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Electron–positron pairs production in a macroscopic charged core

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ABSTRACT

Classical and semi-classical energy states of relativistic electrons bounded by a massive and charged core with the charge-mass radio Q/M and macroscopic radius R_c are discussed. We show that the energies of semi-classical (bound) states can be much smaller than the negative electron mass-energy ($-mc^2$), and energy-level crossing to negative energy continuum occurs. Electron–positron pair production takes place by quantum tunneling, if these bound states are not occupied. Electrons fill into these bound states and positrons go to infinity. We explicitly calculate the rate of pair-production, and compare it with the rates of electron–positron production by the Sauter–Euler–Heisenberg–Schwinger in a constant electric field. In addition, the pair-production rate for the electro-gravitational balance ratio $Q/M = 10^{-19}$ is much larger than the pair-production rate due to the Hawking processes.

1. Introduction

As reviewed in the recent report [1], very soon after the Dirac equation for a relativistic electron was discovered [2,3], Gordon [4] (for all $Z < 137$) and Darwin [5] (for $Z = 1$) found its solution in the point-like Coulomb potential $V(r) = -Z\alpha/r$, they obtained the well-known Sommerfeld's formula [6] for energy spectrum,

$$\mathcal{E}(n,j) = mc^2\left[1+\left(\frac{Z\alpha}{n-|K|+(K^2-Z^2\alpha^2)^{1/2}}\right)^2\right]^{-1/2}, \tag{1}$$

where the fine-structure constant $\alpha = e^2/\hbar c$, the principle quantum number $n = 1, 2, 3, \ldots$ and

$$K = \begin{cases} -(j+1/2) = -(l+1), & \text{if } j = l+\frac{1}{2},\ l \geqslant 0, \\ (j+1/2) = l, & \text{if } j = l-\frac{1}{2},\ l \geqslant 1, \end{cases} \tag{2}$$

$l = 0, 1, 2, \ldots$ is the orbital angular momentum corresponding to the upper component of Dirac bi-spinor, j is the total angular momentum. The integer values n and j label bound states whose energies are $\mathcal{E}(n,j) \in (0, mc^2)$. For the example, in the case of the lowest energy states, one has

$$\mathcal{E}(1S_{\frac{1}{2}}) = mc^2\sqrt{1-(Z\alpha)^2}, \tag{3}$$

$$\mathcal{E}(2S_{\frac{1}{2}}) = \mathcal{E}(2P_{\frac{1}{2}}) = mc^2\sqrt{\frac{1+\sqrt{1-(Z\alpha)^2}}{2}}, \tag{4}$$

$$\mathcal{E}(2P_{\frac{3}{2}}) = mc^2\sqrt{1-\frac{1}{4}(Z\alpha)^2}. \tag{5}$$

For all states of the discrete spectrum, the binding energy $mc^2 - \mathcal{E}(n,j)$ increases as the nuclear charge Z increases. No regular solution with $n = 1$, $l = 0$, $j = 1/2$ and $K = -1$ (the $1S_{1/2}$ ground state) is found for $Z > 137$, this was first noticed by Gordon in his pioneer paper [4]. This is the problem so-called "$Z = 137$ catastrophe".

The problem was solved [7–14] by considering the fact that the nucleus is not point-like and has an extended charge distribution, and the potential $V(r)$ is not divergent when $r \to 0$. The $Z = 137$ catastrophe disappears and the energy-levels $\mathcal{E}(n,j)$ of the bound states $1S$, $2P$ and $2S$, ... smoothly continue to drop toward the negative energy continuum ($E_- < -mc^2$), as Z increases to values larger than 137. The critical values Z_{cr} for $\mathcal{E}(n,j) = -mc^2$ were found [9,11–14,17–19]: $Z_{cr} \simeq 173$ is a critical value at which the lowest energy-level of the bound state $1S_{1/2}$ encounters the negative energy continuum, while other bound states $2P_{1/2}, 2S_{3/2}, \ldots$ encounter the negative energy continuum at $Z_{cr} > 173$, thus energy-level crossings and productions of electron and positron pair takes place, provided these bound states are unoccupied. We refer the readers to [11–19] for mathematical and numerical details.

