Classification of pointed rank one Hopf algebras

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Abstract

We classify pointed rank one Hopf algebras over fields of prime characteristic which are generated as algebras by the first term of the coradical filtration. We obtain three types of Hopf algebras presented by generators and relations. For Hopf algebras with semi-simple coradical only the first and second type appears. We determine the indecomposable projective modules for certain classes of pointed rank one Hopf algebras.

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1. Introduction

In [KrR] Krop and Radford introduce the rank as a measure of complexity for finite-dimensional Hopf algebras whose coradical is a sub Hopf algebra. They classify the finite-dimensional pointed Hopf algebras H of rank one which are generated as algebras by the first term of the coradical filtration H_1 over fields k of characteristic zero. These algebras are para-metrised by tuples (G, χ, a, α), where G is a finite group, χ a linear character of G, $a \in Z(G)$ and $\alpha \in k$. As algebras they are generated by an $(a, 1)$-primitive element x and the group G, subject to the action $gxg^{-1} = \chi(a)x$ and $x^n = \alpha(a^n - 1)$.

In this paper, we classify the finite-dimensional pointed rank one Hopf algebras H over alge-braically closed fields k of prime characteristic p, which are generated as algebras by H_1. The

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list consists of three types of Hopf algebras parametrised by tuples. The type only depends on the conjugation action of the so-called skew point a on the $(a, 1)$-primitive elements. The first type is parametrised by the same tuple as the one in Krop and Radford's paper and is given by similar relations. For Hopf algebras with semi-simple coradical only the first and second type appears. The second and third types are new. Analysing the third type requires new ideas and methods. In every type we analyse when two Hopf algebras parametrised by a different tuple are isomorphic as Hopf algebras. Furthermore, we are able to determine the indecomposable projective modules for certain Hopf algebras in every type. The second and third type provide new examples of pointed rank one Hopf algebras.

By [TaW] and [KrR] we know that a finite-dimensional pointed rank one Hopf algebra has a uniquely determined group-like element a such that there exists an $(a, 1)$-primitive element which does not lie in the coradical. Let X be the set of $(a, 1)$-primitive elements which do not lie in the coradical. The group of group-like elements G together with any $x \in X$ generate the algebra and determine its Hopf algebra structure.

To describe the first type we assume that there exists an $x \in X$ which is an eigenvector with eigenvalue $q \neq 1$ of the conjugation action of a. Then we can show that x is a simultaneous eigenvector of the conjugation action of G. The eigenvalues of this action are given by a linear character χ of G. Similarly to [KrR] we compute n such that $\{gx^i \mid g \in G,\ 0 \leqslant i \leqslant n-1\}$ is a basis of the algebra. We call n the degree of x. We show that n is equal to the multiplicative order of q and that $x^n = \alpha(a^n - 1)$ for an $\alpha \in k$. Thus this Hopf algebra is parametrised by the tuple $R = (G, a, \chi, \alpha)$.

The second type appears only if a has an eigenvector $x \in X$ with eigenvalue 1. Then the degree of x is p and $x^p = \alpha_{-1}(a^p - 1) + \alpha_0 x$ for $\alpha_{-1}, \alpha_0 \in k$. The conjugation action of G on x is described by two functions, a linear character χ of G and a map $c: G \to k$ such that $gxg^{-1} = \chi(g)x + c(g)(a - 1)$. Thus this Hopf algebra is parametrised by the tuple $F = (G, a, \chi, c, (\alpha_{-1}, \alpha_0))$.

Generalising the results in [KrR], we show that if the order of the skew point is not divisible by p then there exists such an eigenvector $x \in X$ and the Hopf algebra is of first or second type. A corollary is that if H has a semi-simple coradical then it is of first or second type. If H is cocommutative it is of second type.

The third type appears if no eigenvector of the conjugation action of a in X exists. Then we can show that there is an $x \in X$ such that $axa^{-1} = x + (a - 1)$. We show that the degree of x is p, that $x^p = x$ and that the conjugation action of G is described by a group homomorphism $c: G \to (k, +)$ such that $gxg^{-1} = x + c(g)(a - 1)$. Thus this Hopf algebra is parametrised by the tuple $E = (G, a, c)$.

Conversely for every tuple R, F and E we show that there exists a uniquely determined Hopf algebra H_R, H_F and H_E which satisfies the given relations. Then we prove that H_R, H_F and H_E are pointed rank one Hopf algebras and determine for which given tuples they are isomorphic.

Following [KrR] we define H_R and H_F to be of nilpotent type if $x^n = 0$. We describe the projective idecomposable modules and the simple modules for Hopf algebras of nilpotent type. We show that H_R and H_F are uniserial if their coradical is semi-simple generalising a result in [KrR] formulated in the case that the group of group-like elements is abelian and the characteristic of the field is zero. Finally we prove that the blocks of Hopf algebras H_R that are not of nilpotent type with semi-simple coradical are Morita equivalent to the Taft algebra $\Gamma_{N, \chi(a)}$ or the matrix ring $M_N(k)$.

2. Preliminaries

We first introduce some basic definitions and facts that will be used in the other sections. Throughout this section H is a finite-dimensional Hopf algebra.

Definition 2.1. (See [Mon, Definitions 5.1.5, 5.2.1].) The coradical H_0 of H is the sum of its simple subcoalgebras. We define inductively $H_i := \Delta^{-1}(H \otimes H_{i-1} + H_0 \otimes H)$ and call $(H_i)_{i \in \mathbb{N}}$ the coradical filtration of H. A coradical filtration is called a Hopf algebra filtration if $H_i H_j \subset H_{i+j}$ and $S(H_i) \subset H_i$ for all $i, j \in \mathbb{N}$.

The following results are well known.

Lemma 2.2. *(See [Mon, Theorem 5.2.2, Lemma 5.2.8].) Let $(H_i)_{i \in \mathbb{N}}$ be the* coradical filtration *of H.*

(1) *The $(H_i)_{i \in \mathbb{N}}$ are a family of subcoalgebras satisfying $H_i \subset H_{i+1}$ for all $i \in \mathbb{N}$ and $\bigcup_{i \in \mathbb{N}} H_i = H$. For the comultiplication of the elements of the filtration we have $\Delta(H_i) \subset \sum_{k=0}^{i} H_k \otimes H_{i-k}$.*
(2) *The coradical filtration $(H_i)_{i \in \mathbb{N}}$ is a Hopf algebra filtration if and only if H_0 is a sub Hopf algebra of H.*

If H is a pointed Hopf algebra, then H_0 is the group algebra generated by the set of group-like elements, which is a sub Hopf algebra of H. Thus the coradical filtration of H is a Hopf filtration.

We introduce variations of primitive elements of a Hopf algebra. We will use certain primitive elements in order to classify pointed rank one Hopf algebras.

Definition 2.3. (See [Mon, Definition 1.3.4].) Let $c \in H$ be an element such that $\Delta(c) = c \otimes g + h \otimes c$ for some group-like elements $g, h \in G(H)$. Then we say that c is a (g, h)-primitive element, and denote the set of those elements by $P_{g,h}$. If $c \in P_{g,1}$, then we say that c is a skew primitive element.

In the following lemma we determine the skew primitive elements in $kG(H)$.

Lemma 2.4. *Let H be a Hopf algebra, $a \in G(H)$ a group-like element and $c \in kG(H)$ a skew primitive element in $P_{a,1}$. Then $c = \alpha(a - 1)$ for some $\alpha \in k$. Thus $P_{a,1} \cap kG(H) = \{\alpha(a - 1) \mid \alpha \in k\}$.*

Proof. The group-like elements are linearly independent by [Swe, 3.2.1]. Thus we can write c as a uniquely determined linear combination of the group-like elements. We take $c = \sum_{g \in G(H)} b_g g$. Then the following equation holds

$$\sum_{g \in G(H)} b_g (g \otimes a + 1 \otimes g) = \Delta(c) = \sum_{g \in G(H)} b_g g \otimes g.$$

As the elements of $G(H) \otimes_k G(H)$ are a basis of $kG(H) \otimes_k kG(H) \cong k(G(H) \otimes_k G(H))$, comparing coefficients of this basis on the left- and right-hand sides gives us $b_g = 0$ for all $g \in G(H) - \{a, 1\}$ and $b_a = -b_1$. $\quad \square$

We define certain kinds of group-like elements.

Definition 2.5. We say that $g \in G(H)$ is a skew point if $P_{g,1} \setminus H_0$ is not empty.

The following lemma shows that every pointed, non-cosemi-simple Hopf algebra H has a skew point $a \in G(H)$. This result can also be found in a similar form in [KrR, Lemma 1, Proposition 1].

Lemma 2.6. *Let H be a pointed Hopf algebra and $(H_i)_{i \in \mathbb{N}}$ its Hopf filtration. The set of primitive elements lies in H_1. Furthermore we have $H_1 = H_0 + \sum_{\{a,b \in G(H)\}} P_{a,b}$. So if $H \neq H_0$ there exists a skew primitive element $x \in P_{a,1} \setminus H_0$ for some $a \in G(H)$.*

Proof. As all group-like elements lie in H_0, the primitive elements are in H_1 by definition. The second part is [TaW, Proposition 2]. $\square$

We introduce the rank of a pointed Hopf algebra and some properties of these Hopf algebras.

Let H be a pointed Hopf algebra and $(H_i)_{i \in \mathbb{N}}$ the coradical filtration of H. Then $(H_i)_{i \in \mathbb{N}}$ is a Hopf filtration and we can view the H_i's as H_0-module. The trivial module k is an H_0-module via the counit map. If $\dim k \otimes_{H_0} H_1 = t + 1$ we call H a rank t pointed Hopf algebra. Using the next remark, we see that H_i is a free left H_0-module for all $i \in \mathbb{N}$.

Remark 2.7. (See [KrR, Section 1].) Let $V_0 \subset V_1 \subset \cdots$ be a chain of subcomodules of H such that V_0 is a sub Hopf algebra, $V_0 V_i \subset V_i$ and $\Delta(V_i) \subset \sum_{t=0}^{i} V_t \otimes V_{i-t}$. Then we can view $Q_i := V_i / V_{i-1}$ as a left V_0-Hopf module via the map $\rho_i : Q_i \to V_0 \otimes Q_i, h + V_{i-1} \mapsto \Delta(h) + V_i \otimes V_{i-1}$. By the Fundamental Theorem of Hopf modules [Swe, Theorem 4.1.1] either $Q_i = \{0\}$, or it is a free left V_0-module on a linear basis of $Q_i{}^{coV_0} = \{z \in Q_i \mid \rho_i(z) = 1 \otimes z\}$. Therefore V_i is a free left V_0-module for all $i \in \mathbb{N}$.

Let a be a skew point of a pointed rank one Hopf algebra and $x \in P_{a,1} \setminus H_0$, then $1, x$ is a basis of H_1 as left H_0-module by the preceding remark.

The next lemma shows that if H is a pointed, rank one Hopf algebra then it has exactly one skew point. This skew point is an element of the centre of $G(H)$. The proof of the following is a combination of [KrR, Lemma 1(c)] and [KrR, Proposition 1(d)]. However the setup is slightly different and therefore we give the details.

