Exact analytical solutions for the local and global buckling of sandwich beam-columns under various loadings

Marc-André Douville, Philippe Le Grogneć

Abstract

Sandwich structures are widely used in many industrial applications, due to the attractive combination of a lightweight and strong mechanical properties. This compromise is realized thanks to the presence of different parts in the composite material, namely the skins and possibly core reinforcements or thin-walled core structure which are both thin/slender and stiff relative to the other parts, namely the homogeneous core material, if any. The buckling phenomenon thus becomes mainly responsible for the final collapse of such sandwiches. In this paper, classical sandwich beam-columns (with homogeneous core materials) are considered and elastic buckling analyses are performed in order to derive the critical values and the associated bifurcation modes under various loadings (compression and pure bending). The two faces are represented by Euler–Bernoulli beams, whereas the core material is considered as a 2D continuous solid. A set of partial differential equations is first obtained from a general bifurcation analysis, using the above assumptions. Original closed-form analytical solutions of the critical loading and mode of a sandwich beam-column are then derived for various loading conditions. Finally, the proposed analytical formulae are validated using 2D linearized buckling finite element computations, and parametric analyses are performed.

1. Introduction

Sandwich composites are plate-like (or shell-like) structures which are normally composed of two thin and stiff skins, separated by a thicker and softer core layer. The core material is often an homogeneous and isotropic foam, which provides the extreme lightweight property of the sandwich. Sometimes, this core material is strengthened using transverse fibrous reinforcements or replaced by a thin-walled core layer (with a honeycomb or corrugated structure, for instance) so as to reinforce the through-thickness behavior. Conversely, the skins and their distance to the middle surface of the composite contribute to the tensile properties and particularly to the flexural rigidity. Sandwich materials are thus commonly used in applications of aerospace, marine or transportation industries, among others, due to this attractive combination of a lightweight and strong mechanical properties.

Due to their practical interest, sandwich materials have been the subject of numerous studies. Lots of theoretical models have been developed in order to predict the overall behavior of such structures under various loadings (see, for instance, Drysdale et al., 1979 or Fazio et al., 1982, the latter including the effects of interlayer slips between the face and core elements). Owing to the geometric configuration (thinness of the skins and possibly of the thin-walled core layer, slenderness of the core reinforcements), and also to the stiffness ratios between the different components of a sandwich material, instabilities are likely to occur under overall or even localized compressive loads. The buckling phenomenon is therefore one of the major causes for the failure of such composite materials, and it has been greatly investigated in the last decades.

This paper is specially about classical sandwiches with an homogeneous foam core. In that case, one usually distinguishes two types of geometric instabilities, namely the global buckling of the sandwich structure under overall compression and the so-called wrinkling (or local buckling) of the faces, which may appear insofar as they undergo compressive stresses. On one hand, the global buckling of a sandwich material can easily be viewed as the classical buckling of an homogeneous structure as soon as the equivalent properties have been properly derived. On the other hand, the local buckling analysis of sandwiches requires the use of advanced models, since classical buckling solutions for beams or plates are no more valid.

The pioneering works dealing with the buckling of sandwich structures dated back to the early 1940s. For about seventy years,
numerous analytical and numerical models characterized by different levels of approximation have been defined in the literature. Lots of contributions (see Hoff and Mautner, 1945 or Allen, 1969, as two of the first leading references in this field) have been proposed for sandwich columns under compression that only consider one type of buckling mode (global or local, symmetric or antisymmetric) or several, but with uncoupled approaches (see Fig. 1 for the different mode types observed in sandwich columns under compression). The predictions for the critical load are often very accurate as far as global buckling is concerned. However, the numerous expressions derived for the critical wrinkling load are very scattered and concern predominantly the symmetric case (see Frostig and Baruch, 1993, as an example). Indeed, many analyses are conducted on a simplified model, considering just one skin (represented by a beam model) resting on an elastic foundation, which is supposed to stand for the core material. Such a model may only be suitable for local symmetric wrinkling (with the neutral axis remaining flat) or antisymmetric wrinkling as long as the core thickness is far larger than the skin one, that is to say when the buckling of both faces do not interact. The foundation is usually described by a one- or two-parameters model (accounting for the extension and possible shear effects) which does not convey the actual behavior of the core layer. Some alternatives have thus been proposed, in order to improve the accuracy of the critical loads. Among others, Niu and Talreja (1999) have represented the core material as an elastic continuous medium (instead of spring distributions) characterized by an Airy’s function, whose determination has been proved however to be rather tricky.