The energetics of this phenomenon can be understood as follows. The energy-level of the bound state $1S_{1/2}$ can be estimated as follows,

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$$\mathcal{E}(1S_{1/2}) = mc^2 - \frac{Ze^2}{\bar{r}} < -mc^2, \tag{6}$$

where $\bar{r}$ is the average radius of the $1S_{1/2}$ state's orbit, and the binding energy of this state $Ze^2/\bar{r} > 2mc^2$. If this bound state is unoccupied, the bare nucleus gains a binding energy $Ze^2/\bar{r}$ larger than $2mc^2$, and becomes unstable against the production of an electron–positron pair. Assuming this pair-production occur around the radius $\bar{r}$, we have energies of electron (ϵ_-) and positron (ϵ_+):

$$\epsilon_- = \sqrt{(c|\mathbf{p}_-|)^2 + m^2c^4} - \frac{Ze^2}{\bar{r}};$$
$$\epsilon_+ = \sqrt{(c|\mathbf{p}_+|)^2 + m^2c^4} + \frac{Ze^2}{\bar{r}}, \tag{7}$$

where $\mathbf{p}_\pm$ are electron and positron momenta, and $\mathbf{p}_- = -\mathbf{p}_+$. The total energy required for a pair production is,

$$\epsilon_{-+} = \epsilon_- + \epsilon_+ = 2\sqrt{(c|\mathbf{p}_-|)^2 + m^2c^4}, \tag{8}$$

which is independent of the potential $V(\bar{r})$. The potential energies $\pm eV(\bar{r})$ of electron and positron cancel each other and do not contribute to the total energy (8) required for pair production. This energy (8) is acquired from the binding energy ($Ze^2/\bar{r} > 2mc^2$) by the electron filling into the bound state $1S_{1/2}$. A part of the binding energy becomes the kinetic energy of positron that goes out. This is analogous to the familiar case that a proton ($Z = 1$) catches an electron into the ground state $1S_{1/2}$, and a photon is emitted with the energy not less than 13.6 eV.

In this Letter, we study classical and semi-classical states of electrons, electron–positron pair production in an electric potential of macroscopic cores with charge $Q = Z|e|$, mass M and macroscopic radius R_c.

2. Classical description of electrons in potential of cores

2.1. Effective potentials for particle's radial motion

Setting the origin of spherical coordinates (r, θ, ϕ) at the center of such cores, we write the vectorial potential $A_\mu = (\mathbf{A}, A_0)$, where $\mathbf{A} = 0$ and A_0 is the Coulomb potential. The motion of a relativistic electron with mass m and charge e is described by its radial momentum p_r, total angular momenta p_ϕ and the Hamiltonian,

$$H_\pm = \pm mc^2\sqrt{1 + \left(\frac{p_r}{mc}\right)^2 + \left(\frac{p_\phi}{mcr}\right)^2} - V(r), \tag{9}$$

where the potential energy $V(r) = eA_0$, and $\pm$ corresponds for positive and negative energies. The states corresponding to negative energy solutions are fully occupied. The total angular momentum p_ϕ is conserved, for the potential $V(r)$ is spherically symmetric. For a given angular momentum $p_\phi = mv_\perp r$, where $v_\perp$ is the transverse velocity, the effective potential energy for electron's radial motion is

$$E_\pm(r) = \pm mc^2\sqrt{1 + \left(\frac{p_\phi}{mcr}\right)^2} - V(r), \tag{10}$$

where $\pm$ indicates positive and negative effective energies, outside the core ($r \geqslant R_c$), the Coulomb potential energy $V(r)$ is given by

$$V_{\rm out}(r) = \frac{Ze^2}{r}. \tag{11}$$

Inside the core ($r \leqslant R_c$), the Coulomb potential energy is given by

$$V_{\rm in}(r) = \frac{Ze^2}{2R_c}\left[3 - \left(\frac{r}{R_c}\right)^2\right], \tag{12}$$

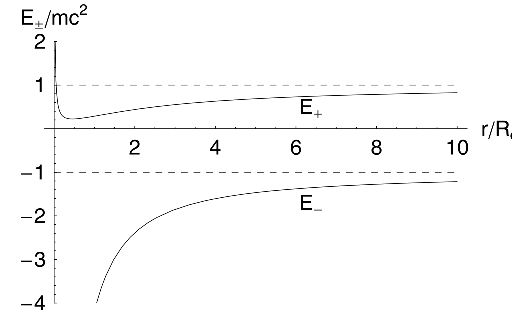

Fig. 1. In the case of point-like charge distribution, we plot the positive and negative effective potential energies $E_\pm$ (10), $p_\phi/(mcR_c) = 2$ and $Ze^2 = 1.95mc^2R_c$, to illustrate the radial location R_L (14) of stable orbits where E_+ has a minimum (15). All stable orbits are described by $cp_\phi > Ze^2$. The last stable orbits are given by $cp_\phi \to Ze^2 + 0^+$, whose radial location $R_L \to 0$ and energy $\mathcal{E} \to 0^+$. There is no any stable orbit with energy $\mathcal{E} < 0$ and the energy-level crossing with the negative energy spectrum E_- is impossible.