Lemma 2.8. *Let H be a pointed, rank one Hopf algebra and $x \in P_{a,1} \setminus H_0$ for some $a \in G(H)$. Then the set of skew primitive elements in $H_1 - H_0$ is given by $P_{a,1} \setminus H_0 := \{sx + t(a-1) \mid s \in k^*, t \in k\}$. Thus $a \in G(H)$ is uniquely determined. We have $a \in Z(G)$ and for each $y \in P_{a,1} \setminus H_0$ there are maps χ and c depending on y such that $gyg^{-1} = \chi(g)y + c(g)(a-1)$ for all $g \in G$. Then $\chi : G \to k^*$ is a linear character and $c : G \to k$ satisfies the condition $c(hg) = \chi(g)c(h) + c(g)$.*

Proof. Suppose that $z \in H - H_0$ is a skew primitive element in $P_{g,1}$ for some $g \in G$. Then $z \in H_1 - H_0$ and has a unique presentation as $z = a_0 + a_1 x$ for $a_0, a_1 \in H_0$ and $a_1 \neq 0$. Thus $\Delta(a_1)(x \otimes a) + \Delta(a_1)(1 \otimes x) + \Delta(a_0) = \Delta(z) = z \otimes g + 1 \otimes z = a_0 \otimes g + 1 \otimes a_0 + (a_1 \otimes g)(x \otimes 1) + (1 \otimes a_1)(1 \otimes x)$. Using the fact that $\{1 \otimes 1, x \otimes 1, 1 \otimes x, x \otimes x\}$ is a basis of $H_1 \otimes H_1$ as an $H_0 \otimes H_0$-module gives us $g = a$ and therefore $\Delta(a_0) = a_0 \otimes a + 1 \otimes a_0$ and $\Delta(a_1) = 1 \otimes a_1$.

Thus by applying Lemma 2.4 we have $a_0 = \alpha(a-1)$ with $\alpha \in k$. Applying the counit to the last equation gives us $a_1 = 1\epsilon(a_1)$. Thus $a_1 \in k^*$.

Let $y \in P_{a,1} \setminus H_0$. The element gyg^{-1} is a $\{gag^{-1}, 1\}$-skew primitive element and lies in $H_1 - H_0$. The previous part gives us $gag^{-1} = a$ for all $g \in G$ and $gyg^{-1} = \chi(g)y + c(g)(a-1)$ for $\chi(g) \in k^*$ and $c(g) \in k$. The equation $\chi(hg)y + c(hg)(a-1) = hgyg^{-1}h^{-1} = \chi(h)\chi(g)y + (\chi(g)c(h) + c(g))(a-1)$ proves the relations for χ and c. $\quad\square$

Finally we state a combinatorial result that we will use to prove some results of the next section.

Lemma 2.9. *(See [Rad2, Corollary 2].) Let k be a field of positive characteristic p, $q \in k$ and $n \in \mathbb{N}$. We set $(n)_q = \sum_{i=0}^{n-1} q^i$ and define*

$$\binom{n}{m}_q := (n)_q!/(n-m)_q!(m)_q!.$$

Suppose $n > 1$. Then $\binom{n}{m}_q = 0$ for all $1 \leqslant m \leqslant n-1$ if and only if $n = Np^r$ with $1 \leqslant N$, $0 \leqslant r$ where N and p are coprime and q is a primitive Nth root of unity.

3. Pointed rank one Hopf algebras of first and second type

In this section we introduce the pointed Hopf algebras H_R and H_F given by tuples R and F and generators and relations similar to [KrR]. We determine under which conditions a pointed, rank one Hopf algebra is isomorphic to H_R or H_F. In particular we classify pointed rank one Hopf algebras with semi-simple coradical and cocommutative pointed rank one Hopf algebras. Throughout this section we will assume that H is a pointed, finite-dimensional Hopf algebra over a field k of positive characteristic p such that $H \neq H_0$. Let a be a skew point of H.

As in characteristic zero [KrR, Proposition 1] we can find for any skew point a whose order is not divisible by p, a skew primitive element in $P_{a,1} \setminus H_0$ which is an eigenvector of the conjugation action of a.

Lemma 3.1. *Let a be a skew point of H, k an algebraically closed field and suppose that p does not divide the order of a. Then there is an element $x \in P_{a,1} - H_0$ such that $axa^{-1} = qx$ for a primitive Nth root of unity $q \in k^*$. If $q \neq 1$ and H is rank one, then x is uniquely determined up to scalar multiplicities by these properties.*

Proof. By definition $P_{a,1} \subset H_1$ contains an element of $H_1 - H_0$ and is therefore not trivial. Let m be the order of a. For the linear transformation $T : P_{a,1} \to P_{a,1}$, $y \mapsto aya^{-1}$ we have then $T^m = \text{id}$. As p does not divide m and as k is algebraically closed, T is diagonalisable. Thus $P_{a,1}$ has a basis of eigenvectors for T whose eigenvalues are roots of unity. As $P_{a,1} \setminus H_0$ is non-trivial, there is an eigenvector $x \in H_1 - H_0$ with eigenvalue q. If H has rank one and $q \neq 1$ the uniqueness of x follows from Lemma 2.8. $\quad\square$

We define the degree of an element in $H_1 - H_0$ as the freeness degree of its powers over H_0.

Definition 3.2. We call the degree of $x \in H_1 - H_0$ the smallest integer n such that $x^n \in \sum_{i=0}^{n-1} H_0 x^i$. Such an n exists as H is finite-dimensional. If H is rank one, then all elements in

$H_1 - H_0$ have the same degree. This holds because $H_1 = H_0 + H_0 y$ for any element $y \in H_1 - H_0$ and therefore the dimension of the subalgebra generated by H_1 is the product of the dimension of H_0 and the degree of y.

Suppose that H is a rank one Hopf algebra which is generated as an algebra by H_1 and has an element x defined as in Lemma 3.1. Then by [KrR, Lemma 1] the degree of x is the smallest integer such that $1, x, \ldots, x^{n-1}$ form a basis for H over H_0.

We show that x is a common eigenvector of the conjugation action of the group-like elements if $q \neq 1$. The analogous result for a field of characteristic zero can be found in [KrR, Proposition 2]. Part (1) and (2) for $q \neq 1$ can be found in [Rad2, pp. 696–699] and part (3) has the same proof as [KrR, Proposition 1(d)].

Theorem 3.3. *Let $a \in G(H)$ be a skew point of H. Suppose there is an $x \in P_{a,1} \setminus H_0$, which is an eigenvector of the conjugation action of a with eigenvalue q, where q is a primitive Nth root of unity.*

(1) *If x has degree n, then $n = N p^r$ for some $r \in \mathbb{N}$ and*

$$x^n = \alpha_{-1}\left(a^n - 1\right) + \sum_{i=0}^{r-1} \alpha_i x^{N p^i}$$

  with $\alpha_i \in k$. In addition if $a^{N(p^r - p^i)} \neq 1$ for $0 \leqslant i$, then $\alpha_i = 0$.
(2) *If H is rank one we have ($r = 0$ and $N > 1$) or ($r = 1$ and $N = 1$) for r and N defined as in (1).*
(3) *Suppose that H is rank one with $N > 1$ and $r = 0$. Then x is a common eigenvector for the conjugation action of the elements in $G(H)$, that is $g x g^{-1} = \chi(g) x$ for all $g \in G$ where $\chi : G(H) \to k^*$ is a linear character.*
(4) *Suppose that H has rank one with $N = 1$ and $r = 1$. All elements of $P_{a,1} \setminus H_0$ are eigenvectors of the conjugation action of a with eigenvalue 1. If $\alpha_0 = 0$ and k is algebraically closed there exist an element $y \in P_{a,1} \setminus H_0$ with $y^p = 0$. If in addition a is not of order p, then y is a common eigenvector of the conjugation action of $G(H)$.*
(5) *Let H be rank one and generated by H_1 as an algebra and A be the sub Hopf algebra of H generated by a and $x \in P_{a,1} \setminus H_0$, where x is an eigenvector of the conjugation action of a. Then A is a kG-module via conjugation. We denote the smash product between A and kG by $A * kG$. The map $A * kG \to H$, $(h, g) \mapsto hg$ is a surjective Hopf algebra homomorphism whose kernel is generated by $a * 1 - 1 * a$.*

Proof. Let n be the degree of x. Then x^n has a unique presentation as $x^n = \sum_{i=0}^{n-1} b_i x^i$ with $b_i \in H_0$. Applying Δ to both sides of the equation and substituting x^n gives us

$$\sum_{i=1}^{n-1} \binom{n}{i}_{q^{-1}} x^i \otimes a^i x^{n-i} + \sum_{i=0}^{n-1} \left(b_i x^i \otimes a^n + 1 \otimes b_i x^i\right)$$

$$= \left(\Delta(x)\right)^n = \Delta\left(x^n\right) = \sum_{i=0}^{n-1} \Delta\left(b_i x^i\right) = \sum_{i=0}^{n-1} \Delta(b_i) \sum_{k=0}^{i} \binom{i}{k}_{q^{-1}} x^k \otimes a^k x^{i-k}.$$

The set $\{x^i \otimes x^j \colon 0 \leqslant i,\ j \leqslant n-1\}$ is a free $H_0 \otimes H_0$-basis of an $H_0 \otimes H_0$-submodule of $H \otimes H$. So we can compare coefficients.

(1) Equating the coefficients of $x^i \otimes x^{n-i}$ gives $\binom{n}{i}_{q^{-1}} = 0$ for $1 \leqslant i \leqslant n-1$,
(2) equating the coefficients of $x^i \otimes 1$ and $1 \otimes x^i$ for $1 \leqslant i \leqslant n-1$ gives $\Delta(b_i) = b_i \otimes a^{n-i} = 1 \otimes b_i$,
(3) equating the coefficients of $x^k \otimes x^{i-k}$ for $1 < i < n$ gives $\Delta(b_i)\binom{i}{k}_{q^{-1}} = 0$ for all $1 \leqslant k \leqslant i-1$,
(4) comparing the coefficients of $1 \otimes 1$ gives $\Delta(b_0) = b_0 \otimes a^n + 1 \otimes b_0$.

By Lemma 2.4 and (4) we have that $b_0 = \alpha_{-1}(a^n - 1)$ for an $\alpha_{-1} \in k$.

Lemma 2.9 and (1) gives us $n = Np^r$. By (3) and Lemma 2.9 we have $\Delta(b_i) = 0$ in the case that $1 < i$ and there is no $s < r$ such that $i = Np^s$. As Δ is injective we have $b_i = 0$. We deduce from (2) that $\epsilon(b_i) = \epsilon(b_i)a^{n-i} = b_i \in k$ for all $1 \leqslant i \leqslant n-1$. In case $b_{Np^s} \neq 0$ we thus have $a^{N(p^r - p^s)} = 1$. Suppose now $b_1 \neq 0$, then $a^{n-1} = a^{Np^r - 1} = 1$. As N divides the order of a it divides $Np^r - 1$. Thus we have $N = 1$ in this case. This proves the degree formula.

From now on let H be rank one. Suppose $(r > 1)$ or $(r = 1$ and $N > 1)$, then we have for $z = x^{Np}$ in the first case that $\Delta(z) = z \otimes a^{Np} + 1 \otimes z$ and for $z = x^N$ in the second case that $\Delta(z) = z \otimes a^N + 1 \otimes z$ using Lemma 2.9. Thus $z \in H_1 - H_0$ and H is not a rank one Hopf algebra as $1, x, z$ are H_0-independent. Therefore x has degree N for $N > 1$ or degree p if $N = 1$.

To prove part (3) we have by Lemma 2.8 that $gxg^{-1} = \chi(g)x + c(g)(a-1)$ for some maps $c, \chi \colon G \to k$. Furthermore we have $gag^{-1} = a$ as $a \in Z(G)$. Conjugating with a gives

$$q\chi(g)x + qc(g)(a-1) = qgxg^{-1} = agxg^{-1}a^{-1} = q\chi(g)x + c(g)(a-1).$$

If $q \neq 1$ this gives us $c(g)(a-1) = 0$ for all $g \in G$ and therefore $c(g) = 0$.

