde{\Gamma}(0, \xi) = 0$;

(ii) $\tilde{\Gamma}(\mu, \xi) \to 0$ *if* $|\mu| + |\xi| \to \infty$.

Proof. Let s and s' be conjugate exponents and let $s > N/(N-2)$. Then $z_0 \in L^s$ and, by the Hölder inequality,

$$\left| \int_{\mathbb{R}^N} h(x)\, z_0^{q+1}\left(\frac{x-\xi}{\mu}\right) dx \right| \leqslant \|h\|_{L^{s'}} \cdot \left(\int_{\mathbb{R}^N} z_0^s \left(\frac{x-\xi}{\mu}\right) dx \right)^{q+1/s}$$

$$= \mu^{N(q+1)/s}\, \|h\|_{L^{s'}} \cdot \|z_0\|_{L^s}^{q+1}, \qquad (\mu > 0).$$

Hence

$$|\tilde{\Gamma}(\mu, \xi)| \leqslant \mu^{N(q+1)/s - \theta}\, \|h\|_{L^{s'}} \cdot \|z_0\|_{L^s}^{q+1}, \qquad (\mu > 0).$$

Taking s such that $N/(N-2) < s < 2N/(N-2)$, one has that $N(q+1)/s > \theta$ and thus $\tilde{\Gamma}(\mu, \xi) \to 0$ as $\mu \to 0^+$.

The proof of (ii) when $\mu \to \bar{\mu} > 0$ (otherwise we use (i)) follows as in the proof of (ii) of Lemma 5.3. Actually the condition $N > 4$ has not been used there. ∎

We also need:

LEMMA 6.2. $\tilde{\Gamma}(\mu, \xi)$ *is not identically zero provided h does.*

Proof. The result is immediate if $z_0^{q+1} \in L^1$ or if $\int_{\mathbb{R}^N} h(x)\, dx \neq 0$. Actually, in the former case one has

$$\lim_{\mu \to 0^+} \frac{\tilde{\Gamma}(\mu, \xi)}{\mu^{N-\theta}} = \frac{h(\xi)}{q+1} \cdot \int_{\mathbb{R}^N} z_0^{q+1}.$$

In the latter, as in Theorem 5.6, we find

$$\int_{\mathbb{R}^N} h(x)\, z_0^{q+1}\left(\frac{x}{\mu}\right) dx \to C_N^{q+1} \int_{\mathbb{R}^N} h(x)\, dx \neq 0, \qquad \text{as} \quad \mu \to \infty.$$

If $z_0^{q+1} \notin L^1$, namely, in dimensions $N = 3, 4$, we argue as follows.

It is well known know that z_0^{q+1} is the Fourier transform of a *positive* L^1 function, say ϕ. By assumption we know that $h \in L^p$ for all $1 \leqslant p \leqslant \infty$ and then

$$\int_{\mathbb{R}^N} h(y+\xi)\, z_0^{q+1}(y)\, dy \in L^2(\mathbb{R}^N)$$

as a function of ξ. Now choose a sequence of tempered functions h_n such that $h_n \to h$ in L^2. Then

$$\tilde{\Gamma}_n(1, \xi) = \int_{\mathbb{R}^N} h_n(y+\xi)\, z_0^{q+1}(y)\, dy$$

$$\to \int_{\mathbb{R}^N} h(y+\xi)\, z_0^{q+1}(y)\, dy \qquad \text{in } L^2(\mathbb{R}^N).$$

Taking the Fourier transform we have

$$\hat{h}_n(\eta)\, \phi(\eta) \to \hat{h}(\eta)\, \phi(\eta) \text{ pointwise.}$$

Then $\tilde{\Gamma}(1, \xi) \equiv 0$ implies $\hat{h}(\eta)\, \phi(\eta) \equiv 0$. Since $\phi > 0$ we get $\hat{h} \equiv 0$ which is a contradiction with the fact that $h \not\equiv 0$. ▮

From the preceding statements we immediately deduce:

THEOREM 6.3. *Assume that* $1 < q < p$ *and let* $h \in L^1(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$ *be not identically equal to* 0. *Then for* $|\varepsilon|$ *small enough problem* (31) *has a positive solution in* $\mathscr{D}^{1,2}(\mathbb{R}^N)$.

Proof. It is clear that $\tilde{\Gamma}$ has either a positive global maximum or a negative global minimum and hence Theorem 2.1(ii) applies. The positivity of the corresponding solution of (31) follows as in Subsection 3.1. ▮

Remark 6.4. (i) The case $0 < q < 1$ can also be handled, although the perturbation $\tilde{\Gamma}$ is no more regular. In such a case one has to modify the abstract setting, following the arguments of [17]. The details will be given in a forthcoming paper.

(ii) Some of the results concerning (31) can be obtained by means of the mountain pass theorem following the ideas of [9] and the *concentration compactness* principle, see [22, 23]. To be short, we will only give an idea of the results one can find and sketch an outline of the arguments one should use. Using the *concentration compactness* principle, one shows that the (PS) condition holds at level c provided $c < S^{N/2}/N$, where S denotes the best Sobolev constant. In such a case, if $q > 1$ and $\varepsilon > 0$ is small it turns out that c is a m-p critical level for $\tilde{f}_\varepsilon$. In general, to show that $c < S^{N/2}/N$ one needs some additional condition on h such as, e.g. $h(x) \geqslant h_0 > 0$ in a ball of $\mathbb{R}^N$. Let us point out that, on the contrary, the result in Theorem 6.3 does not depend on such kind of assumptions.

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Note added in proof. After the paper was completed a new volume by Th. Aubin, "Some Non-linear Problems in Riemannian Geometry," Springer-Verlag, Berlin, has appeared. Among other things, it contains a broad bibliography on the prescribed curvature problems. In particular, we became aware of some results of E. Hebey, see Theorem 6.92 in the forementioned book. These results deal with rotationally symmetric curvatures on S^N corresponding to radially symmetric K on $\mathbb{R}^N$. Our Theorems 4.2 and 4.4(a) improve, in the perturbative case, the Hebey results.

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