where we postulate the charged core has a uniform charge distribution with constant charge density $\rho = Ze/V_c$, and the core volume $V_c = 4\pi R_c^3/3$. Coulomb potential energies outside the core (11) and inside the core (12) are continuous at $r = R_c$. The electric field on the surface of the core,

$$E_s = \frac{Q}{R_c^2} = \frac{\lambda_e}{R_c}E_c, \qquad \beta \equiv \frac{Ze^2}{mc^2R_c} \tag{13}$$

where the electron Compton wavelength $\lambda_e = \hbar/(mc)$, the critical electric field $E_c = m^2c^3/(e\hbar)$ and the parameter β is the electric potential-energy on the surface of the core in unit of the electron mass-energy.

2.2. Stable classical orbits (states) outside the core

Given different values of total angular momenta p_ϕ, the stable circulating orbits R_L (states) are determined by the minimum of the effective potential $E_+(r)$ (10) (see Fig. 1), at which $dE_+(r)/dr = 0$. We obtain stable orbits locate at the radii R_L outside the core,

$$R_L = \left(\frac{p_\phi^2}{Ze^2m}\right)\sqrt{1 - \left(\frac{Ze^2}{cp_\phi}\right)^2}, \quad R_L \geqslant R_c, \tag{14}$$

for different p_ϕ-values. Substituting Eq. (14) into Eq. (10), we find the energy of electron at each stable orbit,

$$\mathcal{E} \equiv \min(E_+) = mc^2\sqrt{1 - \left(\frac{Ze^2}{cp_\phi}\right)^2}. \tag{15}$$

For the condition $R_L \gtrsim R_c$, we have

$$\left(\frac{Ze^2}{cp_\phi}\right)^2 \lesssim \frac{1}{2}\left[\beta(4 + \beta^2)^{1/2} - \beta^2\right], \tag{16}$$

where the semi-equality holds for the last stable orbits outside the core $R_L \to R_c + 0^+$. In the point-like case $R_c \to 0$, the last stable orbits are

$$cp_\phi \to Ze^2 + 0^+, \qquad R_L \to 0^+, \qquad \mathcal{E} \to 0^+. \tag{17}$$

Eq. (15) shows that there are only positive or null energy solutions (states) in the case of a point-like charge, which corresponds to the energy spectra equations (3), (4), (5) in quantum mechanic scenario. While for $p_\phi \gg 1$, radii of stable orbits $R_L \gg 1$ and energies $\mathcal{E} \to mc^2 + 0^-$, classical electrons in these orbits are critically bound for their banding energy goes to zero. We conclude that the energies (15) of stable orbits outside the core must be smaller than mc^2, but larger than zero, $\mathcal{E} > 0$. Therefore, no energy-level crossing with the negative energy spectrum occurs.

2.3. Stable classical orbits inside the core

We turn to the stable orbits of electrons inside the core. Analogously, using Eqs. (10), (12) and $dE_+(r)/dr = 0$, we obtain the stable orbit radius $R_L \leqslant 1$ in the unit of R_c, obeying the following equation,

$$\beta^2\left(R_L^8 + \kappa^2 R_L^6\right) = \kappa^4; \quad \kappa = \frac{p_\phi}{mcR_c}, \tag{18}$$

and corresponding to the minimal energy (binding energy) of these states

$$\mathcal{E} = \frac{Ze^2}{R_c}\left[\left(\frac{cp_\phi}{Ze^2}\right)^2 \frac{1}{R_L^4} - \frac{1}{2}\left(3 - R_L^2\right)\right]. \tag{19}$$

There are 8 solutions to this polynomial equation (18), only one is physical, the solution R_L that has to be real, positive and smaller than one. As example, the numerical solution to Eq. (18) is $R_L = 0.793701$ for $\beta = 4.4 \cdot 10^{16}$ and $\kappa = 2.2 \cdot 10^{16}$. In following, we respectively adopt non-relativistic and ultra-relativistic approximations to obtain analytical solutions.