If we have $r = 1$ and $N = 1$, then $q = 1$. Let $\alpha_0 = 0$ and suppose $\alpha_{-1} \neq 0$. As k is algebraically closed, we have $\alpha_{-1}^{1/p} \in k$. Then $y := x - \alpha_{-1}^{1/p}(a-1)$ is an element in $P_{a,1} \setminus H_0$ and an eigenvector of the conjugation action of a. It satisfies $y^p = 0$. Therefore

$$0 = y^p = gy^p g^{-1} = \left(gyg^{-1}\right)^p = c(g)^p\left(a^p - 1\right).$$

If we assume $a^p \neq 1$ or $a = 1$, then $c(g)(a-1) = 0$ for all $g \in G$.

For the last part the surjectivity follows from the fact that x and H_0 generate H. By direct computation using the basis $(a^l x^i)_{\{0 \leqslant i \leqslant n-1,\, 1 \leqslant l \leqslant |a|\}}$ of A and the basis $(g)_{g \in G}$ of kG we can see that the kernel is generated by $(a * 1 - 1 * a)$ and that this map is a Hopf algebra morphism. $\quad\square$

We introduce Hopf algebras given by a tuple and generators and relations. We use them to classify the pointed rank one Hopf algebras of Theorem 3.3 with $N > 1$ and $r = 0$.

Definition 3.4 (*First type*). Let $R = (G, \chi, a, \alpha)$ be a tuple where G is a finite group, $a \in Z(G)$, $\chi \colon G \to k^*$ is a linear character with $\chi(a) \neq 1$ and $\alpha := (\alpha_{-1}, \alpha_0, \ldots, \alpha_{r-1}) \in k^{r+1}$. Let $\chi(a)$ be a primitive Nth root and $n := p^r N$.

The Hopf algebra H_R is the algebra with basis $(gx^i)_{\{g \in G, 0 \leqslant i \leqslant n-1\}}$ and relations $gxg^{-1} = \chi(g)x$ for all $g \in G$ and $x^n = \alpha_{-1}(a^n - 1) + \sum_{i=0}^{r-1} \alpha_i x^{p^i N}$. The Hopf algebra structure is determined by $\Delta(gx^i) = (g \otimes g)(x \otimes a + 1 \otimes x)^i$, $\epsilon(gx^i) = \delta_{i,0}$ and $S(gx^i) = (-xa^{-1})^i g^{-1}$ for all $g \in G$ and $0 \leqslant i \leqslant n-1$.

We assume that $\alpha_{-1} = 0$ if $a^n - 1 = 0$. In addition we assume that

(1) if $\alpha_i \neq 0$ for some $i \geqslant 0$ then $a^{N(p^r - p^i)} = 1$ and $\chi(g)^{N(p^r - p^i)} = 1$ for all $g \in G$;
(2) if $\alpha_{-1} \neq 0$, then $\chi(g)^{Np^r} = 1$ for all $g \in G$.

The Hopf algebra H_R is uniquely determined by the given relations. In order to show that for any tuple R, satisfying the conditions of Definition 3.4, the Hopf algebra H_R is well defined, we give a construction of H_R in the following lemma.

Lemma 3.5 (*Constructing H_R*). *Let (G, χ, a, α) be a tuple with the properties described in Definition 3.4. We set $q := \chi(a)$, $N := |\chi(a)|$ and $m := |a|$ and let $A := A_{(q,\alpha,N,m)}$ be the Hopf algebra defined in* [Rad2, (C.6)] *which is generated by the group-like element a and the skew primitive element x. Then A is a kG-module algebra by conjugation with the relation $gxg^{-1} = \chi(g)x$ for all $g \in G$. Let $A * kG$ be the smash product of A and kG whose coalgebra structure is given by the tensor coalgebra $A \otimes kG$. Then the quotient $A * kG/(1 * a - a * 1)$ fulfils all conditions for H_R.*

We also analyse for which tuples R and R' the corresponding Hopf algebras H_R and $H_{R'}$ are isomorphic. The arguments are similar to [KrR, Theorem 1(c)].

Lemma 3.6. *Let $R = (G, \chi, a, \alpha)$ and $R' = (G', \chi', a', \alpha')$ be two tuples as in* 3.4. *Then H_R and $H_{R'}$ are isomorphic if and only if there is a group isomorphism $f : G \to G'$ with $f(a) = a'$, $\chi = \chi' \circ f$ and an element $\beta \in k^*$ such that $\beta^{Np^i} \alpha_i = \beta^n \alpha'_i$ for all $0 \leqslant i \leqslant r = r'$ and $\alpha_{-1} = \beta^n \alpha'_{-1}$.*

Proof. First assume there is a map f and a scalar β as above. Let $F : H_R \to H_{R'}$ the linear map with $F(gx^i) = f(g)(\beta x')^i$ with respect to the basis $(gx^i)_{\{g \in G, \, 0 \leqslant i \leqslant n-1\}}$ of H_R. As $r = r'$ and $|\chi(a)| = |\chi'(a')|$ we have $n = n'$ such that F is obviously a vector space isomorphism and direct computation shows that it is a Hopf algebra isomorphism.

Now suppose $F : H_R \to H_{R'}$ is a Hopf algebra isomorphism. Then $N = N'$ and $r = r'$ as H_R and $H_{R'}$ need to have the same dimension. As group-like elements are mapped to group-like elements, $f := F|_G$ is a group isomorphism from G to G'. The map F also maps skew primitive elements to skew primitive elements. Therefore $F(x)$ is a skew primitive element with skew point $f(a)$. Direct computation shows that the skew primitive elements of $H_{R'} - kG'$ are the elements a' with corresponding skew primitive elements $\{\beta x' + \gamma(a' - 1) \mid \beta \in k^*, \ \gamma \in k\}$ and a'^{Np^s} with corresponding skew primitive elements

$$\left\{ \sum_{i=0}^{r-1} \beta_i x'^{Np^i} + \gamma\left(a'^{Np^s} - 1\right) \,\Big|\, \beta_i, \gamma \in k, \ \beta_i = 0 \text{ if } a^{Np^i} \neq a^{Np^s}, \ \exists \beta_i \neq 0 \right\}$$

for $0 \leqslant s \leqslant r - 1$. As $f(a)F(x)f(a)^{-1} = \chi(a)f(x)$ where $\chi(a)$ is a primitive Nth root of unity unequal 1, we have $f(a) = a'$. Then $F(x) = \beta x'$ for some $\beta \in k^*$. As $F(gxg^{-1}) = f(g)\beta x' f^{-1}(g)$, we have $\chi = \chi' \circ f$. Finally $F(x)^p = F(x^p)$ gives $\beta^{Np^i} \alpha_i = \beta^n \alpha'_i$ for all $0 \leqslant i \leqslant r = r'$ and $\alpha_{-1} = \beta^n \alpha'_{-1}$. $\quad\square$

We need to introduce a second type of Hopf algebras in order to classify the case $N = 1, r = 1$ from Theorem 3.3. First we give a typical example of a Hopf algebra of second type which will be defined in 3.8.

Example 3.7. There is a Hopf algebra L over a field of characteristic 2 generated by an elementary abelian group G of order 4 and an element x. Let $a, b \in G$ be the generators of G such that $1, a, b, ab, ax, bx, x, abx$ is a basis of H. The multiplication is defined by the relations $axa = x$, $bxb = x + a - 1$ and $x^2 = 0$. The coalgebra structure is determined by $\Delta(x) = x \otimes a + 1 \otimes x$, $\Delta(a) = a \otimes a$, $\Delta(b) = b \otimes b$. This gives us an example for a Hopf algebra with $\chi(a) = \chi(b) = 1$ and $c(b) = 1, c(a) = 0$ where c and χ are defined as in 2.8. Thus all skew primitive elements which do not lie in L_0 are eigenvectors of the conjugation action of a but not of b.

In the following definition we describe those pointed rank one Hopf algebras in a formal way and characterise them.

Definition 3.8 (*Second type*). Let $F = (G, \chi, c, a, (\alpha_{-1}, \alpha_0))$ be a tuple, where G is a finite group, $a \in Z(G)$, $\chi : G \to k^*$ is a linear character with $\chi(a) = 1$ and $c : G \to k$ is a map with $c(hg) = \chi(g)c(h) + c(g)$ for all $g, h \in G$ and $c(a) = 0$. The Hopf algebra H_F is the algebra with basis $\{gx^i \mid g \in G, \ 0 \leqslant i \leqslant p - 1\}$ and relations $x^p = \alpha_{-1}(a^p - 1) + \alpha_0 x$ where $\alpha_{-1}, \alpha_0 \in k$ and $gxg^{-1} = \chi(g)x + c(g)(a - 1)$ for all $g \in G$.

The Hopf algebra structure is determined by $\Delta(gx^i) = (g \otimes g)(x \otimes a + 1 \otimes x)^i$, $\epsilon(gx^i) = \delta_{i,0}$ and $S(gx^i) = (-xa^{-1})^i g^{-1}$ for all $g \in G$ and $0 \leqslant i \leqslant p - 1$.

We require that $\alpha_0 \in \{0, 1\}$ and that if $\alpha_0 = 0$ then $\alpha_{-1} = 0$. In addition we assume that

(1) if $\alpha_0 = 0$, then $c(g)(a^p - 1) = 0$ for all $g \in G$;
(2) if $\alpha_0 = 1$, then $a^p = a$, $\chi(g) \in F_p^*$ and $(\alpha_{-1}(\chi(g) - 1) + c(g)^p - c(g))(a - 1) = 0$ for all $g \in G$.

The following explains the two conditions $\alpha_0 \in \{0, 1\}$ and if $\alpha_0 = 0$ then $\alpha_{-1} = 0$.

Assume H is rank one and a has an eigenvector x in $P_{a,1} \setminus H_0$ with eigenvalue 1. Then any element of $P_{a,1} \setminus H_0$ is fixed by conjugation with a and has degree p. If $\alpha_0 = 0$ and k algebraically closed, then there exists an element $y \in P_{a,1} \setminus H_0$ with $y^p = 0$ by part (4) of Theorem 3.3.

If $\alpha_0 \neq 0$ then wx with $w = (\alpha_0)^{-1/p-1} \in k^*$ is an element in $P_{a,1} \setminus H_0$ such that $(wx)^p = wx + (w\alpha_0^{-1}\alpha_{-1})(a^p - 1)$. Therefore we can choose an eigenvector in $P_{a,1} \setminus H_0$ of a with $\alpha_0 = \alpha_{-1} = 0$ or with $\alpha_0 = 1$.

The proof of the next lemma shows that H_F is well defined if and only if the conditions (1) and (2) are satisfied.

Lemma 3.9. *The Hopf algebra H_F is well defined.*

Proof. In order to check that the algebra structure is well defined, we need to consider the equation $\chi(g)^p \alpha_0 x + (\chi(g)^p \alpha_{-1} + c(g)^p)(a^p - 1) = (\chi(g)x + c(g)(a - 1))^p = (gxg^{-1})^p = gx^p g^{-1} = g(\alpha_{-1}(a^p - 1) + \alpha_0 x)g^{-1} = \alpha_{-1}(a^p - 1) + \alpha_0 \chi(g)x + \alpha_0 c(g)(a - 1)$ for all $g \in G$. By the first part of Theorem 3.3 we have $a^p = a$ in case that $\alpha_0 = 1$. The relations for c and χ follow immediately by comparing coefficients.