At the same time, many authors have tried to achieve unified models capable of describing both global and local modes (both symmetric and antisymmetric) in order to investigate the possible interaction between these different behaviors. Benson and Mayers (1967) have been the first to suggest a unified approach to solve the overall buckling and wrinkling problems simultaneously, in a linearized framework. Later, Hunt et al. (1988) investigated not only the critical buckling loads and modes but also the non-linear post-buckling behavior of such sandwich structures. They put forward analytical and semi-analytical solutions that emphasized the possible interaction between global (primary) and local (secondary) bifurcation modes. Using a more advanced model, Hunt and Wadee (1998) also studied this interactive buckling phenomenon and captured the localized pattern of the deformed shape corresponding to the unstable post-buckling response of the sandwich structure. The post-critical behavior is all the more unstable that the primary and secondary bifurcation points are close from each other. The latter results were then extended to the case of orthotropic core materials (Wadee and Hunt, 1998) and imperfect panels including periodic or localized defects (Wadee, 2000). As for Wadee et al. (2010), they analytically solved the buckling and advanced post-buckling response of sandwich structures using two different core bending models based on two different beam theories. The same models were finally used to investigate the interactive buckling phenomenon in a sandwich beam-column under both compression and bending (Yiatros and Wadee, 2011).

Recently, in the context of linear buckling analysis, a few interesting contributions have dealt with higher-order models or more specific mechanical behaviors or geometries. Following Benson–Mayers theory, Hadi and Matthews (2000) (see also Hadi, 2001) derived a general approach for the buckling and wrinkling analysis of anisotropic sandwich panels. It is based on energy methods and the core is assumed to be anti-plane (the longitudinal stiffness is neglected). Moreover, only the solution procedure is presented and no closed-form solution is available. Léotoing et al. (2002) also investigated the overall buckling and local wrinkling of sandwich columns with a unified formulation where the core material is represented by a higher-order beam model so as to take into account the shear effects. The same authors next considered the case of an elastoplastic core material and examined the post-buckling response of the sandwich structure using a numerical approach (Léotoing et al., 2002). One of the most recent works in this field is due to Jasion et al. (2012) and deals with analytical, numerical and experimental analyses of the global buckling/local wrinkling of sandwich rectangular beams/circular plates under compression or bending. Again, simplifying assumptions in the core kinematics allow them to define simple expressions for the critical forces or bending moments. In every instance, the analytical solutions have been (or may be) compared to finite element computations. Relative differences between analytical and numerical reference critical values are often about 10–20%, and the analytical predictions are not always conservative. Furthermore, these errors may change the mode sequence, and particularly the type (global/local, symmetric/antisymmetric) and the wave number of the first dominant mode.

This paper particularly deals with sandwich beam-columns and can be seen as the continuation of the latter studies. The idea is to define a new model, capable to provide more accurate buckling and wrinkling loads still in a unified way. At first sight, the buckling response of a sandwich material can be seen as the buckling of one (or both) face(s) on either side of the homogeneous core material. Despite a comparatively low modulus, the core material strongly influences both the critical buckling load and the bifurcation mode of the sandwich structure (generally displaying a stabilizing effect). As a consequence, the classical Euler critical values, obtained without any core material, are no more valid at all and approximated solutions deriving from simplified kinematics for the core layer are not much better. Special attention must be paid to the core material representation so as to estimate the buckling response with a good accuracy and above all properly describe the possible interaction between the buckling behavior of the two skins. In our approach, the main improvement beside previous models leads in the absence of specific hypotheses for the core kinematics. While the faces are classically considered as slender beams, the foam core is represented by a 2D continuous solid, without any simplification regarding the deformation field. This model may lead to exact solutions, in a similar way to that described by Parnes and Chiksis (2002) or Zhang and Latour (1994), among others, in the context of micro-buckling of reinforced composite materials.