First considering the non-relativistic case for those stable orbit states whose kinetic energy term characterized by angular momentum term p_ϕ, see Eq. (10), is much smaller than the rest mass term mc^2, we obtain the following approximate equation,

$$\beta^2 R_L^8 \simeq \kappa^4, \tag{20}$$

and the solutions for stable orbit radii are,

$$R_L \simeq \frac{\kappa^{1/2}}{\beta^{1/4}} = \left(\frac{cp_\phi}{Ze^2}\right)^{1/2} \beta^{1/4} < 1, \tag{21}$$

and energies,

$$\mathcal{E} \simeq \left(1 - \frac{3}{2}\beta + \frac{1}{2}\kappa\beta^{1/2}\right)mc^2. \tag{22}$$

The consistent conditions for this solution are $\beta^{1/2} > \kappa$ for $R_L < 1$, and $\beta \ll 1$ for non-relativistic limit $v_\perp \ll c$, where the transverse velocity $v_\perp = p_\phi/(mR_L)$. As a result, the binding energies (22) of these states are $mc^2 > \mathcal{E} > 0$, are never less than zero. These in fact correspond to the stable states which have large radii closing to the radius R_c of cores and $v_\perp \ll c$.

Second considering the ultra-relativistic case for those stable orbit states whose the kinetic energy term characterized by angular momentum term p_ϕ, see Eq. (10), is much larger than the rest mass term mc^2, we obtain the following approximate equation,

$$\beta^2 R_L^6 \simeq \kappa^2, \tag{23}$$

and the solutions for stable orbit radii are,

$$R_L \simeq \left(\frac{\kappa}{\beta}\right)^{1/3} = \left(\frac{p_\phi c}{Ze^2}\right)^{1/3} < 1, \tag{24}$$

which gives $R_L \simeq 0.7937007$ for the same values of parameters β and κ in above. The consistent condition for this solution is $\beta > \kappa \gg 1$ for $R_L < 1$. The energy levels of these ultra-relativistic states are,

$$\mathcal{E} \simeq \frac{3}{2}\beta\left[\left(\frac{p_\phi c}{Ze^2}\right)^{2/3} - 1\right]mc^2, \tag{25}$$

and $mc^2 > \mathcal{E} > -1.5\beta mc^2$. The particular solutions $\mathcal{E} = 0$ and $\mathcal{E} \simeq -mc^2$ are respectively given by

$$\left(\frac{p_\phi c}{Ze^2}\right) \simeq 1; \qquad \left(\frac{p_\phi c}{Ze^2}\right) \simeq \left(1 - \frac{2}{3\beta}\right)^{3/2}. \tag{26}$$

These in fact correspond to the stable states which have small radii closing to the center of cores and $v_\perp \lesssim c$.

To have the energy-level crossing to the negative energy continuum, we are interested in the values $\beta > \kappa \gg 1$ for which the energy-levels (25) of stable orbit states are equal to or less than $-mc^2$,

$$\mathcal{E} \simeq \frac{3}{2}\beta\left[\left(\frac{p_\phi c}{Ze^2}\right)^{2/3} - 1\right]mc^2 \leqslant -mc^2. \tag{27}$$

As example, with $\beta = 10$ and $\kappa = 2$, $R_L \simeq 0.585$, $\mathcal{E}_{\min} \simeq -9.87mc^2$. The lowest energy-level of electron state is $p_\phi/(Ze^2) = \kappa/\beta \to 0$ with the binding energy,

$$\mathcal{E}_{\min} = -\frac{3}{2}\beta mc^2, \tag{28}$$

locating at $R_L \simeq (p_\phi c/Ze^2)^{1/3} \to 0$, the bottom of the potential energy $V_{\text{in}}(0)$ (12).