The maps Δ and ϵ are algebra homomorphism and provide H_F with a coalgebra structure. An easy computation using the basis shows that S is the antipode. $\quad\square$

The Hopf algebras H_F can be constructed similarly to Lemma 3.5 as the smash product of A and kG with the kG action $gxg^{-1} = \chi(g)x + c(g)(a-1)$ on A modulo the ideal generated by $(a*1 - 1*a)$.

We show that H_R and H_F are pointed and determine under which conditions H_R and H_F are rank one Hopf algebras. This depends only on the degree of x. The following result can also be deduced from [Rad2, Proposition 2].

Proposition 3.10. *The Hopf algebras H_R and H_F are pointed. The Hopf algebra H_R is rank one if and only if $r = 0$. The Hopf algebra H_F is rank one.*

Proof. We use that by [LaR, Corollary 1.74] every bialgebra B over a field generated by skew primitive elements is pointed. Therefore H_R and H_F are pointed.

By Theorem 3.3 we have that if H_R is rank one, then $r = 0$. Suppose now that $r = 0$ for $H = H_R$. We have $r = 1$ for $H = H_F$. Then any element $z \in H$ has a unique presentation of the form $\sum_{i=0}^{n-1} b_i x^i$ with $n = N > 1$ in the first case and $n = p$ in the second case and $b_i \in kG(H)$ for all $0 \leqslant i \leqslant n-1$. The conditions $\Delta(z) = z \otimes g + 1 \otimes z$ and $z \notin H_0$ give the following equation

$$
\sum_{i=1}^{n-1} \Delta(b_i) \sum_{k=0}^{i} \binom{i}{k}_{q^{-1}} x^k \otimes a^k x^{i-k} = \sum_{i=0}^{n-1} \Delta(b_i) \Delta(x)^i
$$

$$
= \Delta(z) = z \otimes g + 1 \otimes z
$$

$$
= \sum_{i=0}^{n-1} (b_i \otimes g)(x^i \otimes 1) + (1 \otimes b_i)(1 \otimes x^i).
$$

This equation is equivalent to

(1) for all $1 \leqslant k \leqslant i-1$ we have $\Delta(b_i)\binom{i}{k}_{q^{-1}} = 0$;
(2) furthermore $\Delta(b_0) = 1 \otimes b_0 + b_0 \otimes g$;
(3) and $\Delta(b_i) = (b_i \otimes g)(1 \otimes a^{-i}) = 1 \otimes b_i$ for all $1 \leqslant i \leqslant n-1$.

By Lemma 2.9 for all $2 \leqslant i \leqslant n-1$ there exists a natural number k in $\{1, \ldots, i-1\}$ such that $\binom{i}{k}_{q^{-1}}$ is not zero. This forces $b_i = 0$ for all $2 \leqslant i \leqslant n-1$ using the first equation. The second equation gives us $b_0 \in H_0 \cap P_{g,1} = \{\gamma(g-1) \mid \gamma \in k\}$ by Lemma 2.4 and the last equation shows $b_1 \in k^*$ and $a^{-1}g = 1$.

Thus $z = \beta x + \gamma(a-1)$ for $\beta \in k^*$, $\gamma \in k$ and $g = a$. For all $h \in P_{b,d} \setminus H_0$ with $b, d \in G(H)$ we have $h = 0$ for $d^{-1}b \neq a$ and else $h = d^{-1}(\beta x + \gamma(a-1))$. As $H_1 = H_0 + \sum_{b,d \in G(H)} P_{b,d} \subset H_0 + H_0 x$ and $x \notin H_0$, the dimension of $k \otimes_{H_0} H_1$ is two. Thus H is a rank one Hopf algebra. $\quad\square$

We classify certain pointed rank one Hopf algebras using H_R and H_F.

Theorem 3.11. *Let H be a rank one Hopf algebra that is generated as an algebra by H_1. Assume there exists an eigenvector in $P_{a,1} \setminus H_0$ of the conjugation action of a with eigenvalue q. If $q \neq 1$*

then H is isomorphic to a Hopf algebra H_R given by a tuple R with $r = 0$. If $q = 1$ and k algebraically closed, then H is isomorphic to a Hopf algebra H_F given by a tuple F.

Proof. This follows directly by Theorem 3.3, Lemmas 3.5, 3.9 and Proposition 3.10. □

We can now classify pointed rank one Hopf algebras with semi-simple coradical.

Corollary 3.12 *(Semi-simple coradical). Let H be a rank one Hopf algebra over an algebraically closed field that is generated as an algebra by H_1 and has a semi-simple coradical. Then H is isomorphic to a Hopf algebra H_R with $r = 0$ or to a Hopf algebra H_F.*

Proof. The coradical of a pointed Hopf algebra is semi-simple if and only if the characteristic p of k does not divide the order of the group of group-like elements. Thus p does not divide the order of the skew point and 3.1 holds. The rest follows then directly by Theorem 3.11. □

Note that this proof only requires that the order of the skew point is not divisible by p.

Corollary 3.13. *Let H be a cocommutative rank one Hopf algebra over an algebraically closed field. Then H is isomorphic to H_F. Furthermore x is a common eigenvector of the conjugation action of G.*

Proof. If H is cocommutative then $a = 1$. Therefore every element of $P_{1,1} \setminus H_0$ is an eigenvector of the conjugation action of a with eigenvalue 1. By Theorem 3.11 we have $H \cong H_F$. Furthermore $c(g)(a - 1) = 0$ for all $g \in G$ which proves the last statement. □

We introduce nilpotent type Hopf algebras which have also been defined similarly by Krop and Radford [KrR].

Definition 3.14 *(Nilpotent type).* Let H be a rank one Hopf algebra as in 3.3. We call H of nilpotent type if there exists an eigenvector x in $P_{a,1} \setminus H_0$ of the conjugation action of a such that $x^n = 0$, where n denotes the degree of x.

Corollary 3.15. *The rank one Hopf algebra H_R is of nilpotent type if and only if $\alpha_{-1} = 0$. The Hopf algebra H_F is of nilpotent type if and only if $\alpha_0 = 0$.*

Proof. As H_R is rank one, we have $r = 0$ by 3.10. Since $\chi(a) \neq 1$ the eigenvector $x \in P_{a,1} \setminus H_0$ of the conjugation action of a is uniquely determined up to scalar multiplicities. By Theorem 3.3 we have $x^N = \alpha_{-1}(a^N - 1)$ and by the definition of α_{-1} we have that $x^N = 0$ if and only if $\alpha_{-1} = 0$.

Now consider H_F. All elements of $P_{a,1} \setminus H_0$ are eigenvectors of the conjugation action of a. An easy computation shows that if $\alpha_0 = 1$, then there exist no element y in $P_{a,1} \setminus H_0$ such that $y^p = 0$. If $\alpha_o = 0$ then by definition $\alpha_{-1} = 0$ and H_F is of nilpotent type. □

For certain orders of the skew point we have the following structure.

Corollary 3.16. *Let H be as in 3.11 and let a be an element of p-power order. Then H is isomorphic to a nilpotent type Hopf algebra H_F for a suitable tuple $F = (G, a, \chi, c, (0,0))$. If a*

does not have order p, then c is identical zero and $x \in H_F$ is therefore a common eigenvector of the conjugation action of G.

Proof. As the order of a is a p-power, we have $\chi(a) = 1$. Then $H \cong H_F$ by Theorem 3.11. As $a^p \neq a$ we have by Lemma 3.9 that $\alpha_0 = 0$ and H_F is therefore of nilpotent type. If $a^p \neq 1$ then it follows from Theorem 3.3(4) that $c(g) = 0$ for all $g \in G$. ☐

We have already determined precisely in 3.6 under which conditions rank one Hopf algebras H_R and $H_{R'}$ are isomorphic and we can now determine when nilpotent type Hopf algebras H_F and H'_F are isomorphic.

Lemma 3.17. *Let $H_{F'}$ and H_F be rank one Hopf algebras. If there exists a group isomorphism $f : G \to G'$ with $f(a) = a'$, $\chi = \chi' \circ f$, $c = c' \circ f$ and $\alpha = \alpha'$, then H_F and $H_{F'}$ are isomorphic. Conversely suppose that $H_{F'}$ and H_F are isomorphic rank one Hopf algebras of nilpotent type and a does not have order p. Then there exists a group isomorphism $f : G \to G'$ with $f(a) = a'$ and $\chi = \chi' \circ f$.*

Proof. We set $\rho(g x^i) := f(g) x'^i$. Then ρ is a Hopf algebra isomorphism.

Conversely suppose now that H_F and $H_{F'}$ are isomorphic via the Hopf morphism $\rho : H_F \to H_{F'}$. Analogously to 3.6 we have that $f := \rho|_G : G \to G'$ is a group isomorphism with $f(a) = a'$ and $\rho(x) = \beta x' + \gamma(a' - 1)$ for $\beta \in k^*$ and $\gamma \in k$. We have $0 = \rho(x^p) = \gamma^p(a'^p - 1)$. Thus $\gamma(a' - 1) = 0$ and $\chi = \chi' \circ f$. ☐

In the following we analyse the representations of the pointed rank one Hopf algebras H_R and H_F. We first consider the non-nilpotent type Hopf algebra H_R.

Suppose H_R is not of nilpotent type and k is algebraically closed. We can assume without loss of generality that $\alpha = 1$. Let $Z \subset G$ be the kernel of χ and let $\{e_i \mid 1 \leqslant i \leqslant s\}$ be the set of central primitive idempotents in kZ corresponding to the blocks of kZ. As H_R is not of nilpotent type $\chi^N(g) = 1$ for all $g \in G$ by 3.4(2). Thus the cyclic group $\{\chi(g) \mid g \in G\}$ is generated by $\chi(a)$ and therefore $G = \bigcup_{i=1}^N a^i Z$. We set $B_i := kZe_i$ for the blocks of kZ. By the definition of Z and as $a \in Z(G)$, the elements e_i commute with G and x. We have $H_R = \bigoplus_{1 \leqslant i \leqslant s} H_i$ where $H_i := H_R e_i$. Let A_i be the subalgebra of H_i generated by $\bar{x} := x e_i$ and $\bar{a} := a e_i$. Then A_i is isomorphic to $A_{\chi(a),1,N,|ae_i|}$ as in [Rad2, (C.6)]. Then $H_i \cong A_i \otimes_k B_i / \langle e_i \otimes \bar{a}^N - \bar{a}^N \otimes e_i \rangle$. We have $Z(A_i) = k\langle \bar{a}^N \rangle$ and therefore $Z(A_i) \subset Z(B_i)$. As e_i is the unique primitive central idempotent in B_i and $e_i \in Z(A_i)$, e_i is primitive in $Z(A_i)$ as well. Therefore $Z(A_i)$ is indecomposable. So $\bar{a}^N$ is either a scalar we_i for some $w \in k$ if $\dim Z(A_i) = 1$ or $Z(A_i) \cong kC_{p^t}$ for some $t \in \mathbb{N}$. As an algebra with indecomposable centre is indecomposable A_i is indecomposable. In the case of $\bar{a}^N = e_i$ we have $(\bar{x})^N = 0$ and $\langle \bar{a} \rangle$ has order N. Thus A_i is isomorphic to the Taft algebra $\Gamma_{N,\chi(a)}$ and $H_i \cong \Gamma_{N,\chi(a)} \otimes_k B_i$. If $\bar{a}^N = we_i$ with $w \neq 1$, then $A_i \cong M_n(k)$ by [Rad1, 1.3] as k is algebraically closed and $H_i \cong M_n(B_i)$. As the tensor product of two indecomposable algebras is indecomposable, H_i is indecomposable if $\bar{a}^N$ is a scalar. In the case that $Z(A_i) \cong kC_{p^t}$, we have that $\bar{a}^{Np^t}$ is a scalar and $\langle 1 \otimes \bar{a}^N - \bar{a}^N \otimes 1 \rangle^{p^t} = 0$. Therefore idempotents can be lifted from $A_i \otimes_k B_i / \langle e_i \otimes \bar{a}^N - \bar{a}^N \otimes e_i \rangle$ to $A_i \otimes_k B_i$. As $A_i \otimes_k B_i$ is an indecomposable algebra, H_i is indecomposable as well.