This work is thus devoted to the prediction of the critical loadings and modes of a sandwich beam-column, using the above-mentioned modeling assumptions. A general 3D elastic bifurcation analysis is carried out, giving rise to original closed-form analytical expressions. The bifurcation equation of the problem is first written in a 3D framework. Next this equation is linearized and particularized, using the appropriate kinematic assumptions and considering a specific pre-critical stress state, which depends on the loading conditions (compression, bending), with small pre-critical displacements. This approach is a general and efficient way to cope with the buckling of structures subjected to a simple pre-crit-

![Fig. 1. Different buckling modes of a compressed sandwich column (Goodier, 1946).](image)
2. Analytical modeling of the buckling behavior of sandwich columns under compression

2.1. Problem definition

Several assumptions must be made in order to be able to derive closed-form expressions for the critical value. A 2D representation of the sandwich column is retained with a unit depth. The total length of the sandwich structure is denoted by \( L \), whereas the thicknesses of the foam core and the two skins are respectively 2\( h_c \) and 2\( h_s \) (see Fig. 2). The two skins are supposed to be identical (in which case the sandwich material is symmetric) like in most practical cases.

The facings are assumed to behave like Euler–Bernoulli beams (transverse shear effects are negligible due to their low thickness-to-length ratio). Whereas the foam core is modeled by a higher-order beam model in Léotoing et al. (2002), it is considered here as a 2D continuous solid satisfying the plane stress hypothesis.

The homogeneous and isotropic foam core is supposed to be linearly elastic (with Young’s modulus \( E_c \) and Poisson’s ratio \( \nu_c \)). As for the two skins, due to the kinematic hypotheses, only their Young’s modulus \( E_s \) will be involved subsequently.

Lastly, the left and right end sections of the sandwich column are respectively subjected to zero and non-zero (compressive) displacement boundary conditions in the longitudinal direction, what leads to buckling (this particular choice of displacement boundary conditions has been retained as it will give rise to piecewise uniform stress/strain states in the sandwich column during the pre-critical stage in further numerical computations). The critical displacements and the associated bifurcation modes are derived from a 3D framework: the theory is developed using a total Lagrangian formulation where the different components of the model are initially seen as 3D bodies (Le Grognc and Le van, 2009; Le Grognc and Le van, 2011; Le Grognc et al., 2012; Lainé et al., 2013).

2.2. Theoretical formulation

The critical loading \( \lambda_{cr} \) and the bifurcation mode \( X \) of a 3D body are obtained by solving the following bifurcation equation:

\[
\nabla \cdot \lambda_{cr} + K(\lambda_{cr}) : \nabla X d \Omega = 0
\]

The fourth-order nominal tangent elastic tensor \( K \) can be written as follows:

\[
K = \frac{\partial \Pi}{\partial F} = F \cdot \frac{\partial \Sigma}{\partial F} F^T + (\Sigma)^T = F \cdot \mathbf{D} \cdot F^T + (\Sigma)^T
\]