3. Semi-classical description

3.1. Bohr–Sommerfeld quantization

In order to have further understanding, we consider the semi-classical scenario. Introducing the Planck constant $\hbar = h/(2\pi)$, we adopt the semi-classical Bohr–Sommerfeld quantization rule

$$\int p_\phi \, d\phi \simeq h\left(l + \frac{1}{2}\right)$$
$$\Rightarrow \quad p_\phi(l) \simeq \hbar\left(l + \frac{1}{2}\right), \quad l = 0, 1, 2, 3, \ldots, \tag{29}$$

which are discrete values selected from continuous total angular momentum p_ϕ in the classical scenario. The variation of total angular momentum $\Delta p_\phi = \pm\hbar$ in th unit of the Planck constant $\hbar$, we make substitution

$$\left(\frac{p_\phi c}{Ze^2}\right) \Rightarrow \left(\frac{2l+1}{2Z\alpha}\right), \quad \alpha = \frac{e^2}{(\hbar c)}, \tag{30}$$

in classical solutions that we obtained in Section 2.

1. The radii and energies of stable states outside the core (14) and (15) become:

$$R_L = \lambda\left(\frac{2l+1}{Z\alpha}\right)\sqrt{1 - \left(\frac{2Z\alpha}{2l+1}\right)^2}, \tag{31}$$

$$\mathcal{E} = mc^2\sqrt{1 - \left(\frac{2Z\alpha}{2l+1}\right)^2}, \tag{32}$$

   where the electron Compton length $\lambda = \hbar/(mc)$.

2. The radii and energies of non-relativistic stable states inside the core (21) and (22) become:

$$R_L \simeq \left(\frac{2l+1}{2Z\alpha}\right)^{1/2} \beta^{1/4}, \tag{33}$$

$$\mathcal{E} \simeq \left(1 - \frac{3}{2}\beta + \frac{\lambda(2l+1)}{4R_c}\beta^{1/2}\right)mc^2. \tag{34}$$

3. The radii and energies of ultra-relativistic stable states inside the core (24) and (25) become:

$$R_L \simeq \left(\frac{2l+1}{2Z\alpha}\right)^{1/3}, \tag{35}$$

$$\mathcal{E} \simeq \frac{3}{2}\beta\left[\left(\frac{2l+1}{2Z\alpha}\right)^{2/3} - 1\right]mc^2. \tag{36}$$

Note that radii R_L in the second and third cases are in unit of R_c.

3.2. Stability of semi-classical states

When these semi-classical states are not occupied as required by the Pauli principle, the transition from one state to another with different discrete values of total angular momentum l (l_1, l_2 and $\Delta l = l_2 - l_1 = \pm 1$) undergoes by emission or absorption of a spin-1 ($\hbar$) photon. Following the energy and angular-momentum conservations, photon emitted or absorbed in the transition have angular momenta $p_\gamma = p_\phi(l_2) - p_\phi(l_1) = \hbar(l_2 - l_1) = \pm\hbar$ and energy $\mathcal{E}_\gamma = \mathcal{E}(l_2) - \mathcal{E}(l_1)$. In this transition of stable states, the variation of radius is $\Delta R_L = R_L(l_2) - R_L(l_1)$.

We first consider the stability of semi-classical states against such transition in the case of point-like charge, i.e., Eqs. (31), (32) with $l = 0, 1, 2, \ldots$. As required by the Heisenberg indeterminacy principle $\Delta\phi\Delta p_\phi \simeq 4\pi p_\phi(l) \gtrsim h$, the absolute ground state for minimal energy and angular momentum is given by the $l = 0$ state, $p_\phi \sim \hbar/2$, $R_L \sim \lambda(Z\alpha)^{-1}(1 - (2Z\alpha)^2)^{1/2} > 0$ and $\mathcal{E} \sim mc^2(1 - (2Z\alpha)^2)^{1/2} > 0$, which corresponds to the last stable orbit (17) in the classical scenario. Thus the stability of all semi-classical states $l > 0$ is guaranteed by the Pauli principle. This is only case for $Z\alpha \leqslant 1/2$. While for $Z\alpha > 1/2$, there is not an absolute ground state in the semi-classical scenario. This can be understood by examining how the lowest energy states are selected by the quantization rule in the semi-classical scenario out of the last stable orbits (17) in the classical scenario. For the case of $Z\alpha \leqslant 1/2$, equating p_ϕ in Eq. (17) to $p_\phi = \hbar(l + 1/2)$ (29), we find the selected state $l = 0$ is only possible solution so that the ground state $l = 0$ in the semi-classical scenario corresponds to the last stable orbits (17) in the classical scenario. While for the case of $Z\alpha > 1/2$, equating p_ϕ in Eq. (17) to $p_\phi = \hbar(l + 1/2)$ (29), we find the selected semi-classical state

$$\tilde{l} = \frac{Z\alpha - 1}{2} > 0, \tag{37}$$

in the semi-classical scenario corresponds to the last stable orbits (17) in the classical scenario. This state $l = \tilde{l} > 0$ is not protected by the Heisenberg indeterminacy principle from quantum-mechanically decaying in $\hbar$-steps to the states with lower angular momenta and energies (correspondingly smaller radius R_L (31)) via photon emissions. This clearly shows that the "$Z = 137$-catastrophe" corresponds to $R_L \to 0$, falling to the center of the Coulomb potential and all semi-classical states (l) are unstable.