The proof of [KrR, Proposition 4] works in the following case.

Theorem 3.18. *Let H_R be a pointed rank one Hopf algebra over an algebraically closed field which is not of nilpotent type and which has a semi-simple coradical. Then its blocks are Morita equivalent to the Taft algebra $\Gamma_{N,\chi(a)}$ or to the matrix ring $M_N(k)$.*

Proof. We fix i and set $B = B_i$, $e = e_i$, $\bar{x} = xe$, $\bar{a} = ae$ and $A = A_i$. As kG is semi-simple and k algebraically closed we have $B \cong M_n(k)$ for some $n \in \mathbb{N}$ by the Wedderburn–Artin Theorem. The element $\bar{a}^N$ lies in the centre of B, hence $\bar{a}^N = we$ for some $w \in k$. Therefore $He \cong A \otimes_k B$ and $\bar{x}^N = (w-1)e$. If $w = 1$, then A is isomorphic to the Taft algebra $\Gamma_{N,\chi(a)}$. Therefore $He \cong \Gamma_{N,\chi(a)} \otimes M_n(k)$ is Morita equivalent to $\Gamma_{N,\chi(a)}$ by [ASS, 6.11]. If $w \neq 1$, then $A \cong M_N(k)$ and He is Morita equivalent to the full matrix ring. $\square$

The representations of Hopf algebras H of nilpotent type can be described entirely in terms of those of the underlying group algebra $kG(H)$. We generalise a result in [KrR, Proposition 3] which has only been proven for the nilpotent type in characteristic zero and G an abelian group.

Theorem 3.19. *Suppose H is a pointed rank one Hopf algebra of nilpotent type so that $H \cong H_R$ for some tuple R or $H \cong H_F$ for some tuple F.*

(a) *The simple H-modules are the simple kG-modules on which x acts as zero.*
(b) *Assume that the coradical of H is semi-simple. Let L_λ be the simple module where the restriction to kG has character λ, and let e_λ be the primitive idempotent of kG associated to λ. Then He_λ is an indecomposable projective H-module, and every indecomposable projective H-module is isomorphic to some He_λ.*
(c) *The module He_λ is uniserial, with composition factors*

$$L_\lambda, L_{\lambda\chi}, L_{\lambda\chi^2}, \ldots, L_{\lambda\chi^{n-1}}.$$

Hence two projectives He_λ, He_μ are in the same block if and only if $\mu = \lambda\chi^i$ for some i.

Proof. We will prove this theorem separately for the case $H \cong H_R$ and $H \cong H_F$. We suppose first that $H \cong H_R$. (a) As H_R is of nilpotent type, (x) is a nilpotent ideal. Therefore $(x) \subset J(H_R)$. Note that the quotient $H_R/(x)$ is isomorphic to kG. Thus all simple modules are isomorphic to simple kG-modules on which x acts as zero and a as the identity.

(b) and (c) We can take $L_\lambda = kGe_\lambda$. Consider the module He_λ, define subspaces

$$M^i := \bigoplus_{j \geqslant i} x^j(kGe_\lambda)$$

for $j = 0, 1, \ldots, n-1$. These are H-submodules of He_λ and $M^{i+1} \subseteq M^i$ for each i. The action of H can be calculated explicitly on the quotient, and one sees that M^i/M^{i+1} is isomorphic to the simple module with character $\lambda\chi^i$. We claim that He_λ is uniserial. Let M be any submodule, then there is i such that M is contained in M^i but not in M^{i+1}. Then M contains some element of the form

$$\omega = \sum_{j=i}^{n-1} x^j m_j$$

with $m_j \in kGe_\lambda$ where $m_i \neq 0$ (note that the m_j are unique). A standard argument using the nilpotency of x shows that then $M = M^i$.

Since He_λ is uniserial, it is indecomposable. The group algebra kG is a direct sum of simple modules $kG = \bigoplus kGe_j$ where the e_j are orthogonal primitive idempotents and each kGe_j is isomorphic to kGe_λ for some λ. It follows that $H = \bigoplus He_j$, and hence each indecomposable projective H-module is isomorphic to He_λ for some λ. All other statements of (b) and (c) follow for $H \cong H_R$.

We suppose now that $H \cong H_F$. If c is identically zero, (x) is nilpotent of index p and $(x) \subset J(H)$. In this case $H/(x) \cong kG$. If a does not have order p, then by the proof of Theorem 3.3 part (4) the map c is identically zero.

Suppose that a has order p and c is not identically zero. Since a is a central element of order p, the ideal $(a-1)H$ is nilpotent of index p. As c is not identically zero we have $(a-1)H \subset (x)$. Modulo $(a-1)H$ the elements $g \in G$ act on x by multiplication through $\chi(g)$. Therefore $(x)/(a-1)H$ is nilpotent of index p. Thus $(x)^{p^2} = 0$. This proves $(x) \subset J(H)$. In this case the quotient $H_F/(x)$ is isomorphic to $kG/(a-1)kG$ as $(x) \cap kG = (a-1)kG$. Thus all simple modules are isomorphic to simple kG-modules on which x acts as zero and a as identity.

(b) and (c) If kG is semi-simple, a does not have order p and x is therefore a common eigenvector of the conjugation action of G. The rest of the proof is the same then the proof of (b) and (c) in the case that $H \cong H_R$. □

4. Pointed rank one Hopf algebras of third type

Throughout this section H is a finite-dimensional pointed rank one Hopf algebra over a field k of positive characteristic p which is generated as an algebra by H_1. We assume that $H \neq H_0$. Let a be the skew point of H.

In this section we classify the pointed rank one Hopf algebras which do not have an eigenvector of the conjugacy action of a in $P_{a,1} \setminus H_0$.

We first analyse how the order of a and its conjugation action determine the type of Hopf algebra and introduce the third type of pointed rank one Hopf algebras.

Theorem 4.1. *Let a be the skew point of H and k algebraically closed. Then precisely one of the following holds*:

(i) *There is some $x \in P_{a,1} \setminus H_0$ such that $axa^{-1} = qx$ where q is a primitive N-root of unity.*
(ii) *There is some $x \in P_{a,1} \setminus H_0$ with $axa^{-1} = x + (a-1)$.*

Suppose (i) *holds. If $q \neq 1$ then H is isomorphic to H_R, with $r = 0$. If* (i) *holds and $q = 1$, then H is isomorphic to an algebra H_F as in* 3.8.

Suppose (ii) *holds, then p divides the order of a.*

Proof. If x is an eigenvector of a then Theorem 3.3 and Proposition 3.10 give the result.

Suppose now that a does not have an eigenvector in $P_{a,1} \setminus H_0$. By 3.1 the order of a must be divisible by p. By Lemmas 2.8 and 2.4 we know that the vector space of skew primitive elements $P_{a,1}$ is $\{cx + d(a-1) \mid c, d \in k\}$. Thus it has dimension two. As k is algebraically closed, the conjugation action of a on $P_{a,1}$ is represented by a two-dimensional Jordan block. As no element of $P_{a,1} \setminus H_0$ is an eigenvector, any eigenvector must be of the form $d(a-1) \in H_0$ for some $d \in k^*$. As those elements are eigenvectors with eigenvalue one, the entries on

the diagonal of the Jordan block must be one. Thus there is an element $x \in P_{a,1} \setminus H_0$ so that $axa^{-1} = x + (a-1)$. $\quad \square$

Considering the previous theorem we can introduce the following notations to classify H.

Definition 4.2. Suppose there is some $x \in P_{a,1} \setminus H_0$ with $axa^{-1} = qx$. If $q \neq 1$, we call H of first type; and if $q = 1$ of second type. If $P_{a,1} \setminus H_0$ contains an element x with $axa^{-1} = x + (a-1)$ then we call H of third type.

Note that all three cases are different. If k is algebraically closed then we have by the previous theorem that H is either of first, second or third type. We have already completely described the first and second type by generators and relations.

The conjugation action of the third type can be described as follows.

Corollary 4.3. Let H be of third type. Then the action of $G(H)$ on H is given by $gxg^{-1} = x + c(g)(a-1)$ where $c : G \rightarrow (k, +)$ is a group homomorphism.

Proof. By Lemma 2.8 the conjugation action of G can be described via the maps c and χ with $c(hg) = \chi(g)c(h) + c(g)$ for all $g, h \in G(H)$. We have $c(h) + 1 = c(ha) = c(ah) = \chi(h) + c(h)$ for all $h \in G$. Thus $\chi = 1$ and $c(hg) = c(h) + c(g)$ for all $g, h \in G(H)$. $\quad \square$

In order to show that third type Hopf algebras do exist we give an example in characteristic 2 and 3.

Example 4.4 (*Third type*). Consider the four-dimensional Hopf algebra A_2 over a field of characteristic 2 generated by a, x with basis $\{1, a, ax, x\}$ given by the following relations: $a^2 = 1$, $x^2 = x$ and $axa = x + (a+1)$. The following linear maps define the Hopf algebra structure on A:

- $\Delta(a) = a \otimes a$, $\Delta(x) = x \otimes a + 1 \otimes x$, $\Delta(ax) = ax \otimes 1 + a \otimes ax$,
- $\epsilon(a) = \epsilon(1) = 1$, $\epsilon(x) = \epsilon(ax) = 0$,
- $S(a) = a$, $S(1) = 1$, $S(x) = xa$, $S(ax) = x$.

Via direct computation we can check that Δ and ϵ fulfil the relations and are therefore algebra homomorphisms. The other axioms of a Hopf algebra can be checked on the basis elements.

This Hopf algebra is pointed rank one such that by Lemma 2.8 no skew primitive element is an eigenvector of the conjugation action of a. Thus it is not isomorphic to a Hopf algebra H_R or H_F.

As a left module A_2 is the direct sum of the indecomposable projective modules $P_1 = A_2x = \langle x, ax \rangle$ and $A_2(x+1) = \langle x+1, a+ax \rangle$. The algebra A_2 has two simple one-dimensional modules S_1 and S_2 where a acts on both as the identity and x acts on S_1 as the identity and on S_2 as zero. Then soc $P_1 = \langle x + ax \rangle \cong S_2$, $P_1/\operatorname{rad} P_1 \cong S_1$ and soc $P_2 = \langle 1 + ax + a + x \rangle \cong S_1$, $P_2/\operatorname{rad} P_2 \cong S_2$.

Example 4.5. Let A_p be the algebra over a field of characteristic p generated by a and x with basis $(x^i a^j)_{0 \leqslant i,j < p}$ and the relations $axa^{-1} = x + (a-1)$, $x^p = x$ and $a^p = 1$. Then we define linear maps by:

- $\Delta(a) = a \otimes a$, $\Delta(x) = x \otimes a + 1 \otimes x$, $\Delta(x^i a^j) = \Delta(x)^i \Delta(a)^j$ for all $0 \leqslant j, i \leqslant p - 1$,
- $\epsilon(x^i a^j) = 0$ for $i > 0$ and equal to 1 else,
- $S(a) = a$, $S(x) = -xa^{-1}$ and $S(x^i a^j) = S(a)^i S(x)^j$ for all $0 \leqslant j, i \leqslant p - 1$.