Then we consider the stability of semi-classical states against such transition in the case of charged cores $R_c \neq 0$. Substituting p_ϕ in Eq. (29) into Eq. (16), we obtain the selected semi-classical state $\tilde{l}$ corresponding to the last classical stable orbit outside the core,

$$\tilde{l} = \sqrt{2}\left(\frac{R_c}{\lambda}\right)\left[\left(\frac{4R_c}{Z\alpha\lambda} + 1\right)^{1/2} - 1\right]^{-1/2}$$
$$\approx (Z\alpha)^{1/4}\left(\frac{R_c}{\lambda}\right)^{3/4} > 0. \tag{38}$$

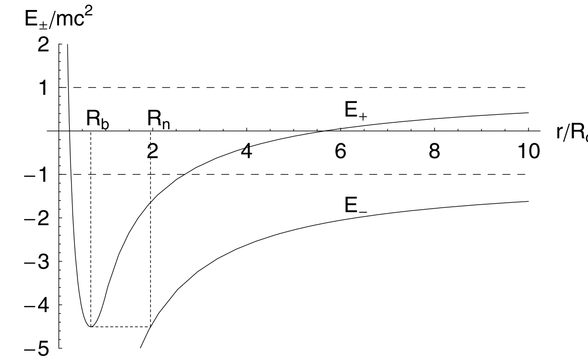

Fig. 2. For the core $\kappa = 2$ and $\beta = 6$, we plot the positive and negative effective potentials $E_\pm$ (10), in order to illustrate the radial location (24) $R_L < R_c$ of stable orbit, where E_+'s minimum (25) $\mathcal{E} < mc^2$ is. All stable orbits inside the core are described by $\beta > \kappa > 1$. The last stable orbit is given by $\kappa/\beta \to 0$, whose radial location $R_L \to 0$ and energy $\mathcal{E} \to \mathcal{E}_{\min}$ (28). We indicate that the energy-level crossing between bound state (stable orbit) energy at $R_L = R_b$ and negative energy spectrum E_- (25) at the turning point R_n.

Analogously to Eq. (37), the same argument concludes the instability of this semi-classical state, which must quantum-mechanically decay to states with angular momentum $l < \tilde{l}$ inside the core, provided these semi-classical states are not occupied. This conclusion is independent of $Z\alpha$-value.

We go on to examine the stability of semi-classical states inside the core. In the non-relativistic case ($1 \gg \beta > \kappa^2$), the last classical stable orbits locate at $R_L \to 0$ and $p_\phi \to 0$ given by Eqs. (21), (22), corresponding to the lowest semi-classical state (33), (34) with $l = 0$ and energy $mc^2 > \mathcal{E} > 0$. In the ultra-relativistic case ($\beta > \kappa \gg 1$), the last classical stable orbits locate at $R_L \to 0$ and $p_\phi \to 0$ given by Eqs. (24), (25), corresponding to the lowest semi-classical state (35), (36) with $l = 0$ and minimal energy,

$$\mathcal{E} \simeq \frac{3}{2}\beta\left[\left(\frac{1}{2Z\alpha}\right)^{2/3} - 1\right]mc^2 \approx -\frac{3}{2}\beta mc^2. \tag{39}$$

This concludes that the $l = 0$ semi-classical state inside the core is an absolute ground state in both non- and ultra-relativistic cases. The Pauli principle assures that all semi-classical states $l > 0$ are stable, provided all these states accommodate electrons. The electrons can be either present inside the neutral core or produced from the vacuum polarization, later will be discussed in details.

We are particular interested in the ultra-relativistic case $\beta > \kappa \gg 1$, i.e., $Z\alpha \gg 1$, the energy-levels of semi-classical states can be profound than $-mc^2$ ($\mathcal{E} < -mc^2$), energy-level crossings and pair-productions occur if these states are unoccupied, as discussed in introductory section.

4. Production of electron–positron pair

When the energy-levels of semi-classical (bound) states $\mathcal{E} \leqslant -mc^2$ (27), energy-level crossings between these energy-levels (25) and negative energy continuum (10) for $p_r = 0$, as shown in Fig. 2. The energy-level crossing indicates that $\mathcal{E}$ (25) and E_- (10) are equal,

$$\mathcal{E} = E_-, \tag{40}$$

where angular momenta p_ϕ in $\mathcal{E}$ (36) and E_- (10) are the same for angular-momentum conservation. The production of electron–positron pairs must takes place [20–22], provided these semi-classical (bound) states are unoccupied. The phenomenon of pair production can be understood as a quantum-mechanical tunneling process of relativistic electrons. The energy-levels $\mathcal{E}$ of semi-classical (bound) states are given by Eq. (36) or (27). The probability amplitude for this process can be calculated by a semi-classical WKB method [19]:

$$W_{\rm WKB}(|\mathbf{p}_\perp|) \equiv \exp\left\{-\frac{2}{\hbar}\int_{R_b}^{R_n} p_r\, dr\right\}, \tag{41}$$

where $|\mathbf{p}_\perp| = p_\phi/r$ is transverse momenta and the radial momentum,

$$p_r(r) = \sqrt{\left(c|\mathbf{p}_\perp|\right)^2 + m^2c^4 - \left[\mathcal{E} + V(r)\right]^2}. \tag{42}$$

The energy potential $V(r)$ is either given by $V_{\rm out}(r)$ (11) for $r > R_c$, or $V_{\rm in}(r)$ (12) for $r < R_c$. The limits of integration (41): $R_b = R_L < R_c$ (24) or (35) indicating the location of the classical orbit (classical turning point) of semi-classical (bound) state; while another classical turning point R_n is determined by setting $p_r(r) = 0$ in Eq. (42). There are two cases: $R_n < R_c$ and $R_n > R_c$, depending on β and κ values.

To obtain a maximal WKB-probability amplitude (41) of pair production, we only consider the case that the charge core is bare and

- the lowest energy-levels of semi-classical (bound) states: $p_\phi/(Ze^2) = \kappa/\beta \to 0$, the location of classical orbit (24) $R_L = R_b \to 0$ and energy (25) $\mathcal{E} \to \mathcal{E}_{\rm min} = -3\beta mc^2/2$ (28);
- another classical turning point $R_n \leqslant R_c$, since the probability is exponentially suppressed by a large tunneling length $\Delta = R_n - R_b$.

In this case ($R_n \leqslant R_c$), Eq. (42) becomes

$$p_r = \sqrt{\left(c|\mathbf{p}_\perp|\right)^2 + m^2c^4}\sqrt{1 - \frac{\beta^2 m^2c^4}{4[(c|\mathbf{p}_\perp|)^2 + m^2c^4]}\left(\frac{r}{R_c}\right)^4}, \tag{43}$$

and $p_r = 0$ leads to

$$\frac{R_n}{R_c} = \left(\frac{2}{\beta mc^2}\right)^{1/2}\left[\left(c|\mathbf{p}_\perp|\right)^2 + m^2c^4\right]^{1/4}. \tag{44}$$

Using Eqs. (41), (43), (44), we have

$$\begin{aligned} W_{\rm WKB}(|\mathbf{p}_\perp|) &= \exp\left\{-\frac{2^{3/2}[(c|\mathbf{p}_\perp|)^2 + m^2c^4]^{3/4}R_c}{c\hbar(mc^2\beta)^{1/2}}\int_0^1 \sqrt{1-x^4}\, dx\right\} \\ &= \exp\left\{-0.87\frac{2^{3/2}[(c|\mathbf{p}_\perp|)^2 + m^2c^4]^{3/4}R_c}{c\hbar(mc^2\beta)^{1/2}}\right\}. \end{aligned} \tag{45}$$