The maps Δ and ϵ satisfy the first relation and ϵ fulfils both relations and is therefore an algebra homomorphism. In the case $p = 2$ this is just the Hopf algebra given in Example 4.4. For $p = 3$ an easy calculation shows that the map Δ satisfies the second relation $\Delta(x)^3 = \Delta(x)$ and is therefore an algebra homomorphism. In this case the axioms of a Hopf algebra hold and A_3 is a Hopf algebra.

In order to classify third type Hopf algebras we first show that x has degree p and $x^p = x + \alpha_0(a^p - 1)$ with $\alpha_0 \in k$. To this end we will compute the coefficients of elements $x^i \otimes x^z a^j$ with $0 \leqslant i, z, j \leqslant p$ in $\Delta(x^p)$. Throughout this section p is a prime number.

Lemma 4.6 *(Fermat's Little Theorem). In a field of characteristic p we have $\prod_{m=1}^{p-1}(t - m) = t^{p-1} - 1$ where t is a variable.*

Furthermore we have to define two maps.

Lemma 4.7. *For $j, l \in \mathbb{N}$ we define*

$$f(j, l) = \sum_{1 \leqslant j_1 \leqslant \cdots \leqslant j_l \leqslant j} \prod_{u=1}^{l} j_u$$

and for $r, q \in \mathbb{N}$ with $0 < q < r$

$$g(r, q) = \sum_{0 < j_1 < \cdots < j_q < r} \prod_{u=1}^{q} j_u.$$

Then $f(j, p - j) = g(p, p - j) = 0 \bmod p$ for $1 < j < p$. Furthermore $f(1, p - 1) = 1$ and $g(p, p - 1) = (p - 1)! = -1 \bmod p$.

Proof. The value of $f(j, p - j)$ is the coefficient of t^{p-j} in the series expansion of $\prod_{i=1}^{j} 1/(1 - it)$, where t is a variable. Using Fermat's Little Theorem we have $\prod_{i=1}^{p-1} 1/(1 - it) = 1/(1 - t^{p-1})$. Thus

$$\prod_{i=1}^{j} 1/(1 - it) = 1/(1 - t^{p-1}) * \prod_{i=j+1}^{p-1}(1 - it) = \left(\sum_{i=0}^{\infty} t^{i(p-1)}\right) \prod_{i=j+1}^{p-1}(1 - it).$$

As $\prod_{i=j+1}^{p-1}(1 - it)$ is a polynomial of degree $p - j - 1$ the coefficients of t^{p-j} are zero for $1 < j < p$ and the coefficient of t^{p-1} is 1.

The value of $g(p, p - j)$ is just the coefficient of t^{j-1} up to sign in the polynomial $\prod_{m=1}^{p-1}(t - m)$. Thus by Fermat's little Theorem we have $g(p, p - j) = 0 \bmod p$ for $1 < j < p$ and $g(p, p - 1) = -1$. $\square$

We set $f(j,0) = g(j,0) = 1$ and $f(j,l) = g(j,l) = 0$ for $l < 0$ and $j > 0$. As one can easily see the following relations are true.

Lemma 4.8. *We have* $jf(j, l-1) + f(j-1, l) = f(j, l)$ *and* $(j-1)g(j-1, m-1) + g(j-1, m) = g(j, m)$ *for all* m, l *and* $j > 0$.

The next lemma and corollary work in an algebra A over a field of characteristic p generated by an element x and an element a such that the set $\{x^i a^j \mid i, j \in \mathbb{N}\}$ is linearly independent and $ax = xa + a^2 - a$. We also fix a scalar $c(g) \in k$.

Lemma 4.9. *We write* $(x + c(g)a)^k$ *as a linear combination of elements in* $\{x^i a^j \mid 0 \leqslant i,$ $j \leqslant k\}$. *Let* $s^k_{i,j}$ *be the coefficient of* $x^i a^j$. *We define* $l := k - (i + j)$. *Then we have* $s^k_{i,j} = (-1)^l \binom{k}{i} f(j, l) \prod_{u=1}^{j} [c(g) - 1 + u]$ *for* $j > 0$ *and* $s^k_{i,0} = \delta_{i,k}$.
We write $(x \otimes a + 1 \otimes x)^k$ *in* $A \otimes_k A$ *as linear combination of elements in* $\{x^i \otimes x^z a^j \mid 0 \leqslant i,$ $j, z \leqslant k\}$. *Let* $h^k_{i,z,j}$ *be the coefficient of* $x^i \otimes x^z a^j$. *We define* $l := k - (z + j)$ *and* $m := j - i$. *Then we have* $h^k_{i,z,j} = (-1)^l \binom{k}{z} f(j, l) g(j, m)$ *for* $0 < i, j$, $h^k_{0,z,j} = \delta_{j,0} \delta_{k,z}$ *and* $h^k_{i,z,0} = \delta_{i,0} \delta_{k,z}$.
We write $(xa^{-1})^k$ *as a linear combination of elements in* $\{x^i a^{-j} \mid 0 \leqslant i, j \leqslant k\}$. *Let* $c^k_{i,j}$ *be the coefficient of* $x^i a^{-j}$. *Then we have* $c^k_{i,j} = (-1)^{k-j} g(j, j-i) f(j, k-j)$ *for* $1 \leqslant i, j \leqslant k$ *and* $c^k_{0,j} = c^k_{i,0} = 0$.

Proof. By definition it is clear that $a^j x = xa^j + j(a^{j+1} - a^j)$ for all $j \geqslant 0$. If we set $s^k_{v,w} = 0$ for $v < 0$ or $w < 0$, then the coefficients $s^k_{i,j}$ of $x^i a^j$ are given by the recursion

$$s^k_{i,j} := s^{k-1}_{i-1,j} + (c(g) + j - 1)s^{k-1}_{i,j-1} - js^{k-1}_{i,j},$$

where $s^1_{0,1} = c(g)$, $s^1_{1,0} = 1$ and the other coefficients $s^1_{i,j}$ are zero. This follows directly from the fact that $x^i a^j (x + c(g)a) = x^{i+1} a^j + (c(g) + j)x^i a^{j+1} - jx^i a^j$. First one checks that $s^k_{i,j} = 0$ if $i + j > k$ by a straightforward induction using recursion. We will show by induction on k that the stated formulae hold for the remaining case that $i + j \leqslant k$. For $k = 1$ they are true. Suppose now that the formulae are true for $k - 1$. If $j = 0$ we have by the recursion $s^k_{i,0} = s^{k-1}_{i-1,0} = \delta_{i-1,k-1} = \delta_{i,k}$.
We have $0 \leqslant k - (i + j) =: l$. If $j = 1$ and $i < k - 1$ we have

$$s^k_{i,1} = (-1)^l \binom{k-1}{i-1} f(1, l)c(g) - (-1)^{l-1} \binom{k-1}{i} f(1, l-1)c(g) = (-1)^l \binom{k}{i} c(g)$$

using that $f(1, r) = 1$ for all $0 \leqslant r$. If $j = 1$ and $i = k - 1$ we have

$$s^k_{k-1,1} = (k-1)f(1,0)c(g) + c(g) - (-1)f(1,-1)c(g) = kc(g).$$

If $j > 1$ we have

$$s_{i,j}^k = (-1)^l \binom{k-1}{i-1} f(j,l) \prod_{u=1}^{j} [c(g) - 1 + u]$$

$$+ (j - 1 + c(g)) \prod_{u=1}^{j-1} [c(g) - 1 + u](-1)^l \binom{k-1}{i} f(j-1, l)$$

$$- j(-1)^{l-1} \binom{k-1}{i} f(j, l-1) \prod_{u=1}^{j} [c(g) - 1 + u]$$

$$= (-1)^l \binom{k-1}{i-1} f(j,l) \prod_{u=1}^{j} [c(g) - 1 + u]$$

$$+ (-1)^l \binom{k-1}{i} f(j,l) \prod_{u=1}^{j} [c(g) - 1 + u]$$

$$= (-1)^l \binom{k}{i} f(j,l) \prod_{u=1}^{j} [c(g) - 1 + u]$$

using the first identity of Lemma 4.8. This proves the coefficient formula for s.

If we set $h_{r,s,t}^k = 0$ for all k and $r, 0$ or $s < 0$ or $t < 0$, then the coefficients $h_{i,z,j}^k$ of $x^i \otimes x^z a^j$ are given by the recursion

$$h_{i,z,j}^k := h_{i-1,z,j-1}^{k-1} + h_{i,z-1,j}^{k-1} + (j-1)h_{i,z,j-1}^{k-1} - jh_{i,z,j}^{k-1},$$

where $h_{0,1,0}^1 = h_{1,0,1}^1 = 1$ and zero else. This follows from the fact that $(x^i \otimes x^z a^j)(x \otimes a + 1 \otimes x) = x^{i+1} \otimes x^z a^{j+1} + x^i \otimes x^{z+1} a^j + jx^i \otimes x^z a^{j+1} - jx^i \otimes x^z a^j$. By an easy induction we see that $h_{i,j,z}^k$ is zero if $j - i < 0$ or $z + j > k$. We show by induction on k that the formula also holds for the other cases. We set $0 \leqslant j - i =: m$ and $0 \leqslant k - z - j =: l$.

Suppose $i = 0$, then

$$h_{0,z,j}^k = h_{0,z-1,j}^{k-1} + (j-1)h_{0,z,j-1}^{k-1} - jh_{0,z,j}^{k-1}$$

$$= \delta_{j,0}\delta_{k-1,z-1} + (j-1)\delta_{j-1,0}\delta_{k-1,z} - j\delta_{j,0}\delta_{k-1,z}$$

$$= \delta_{j,0}\delta_{k,z}.$$

Suppose that $j = 0$, then $h_{i,z,0}^k = h_{i,z-1,0}^{k-1} = \delta_{i,0}\delta_{k-1,z-1} = \delta_{j,0}\delta_{k,z}$. Now we assume that $1 < i \leqslant j$, then

$$h_{i,z,j}^k = (-1)^l \binom{k-1}{z} f(j-1, l)g(j-1, m) + (-1)^l \binom{k-1}{z-1} f(j, l)g(j, m)$$

$$+ (j-1)(-1)^l \binom{k-1}{z} f(j-1, l)g(j-1, m-1)$$

$$- j(-1)^{l-1} \binom{k-1}{z} f(j, l-1)g(j, m)$$

$$= (-1)^l \binom{k-1}{z} f(j-1,l)g(j,m) + (-1)^l \binom{k-1}{z-1} f(j,l)g(j,m)$$

$$- j(-1)^{l-1} \binom{k-1}{z} f(j,l-1)g(j,m)$$

$$= (-1)^l \binom{k-1}{z} f(j,l)g(j,m) + (-1)^l \binom{k-1}{z-1} f(j,l)g(j,m)$$

$$= (-1)^l \binom{k}{z} f(j,l)g(j,m)$$

using the second and then the first identity of Lemma 4.8.

In the case $i = 1$, $j > 1$ we have

$$h_{1,z,j} = \delta_{j,0}\delta_{k-1,z} + (-1)^l \binom{k-1}{z-1} f(j,l)g(j,j-1)$$

$$+ (j-1)(-1)^l \binom{k-1}{z} f(j-1,l)g(j-1,j-2)$$

$$- j(-1)^{l-1} \binom{k-1}{z} f(j,l-1)g(j,j-1)$$

$$= (-1)^l \binom{k}{z} f(j,l)g(j,j-1),$$

if we use first that $(j-1)g(j-1,j-2) = g(j,j-1)$ and then the first identity of Lemma 4.8. Finally we consider the case $i = j = 1$. Then

$$h^k_{1,z,1} = \delta_{k,z} + (-1)^l \binom{k-1}{z-1} f(1,l)g(1,0)$$

$$- (-1)^{l-1} \binom{k-1}{z} f(1,l-1)g(1,0)$$

$$= (-1)^l \binom{k}{z}$$

because we have $k > z$, since otherwise $l < 0$ which contradicts our assumptions. This proves the statement.