Dividing this probability amplitude by the tunneling length $\Delta \simeq R_n$ and time interval $\Delta t \simeq 2\pi\hbar/(2mc^2)$ in which the quantum tunneling occurs, and integrating over two spin states and the transverse phase-space $2\int d\mathbf{r}_\perp d\mathbf{p}_\perp/(2\pi\hbar)^2$, we approximately obtain the rate of pair-production per the unit of time and volume,

$$\Gamma_{\rm NS} \equiv \frac{d^4N}{dt\, d^3x} \simeq \frac{1.15}{6\pi^2}\left(\frac{Z\alpha}{\tau R_c^3}\right)\exp\left\{-\frac{2.46}{(Z\alpha)^{1/2}}\left(\frac{R_c}{\lambda}\right)^{3/2}\right\}, \tag{46}$$

$$= \frac{1.15}{6\pi^2}\left(\frac{\beta}{\tau\lambda R_c^2}\right)\exp\left\{-\frac{2.46 R_c}{\beta^{1/2}\lambda}\right\}, \tag{47}$$

$$\begin{aligned} &= \frac{1.15}{6\pi^2}\left(\frac{1}{\tau\lambda^2 R_c}\right)\left(\frac{E_s}{E_c}\right) \\ &\quad\times \exp\left\{-2.46\left(\frac{R_c}{\lambda}\right)^{1/2}\left(\frac{E_c}{E_s}\right)^{1/2}\right\}, \end{aligned} \tag{48}$$

where $E_s = Ze/R_c^2$ is the electric field on the surface of the core and the Compton time $\tau = \hbar/mc^2$.

To have the size of this pair-production rate, we consider a macroscopic core of mass $M = M_\odot$ and radius $R_c = 10$ km, and the electric field on the core surface E_s (13) is about the critical field ($E_s \simeq E_c$). In this case, $Z = \alpha^{-1}(R_c/\lambda)^2 \simeq 9.2 \cdot 10^{34}$, $\beta = Z\alpha\lambda/R_c = R_c/\lambda \simeq 2.59 \cdot 10^{16}$, and the rate (47) becomes

$$\Gamma_{\rm NS} \equiv \frac{d^4N}{dt\, d^3x} \simeq \frac{1.15}{6\pi^2}\left(\frac{1}{\tau\lambda^3}\right)\left(\frac{\lambda}{R_c}\right)\exp\left\{-2.46\left(\frac{R_c}{\lambda}\right)\right\}, \tag{49}$$

which is exponentially small for $R_c \gg \lambda$. In this case, the charge-mass radio $Q/(G^{1/2}M) = 2 \cdot 10^{-6}|e|/(G^{1/2}m_p) = 8.46 \cdot 10^{-5}$, where G is the Newton constant and proton's charge-mass radio $|e|/(G^{1/2}m_p) = 1.1 \cdot 10^{18}$.

It is interesting to compare this rate of electron–positron pair-production with the rate given by the Hawking effect. We take $R_c = 2GM/c^2$ and the charge-mass radio $Q/(G^{1/2}M) \simeq 10^{-19}$ for a naive balance between gravitational and electric forces. In this case $\beta = \frac{1}{2}(Q/G^{1/2}M)(|e|/G^{1/2}m) \simeq 10^2$, the rate (47) becomes,

$$\Gamma_{\rm NS} = \frac{1.15}{6\pi^2}\left(\frac{25}{\tau\lambda^3}\right)\left(\frac{1}{mM}\right)\exp\left\{-0.492(mM)\right\}, \tag{50}$$

where $mM = R_c/(2\lambda)$. This is much larger than the rate of electron–positron emission by the Hawking effect [23],

$$\Gamma_{\rm H} \sim \exp\left\{-8\pi(mM)\right\}, \tag{51}$$

since the exponential factor $\exp\{-0.492(mM)\}$ is much larger than $\exp\{-8\pi(mM)\}$, where $2mM = R_c/\lambda \gg 1$.

5. Summary and remarks

In this Letter, analogously to the study in atomic physics with large atomic number Z, we study the classical and semi-classical (bound) states of electrons in the electric potential of a massive and charged core, which has a uniform charge distribution and macroscopic radius. We have found negative energy states of electrons inside the core, whose energies can be smaller than $-mc^2$, and the appearance of energy-level crossing to the negative energy spectrum. As a result, quantum tunneling takes place, leading to electron–positron pairs production, electrons then occupy these semi-classical (bound) states and positrons are repelled to infinity. Assuming that massive charged cores are bare and non of these semi-classical (bound) states are occupied, we analytically obtain the maximal rate of electron–positron pair production in terms of the core radius, charge and mass. We find that this rate is much larger than the rate of electron–positron pair-production by the Hawking effect, even for very small charge-mass radio of the core given by the naive balance between gravitational and electric forces.

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