By definition $a^{-j}x = xa^{-j} + j(a^{-j} - a^{-j+1})$ for $j \in \mathbb{N}$. Therefore $x^i a^{-j}xa^{-1} = x^{i+1}a^{-j-1} + jx^i a^{-j-1} - jx^i a^{-j}$. So in the last case the recursion formula is given by $c^k_{i,j} = c^{k-1}_{i-1,j-1} + (j-1)c^{k-1}_{i,j-1} - jc^{k-1}_{i,j}$ with $c^1_{i,j} = \delta_{i,1}\delta_{j,1}$. Then the formula for the coefficients follows as in the previous step by induction using the identities given in Lemma 4.8.

Suppose $i = 0$ or $j = 0$, then by the recursion formula $c^k_{0,j} = c^k_{i,0} = 0$ and $c^k_{i,j} = 0$ for $i > j$. Suppose now that $i = 1$ and $j = 1$. Then $c^k_{1,1} = -1c^{k-1}_{1,1} = (-1)^{k-1}$. Next suppose that $i = 1$ and $j > 1$. Then $c^k_{1,j} = (j-1)c^{k-1}_{1,j-1} - jc^{k-1}_{1,j} = (j-1)(-1)^{k-j}g(j-1,j-2)f(j-1,k-j) + j(-1)^{k-j}g(j,j-1)f(j,k-j-1) = (-1)^{k-j}(j-1)![f(j-1,k-j) + jf(j,k-j-1)] = (-1)^{k-j}g(j,j-1)f(j,k-j)$ by the second equation of 4.8. Finally we can assume $1 < i \leqslant j \leqslant k$. Then $c^k_{i,j} = (-1)^{k-j}[g(j-1,j-i)f(j-1,k-j) + (j-1)g(j-1,j-i-1)f(j-1,$

$k - j) + jg(j, j - i) f(j, k - j - 1)] = (-1)^{k-j} [g(j, j - i) f(j - 1, k - j) + jg(j, j - i) f(j, k - j - 1)] = (-1)^{k-j} g(j, j - i) f(j, k - j)$ using first the identity on g in 4.8 and then the identity on f in 4.8. $\quad\square$

Using the result of the previous lemma and the same setup we can now determine the coefficients for the pth power.

Corollary 4.10. *We have* $(x + c(g)a)^p = x^p + c(g)a + (c(g)^p - c(g))a^p$, $(x \otimes a + 1 \otimes x)^p = x^p \otimes a^p + (p - 1)! x \otimes a^p + x \otimes a + 1 \otimes x^p$ *and* $(xa^{-1})^p = x^p a^{-p} - xa^{-p} + xa^{-1}$.

Proof. We have to show that $s^p_{p,0} = 1$, $s^p_{0,1} = c(g)$ and $s^p_{0,p} = c(g)^p - c(g)$ and that all other coefficients are zero by using the formula provided in Lemma 4.9. If we take $0 < j, i < p$, we get $s^p_{i,j} = 0$ mod p as $\binom{p}{i} = 0$ mod p. If $i = 0$ and $0 < j \leqslant p$, then we have $s^p_{0,j} = (-1)^{p-j} f(j, p - j) \prod_{u=1}^{j} [c(g) - 1 + u]$. Lemma 4.7 proves that $s^p_{0,j} = 0$ for $1 < j < p$ and $s^p_{0,1} = c(g)$. By Fermat's Little Theorem we get $s^p_{0,p} = \prod_{u=1}^{p} [c(g) - 1 + u] = c(g) \prod_{u=1}^{p-1} [c(g) - u] = c(g)(c(g)^{p-1} - 1) = c(g)^p - c(g)$. Furthermore we have $s^p_{p,j} = \delta_{j,0}$ since if $j > 0$ then $l < 0$ and $s^p_{i,0} = \delta_{i,p}$ by Lemma 4.9. Finally $s^p_{i,p} = 0$ for $i > 0$ since then $l < 0$. This proves the first identity.

In order to prove the second identity, we have to show that $h^p_{0,p,0} = 1$, $h^p_{p,0,p} = 1$, $h^p_{1,0,p} = (p - 1)!$ and $h^p_{1,0,1} = 1$ and all other coefficients are zero. We first consider the case $j, i > 0$. If $0 < z < p$ we get $h^p_{i,z,j} = 0$ mod p as $\binom{p}{z} = 0$ mod p. If $z = 0$ and $i, j > 0$ we have $h^p_{i,0,j} = (-1)^{p-j} f(j, p - j) g(j, j - i)$. Lemma 4.7 gives us $h^p_{i,0,j} = 0$ for $1 < j < p$ and $h^p_{1,0,1} = (-1)^{p-1} f(1, p - 1) g(1, 0) = 1$. By Lemma 4.7 we have $h^p_{i,0,p} = g(p, p - i) = 0$ mod p for $1 < i < p$, $h^p_{p,0,p} = 1$ and $h^p_{1,0,p} = g(p, p - 1) = (p - 1)!$. If $z = p$ then $h(i, p, j) = 0$ because $l < 0$.

The values of the coefficients for $i = 0$ or $j = 0$ follow immediately from 4.9.

By 4.9 we have $c^p_{0,j} = c^p_{i,0} = 0$ and $c^p_{i,j} = 0$ for all $1 < i < p$ using 4.7. Furthermore $c^p_{i,1} = \delta_{1,i}$ as $g(1, 1 - i) = 0$ for $i > 1$ and $c^p_{i,p} = 0$ for $1 < i < p$, $c^p_{1,p} = -1$ by 4.7 and $c^p_{p,p} = 1$. Therefore the equation holds. $\quad\square$

One can easily see that 4.9 and 4.10 can be applied to our Hopf algebra H in order to deduce some necessary conditions for the degree n of x.

Similarly to [KrR, Lemma 1] we prove the following property of the degree n of x. If we set $V_0 = H_0$ and $V_k := \sum_{t=0}^{k} H_0 x^t$, then all conditions of Remark 2.7 are satisfied and $\rho_k(x^k + V_{k-1}) = 1 \otimes (x^k + V_{k-1})$ as $h^k_{j,z,i} = \delta_{i,0} \delta_{j,0} \delta_{k,z}$ for $z \geqslant k$ and $x^t H_0 \subset V_t$ by 4.9. We have $Q_k \neq \{0\}$ for all $k < n$ and $Q_n = \{0\}$. As $x^k + V_{k-1}$ forms a basis for Q_k as H_0-module, n is the smallest integer such that $1, x, \ldots, x^{n-1}$ form a basis for H over H_0.

Lemma 4.11. *Let H be a pointed rank one Hopf algebra of third type. Let n be the degree of x and $x^n = \sum_{i=0}^{n-1} \alpha_i x^i$ with $\alpha_i \in kG$. Then the coefficients α_i with $1 \leqslant i$ are in k and $\alpha_0 = \alpha(a^n - 1)$ for some $\alpha \in k$. Furthermore $n = p^q$ for a positive integer q.*

Proof. We use that $x^k \otimes x^j$ with $0 \leqslant k, j \leqslant n - 1$ is free over $kG \otimes_k kG$ and that the set $\{x^k g \mid g \in G, \ 0 \leqslant k \leqslant n - 1\}$ is linearly independent. We have $(x \otimes a + 1 \otimes x)^n = \Delta(x)^n = \sum_{i=0}^{n-1} \Delta(\alpha_i) \Delta(x)^i$. If we compare the coefficients in $H_0 \otimes H_0$ of $1 \otimes x^i$ for all $0 \leqslant i \leqslant n - 1$ on

both sides this gives us $1 \otimes \alpha_i = \Delta(\alpha_i)$ for $1 \leqslant i \leqslant n-1$ and $\alpha_0 \otimes a^n + 1 \otimes \alpha_0 = \Delta(\alpha_0)$. Thus $\alpha_i \in k$ for $1 \leqslant i \leqslant n-1$ and $\alpha_0 = \alpha(a^n - 1)$ for an $\alpha \in k$. For the second part we compare the coefficients of $x^r \otimes x^{n-r} a^r$ for $1 \leqslant r \leqslant n-1$ using the coefficients computed in Lemma 4.9. We have $h_{r,n-r,r}^n = \binom{n}{r}$ on the left-hand side and $h_{r,n-r,r}^i = 0$ for all $1 \leqslant i \leqslant n-1$ on the right-hand side. Thus $\binom{n}{r} = 0 \bmod p$ for all $1 \leqslant r \leqslant n-1$ and n has to be a p-power. $\quad\square$

Now we show that $n = p$ using the results for the coefficients of the term $\Delta(x)^p$ and the previous lemma.

Lemma 4.12. *Let H be a pointed rank one algebra of third type. Then $x^p \in H_1$ and p is the degree of x. We have $x^p = x + \alpha(a^p - 1)$ for some $\alpha \in k$.*

Proof. We have $\Delta(x^p) = x^p \otimes a^p + (p-1)! x \otimes a^p + x \otimes a + 1 \otimes x^p$ from Corollary 4.10. Thus $x^p \in H_1$ by the definition of H_1. Using Lemma 4.11 we know that the degree of x is p. By part two of Lemma 4.11 we have $x^p = \alpha_1 x + \alpha(a^p - 1)$ with $\alpha_1, \alpha \in k$. Comparing the coefficients of $x \otimes a$ in $\Delta(x)^p$ with the coefficient in $\Delta(\alpha_1 x + \alpha(a^p - 1))$ gives $\alpha_1 = 1$ if $a^p \neq a$. By 4.1 we know that the order of a is divisible by p and therefore $a^p \neq a$. $\quad\square$

If a has order p, then $\alpha = 0$. If k is algebraically closed, then we can choose $\beta \in k$ such that $\beta - \beta^p = \alpha$ and set $\bar{x} = x + \beta(a-1)$. In this case $\bar{x} \in P_{a,1} \setminus H_0$ and $g\bar{x}g^{-1} = \bar{x} + c(g)(a-1)$ as a is central. Furthermore $\bar{x}^p = (x + \beta a)^p - \beta^p = x^p + \beta a + (\beta^p - \beta)a^p - \beta^p = \bar{x}$. We can therefore assume without loss of generality that $\alpha = 0$.

In the next definition we introduce Hopf algebras parametrised by a tuple E.

Definition 4.13 (*third type*). Let $E := (G, a, c, \alpha)$ be a tuple, where G is a finite group, $a \in Z(G)$ and a has an order divisible by p and $c : G \to (k, +)$ is a group homomorphism. The Hopf algebra H_E is the algebra with basis $(gx^i)_{\{g \in G, 0 \leqslant i \leqslant p-1\}}$ and relations $gxg^{-1} = x + c(g)(a-1)$ and $x^p = x + \alpha(a^p - 1)$. The Hopf algebra structure is determined by the linear maps Δ, ϵ and S with $\Delta(gx^i) = (g \otimes g)(x \otimes a + 1 \otimes x)^i$, $\epsilon(gx^i) = \delta_{i,0}$ and $S(gx^i) = (-xa^{-1})^i g^{-1}$ for all $g \in G$ and $0 \leqslant i \leqslant p-1$. If $a^p \neq 1$ we assume that $c(g) \in F_p$ for all $g \in G$.

The proof of the next lemma shows that H_E with $a^p \neq 1$ is well defined if and only if $c(g) \in F_p$ for all $g \in G$.

Lemma 4.14. *The Hopf algebra H_E is well defined.*

Proof. Using the results of Corollary 4.10, we get the following equation

$$
\begin{aligned}
x^p + c(g)(a-1) = gx^p g^{-1} &= \left(gxg^{-1}\right)^p \\
&= \left(x + c(g)a - c(g)\right)^p = \left(x + c(g)a\right)^p - c(g)^p \\
&= x^p + c(g)a + \left(c(g)^p - c(g)\right)a^p - c(g)^p.
\end{aligned}
$$

If $a^p = 1$, then the equation holds for any value of $c(g)$. If $a^p \neq 1$, then $gx^p g^{-1} = (gxg^{-1})^p$ is true if and only if $c(g)^p = c(g)$, which is the case if and only if $c(g) \in F_p$ for all $g \in G$. Then H_E is well defined as an algebra. Using Theorem 4.12 and the expression for $\Delta(x)^p$ in Corollary 4.10, we see that Δ and ϵ are algebra homomorphism.

Let S be a linear map with $S(gx^i) = (-xa^{-1})^i g^{-1}$ on the basis $\{gx^i \mid g \in G, \ 0 \leqslant i \leqslant p-1\}$. By 4.10 and 4.12 we have $S(x)^p = S(x) + \alpha(a^{-p} - 1)$ and $S(gxg^{-1}) = S(g^{-1})S(x)S(g)$ for all $g \in G$. Therefore S is an anti-algebra homomorphism. It is easy to prove that the axiom is satisfied for all elements of this basis. $\square$

We can now classify Hopf algebras of third type.

Theorem 4.15. *Let H be a third type, then H is isomorphic to a Hopf algebra H_E for a tuple $E = (G(H), a, c, \alpha)$.*

Proof. This follows immediately from Lemma 4.12, Definition 4.13 and Corollary 4.3. $\square$

The converse is also true.

Lemma 4.16. *The Hopf algebras H_E are pointed, rank one and of third type.*

Proof. Exactly as in 3.10 we can see that H_E is pointed, as it is generated by skew primitive elements. We want to show that it is rank one. Let $z \in H_E \setminus kG$ be an element with $\Delta(z) = z \otimes g + 1 \otimes z$ for some $g \in G$. Then z has a unique presentation as $z = \sum_{t=0}^{p-1} x^t b_t$ with $b_t \in kG$ for all $0 \leqslant t \leqslant p-1$. These conditions give the following equation

$$\sum_{t=0}^{p-1} \sum_{i,z,j=0}^{t} h_{i,z,j}^t \left(x^i \otimes x^z a^j \right) \Delta(b_t) = \sum_{t=0}^{p-1} \Delta(x)^t \Delta(b_t)$$

$$= \Delta(z) = z \otimes g + 1 \otimes z$$

$$= \sum_{t=0}^{p-1} \left(x^t \otimes 1 \right) (b_t \otimes g) + \left(1 \otimes x^t \right)(1 \otimes b_t).$$

We compare the coefficients of $1 \otimes x^z$ for $0 \leqslant z \leqslant p-1$ on both sides. We have $h_{0,z,j}^t = \delta_{j,0}\delta_{t,z}$ for all $0 \leqslant t \leqslant p-1$. Therefore $\Delta(b_z) = 1 \otimes b_z$ for $1 \leqslant z \leqslant p-1$. This forces $b_z \in k$ for all $1 \leqslant z \leqslant p-1$. In the case $z = 0$ we have $\Delta(b_0) = b_0 \otimes g + 1 \otimes b_0$ and therefore $b_0 = b(g-1)$ for some $b \in k$. Let n be maximal with $b_n \neq 0$. We compare now the coefficients of $x^n \otimes 1$. We have $h_{n,0,j}^t = \delta_{j,n}\delta_{t,n}$ for $t \leqslant n$ by 4.9. This gives us $(1 \otimes a^n)\Delta(b_n) = b_n \otimes g$. Therefore $g = a^n$. Suppose $n > 1$. The coefficient of $x \otimes x^{n-1}a$ on the left-hand side is $\Delta(b_n)n \neq 0$ as $h_{1,n-1,1}^t = \delta_{n,t}\binom{n}{n-1}$ for $t \leqslant n$. As $x \otimes x^{n-1}a$ does not appear on the right-hand side, this is a contradiction. Thus $z = b_1 x + b(a-1)$ for $b_1 \in k^*$ and $b \in k$.

For all $h \in P_{v,u} \setminus H_0$ with $v, u \in G$ we have $h = 0$ for $u^{-1}v \neq a$ and else $h = u(b_1 x + b(a-1))$. As $H_1 = H_0 + \sum_{v,u \in G(H)} P_{v,u} \subset H_0 + H_0 x$ and $x \notin H_0$, the dimension of $k \otimes_{H_0} H_1$ is two. Thus H_E is a rank one Hopf algebra. It is clearly of third type as the skew point a does not have an eigenvector in $P_{a,1} \setminus H_0$ of the conjugation action. $\square$

In the following lemma we determine precisely under which conditions two third type Hopf algebras presented by different tuples are isomorphic.

Lemma 4.17. *Let H_E and $H_{E'}$ be two Hopf algebras of third type over an algebraically closed field given by tuples E and E'. Then they are isomorphic if and only if there is a group isomorphism from $f : G \to G'$ such that $f(a) = a'$ and $c' \circ f = c$.*

Proof. Let $f : G \to G'$ be a group isomorphism with $f(a) = a'$ and $c' \circ f = c$. We define the linear map $F : H_E \to H_{E'}$, $gx^i \mapsto f(g)x'^i$ which is bijective. This gives a Hopf algebra isomorphism.

Conversely, we suppose now that H_E and $H_{E'}$ are isomorphic as Hopf algebras via a map F. As group-like elements are mapped to group-like elements, the map $f := F|_G : G \to G'$ is a group isomorphism. Since elements in $H_1 \setminus H_0$ are mapped to elements in $H_1' \setminus H_0'$ and skew primitives to skew primitives, we have $f(a) = a'$ and $F(x) = \gamma x' + \beta(a' - 1)$ for some $\beta, \gamma \in k$. We have $\gamma x' + (\beta + \gamma)(a' - 1) = a'F(x)a'^{-1} = F(axa^{-1}) = \gamma x' + (\beta + 1)(a' - 1)$. Thus $\gamma = 1$. We have $x' + c'(f(g))(a' - 1) + \beta(a' - 1) = f(g)F(x)f(g)^{-1} = F(gxg^{-1}) = x' + c(g)(a' - 1) + \beta(a' - 1)$. As $a' \neq 1$ we have $c(g) = c'(f(g))$ for all $g \in G$.   □

In order to analyse the representation theory of third type Hopf algebras we first analyse its subalgebra $k[x]/\langle x^p - x \rangle$.

Lemma 4.18. *Let $k[x]_p := k[x]/\langle x^p - x \rangle$ be a polynomial ring with indeterminate x and relation $x^p = x$. Let k be a field of characteristic p. Then $k[x]_p$ viewed as a left $k[x]_p$-module is the direct sum of p simple one-dimensional modules S_c on which x acts as scalar c with $c \in F_p$.*

Proof. We have $x^p - x = \prod_{c \in F_p}(x - c)$. Therefore $k[x]/\langle x^p - x \rangle$ is isomorphic to $\prod_{x \in F_p} k[x]/\langle x - c \rangle$ by the Chinese Reminder Theorem. Notice that $k[x]/\langle x - c \rangle$ is isomorphic to k for any $c \in F_p$. Therefore $k[x]_p$ is semi-simple and the simple modules are as stated.   □

Let $\{e_c \mid c \in F_p\}$ be the set of orthogonal central primitive idempotents in $k[x]_p$ such that $k[x]_p \cong \bigoplus_{c \in F_p} k[x]_p e_c$ and $k[x]_p e_c \cong S_c$. We can now describe the projective and simple modules of certain third type Hopf algebras.

Lemma 4.19. *Let A be of third type, with $G = \langle a \rangle$ cyclic of p-power order and $\alpha = 0$. Then A is indecomposable and its simple modules are one-dimensional. There are p simple modules T_c indexed by $c \in F_p$ on which x acts by scalar multiplication with c and a as the identity. The projective indecomposable modules of A are Ae_c.*

Proof. We have $A = \bigoplus_{c \in F_p} Ae_c$. The submodules Ae_c are clearly indecomposable as left $k\langle a \rangle$-modules. We set $y := 1 - a$. Then a composition series of Ae_c is $P_c^{p^r-1} \subset P_c^{p^r-2} \subset \cdots \subset P_c^0$ where $P_c^i := \langle y^j e_c \mid i \leqslant j \leqslant p^r - 1 \rangle$. The quotients are one-dimensional and a acts on them as the identity. We have $xy^i e_c = y^i xe_c + i(y^{i+1} - y^i)e_c$ and therefore x acts on P_c^i / P_c^{i+1} by scalar multiplication with $c - i$. As any two projective indecomposable modules have a common composition factor, the algebra A is indecomposable.   □

If a has order p, the algebra in 4.19 is isomorphic to a restricted p-Lie algebra constructed in the following way: We choose $L := \langle u, v \rangle$ and $[u, v] = -v$ and define the p-map $u^{[p]} := u$ and $v^{[p]} := 0$. It is easy to see that $(\mathrm{ad}u)^p = \mathrm{ad}u^{[p]} = \mathrm{ad}u$ and $(\mathrm{ad}v)^p = \mathrm{ad}v^{[p]} = 0$. In $U^{[p]}(L)$ we have $u^p = u$ and $v^p = 0$. If we set $x := u$ and $a^{-1} := 1 - v$ we have $a^p = 1$ and $axa^{-1} = x + (a - 1)$. Therefore $U^{[p]}(L)$ is isomorphic to A.

In general we can describe third type Hopf algebras as follows.

Let H be a third type Hopf algebra. The group G acts on the two-dimensional subspace A spanned by a and x via conjugation. This gives by 4.3 the following representation $\rho: G \to Gl_n(A)$, $g \mapsto \begin{pmatrix} 1 & c(g) \\ 0 & 1 \end{pmatrix}$.

Let $Z := \{g \in G \mid c(g) = 0\}$ be the kernel of ρ. The image of c is isomorphic to a finite additive subgroup of k. Furthermore all elements of G/Z have order p. Therefore G/Z is an elementary abelian p-group. If a does not have order p, then $|\text{Im}\, c| = p$ and therefore G/Z is cyclic of order p and generated by aZ. Thus $G \cong Z \times_{\langle a^p \rangle} \langle a \rangle \cong Z \times \langle a \rangle / \langle b \times a^p \rangle$ with $b = a^{-p} \in Z$. Let $\{e_i \mid 1 \leqslant i \leqslant s\}$ be the set of orthogonal central primitive idempotents in kZ and $B_i := kZe_i$ its blocks. Then e_i commutes with all elements of H. Therefore $H = \bigoplus_{i=1}^{s} He_i$ is a decomposition into subalgebras $H_i := He_i$. Set $\bar{a} := ae_i$, $\bar{x} := xe_i$ and let A_i denote the subalgebra generated by $\bar{a}$ and $\bar{x}$. Then $H_i \cong A_i \otimes_k B_i / \langle e_i \otimes_k \bar{a}^p - \bar{a}^p \otimes_k e_i \rangle$. If $\bar{a}^p = e_i$, then the algebra A_i is isomorphic to the indecomposable algebra in 4.19. In this case $H_i \cong A_i \otimes_k B_i$. The algebra structure of B_i can be arbitrarily complicated.

By an argument similar to the argument before 3.18 the algebra H_i is indecomposable.

Acknowledgments

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