In the above equation, \( E \) denotes the Green strain tensor and \( \Sigma \) the second Kirchhoff stress tensor (symmetric). \( F \) is the deformation gradient and \( \Pi = F \cdot \Sigma \) the first Kirchhoff stress tensor (non-symmetric). \( \Sigma \) represents the fourth-order unit tensor \( (l_{ijkl} = \delta_{ij}\delta_{kl}) \) and the superscript \( ^T \) the transposition of a second-order tensor and the major transposition of a fourth-order tensor \( (A^T)^{ijkl} = A_{ijkl} \), respectively. The fourth-order material tangent elastic tensor \( D \) of an isotropic material can be defined by its components in an ortho-normal basis \( \delta_{ijkl} = \Lambda_{ijkl} + \mu(\delta_{ij}\delta_{kl} + \delta_{ik}\delta_{jl}) \), where \( \delta_{ij} \) is the Kronecker symbol, and \( \Lambda \) and \( \mu \) are the Lamé constants. Use is also made of the Young’s modulus \( E \), the Poisson’s ratio \( \nu \) and the shear modulus \( G \) related to \( \Lambda \) and \( \mu \) by the standard relations \( \Lambda = \frac{E(1 - \nu^2)}{(1 + \nu)(1 - 2\nu)} \) and \( \mu = G = \frac{E}{2(1 + \nu)} \).

We shall now derive more explicit expressions of the above tensors by exploiting the uniaxial stress state in the body. The uniform compressive displacement applied on the right-hand side of the sandwich column leads to compressive stresses in the facings as well as in the foam core. Owing to the respective moduli of the two materials, the uniaxial stresses in the pre-critical state are far larger in the skins than in the foam, so that they will be neglected in the core material.

On one hand, the skins are subjected to a nominal axial compressive stress \( \Pi_{XX} = -P < 0 \) in their longitudinal direction, so that the first Kirchhoff stress tensor \( \Pi \) is expressed in the orthonormal basis \((e_x, e_y, e_z)\) as:

\[
\Pi = -P e_x \otimes e_x = \begin{bmatrix} -P & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} (P > 0)
\]

Let us make the assumption that the pre-critical deformations are small, which is usually satisfied in practice:

\[
||\nabla \mathbf{U}|| < 1
\]

Thus, the stress tensor \( \Sigma \) writes:
\[ \Sigma = F^{-1} \cdot \Pi \cong \Pi \]

The nominal tangent elastic tensor in Eq. (2) becomes then:

\[ \mathbf{K} \equiv \frac{\partial \Sigma}{\partial \mathbf{E}} = \mathbf{D} - \mathbf{P} \mathbf{e}_b \otimes \mathbf{e}_b \otimes \mathbf{e}_b \otimes \mathbf{e}_b, \]

which is independent of the spatial coordinates (the implicit summation convention on repeated indices is used with \( i = X, Y, Z \)). Furthermore, when dealing with 1D models like beams, ad hoc assumptions are usually added in order to enforce some specific stress state in the body. Namely, the transverse normal material stresses are assumed to be zero: \( \Sigma_T = \Sigma_Z = 0 \). Taking into account these assumptions leads one to replace tensor \( \mathbf{D} \) with the reduced tensor \( \mathbf{C} \) defined as:

\[ C_{ijkl} = D_{ijkl} + (D_{ijkl}D_{klmn} - D_{klmn}D_{ijkl}) + D_{ijklmn} + D_{ijklmn} \]

\[ (i,j) \neq (Y,Y),(Z,Z) \quad (k,l) \neq (Y,Y),(Z,Z) \]

It can be readily checked that tensor \( \mathbf{C} \) has the major and both minor symmetries. Subsequently, we only need the following reduced moduli (and their equivalents obtained by major or minor symmetries):

\[ C_{XXX} = C_{YYY} = C_{XXZ} = C_{YYZ} = G \]

where only \( E = E_i \) will explicitly appear in the final bifurcation equation.

On the other hand, as the existing initial stresses in the foam at the critical point have numerically shown to produce no significant effect on the buckling behavior, they are not introduced in our model, for simplicity purposes. Therefore, the nominal tangent elastic tensor in this case simply writes:

\[ \mathbf{K} \equiv \frac{\partial \Sigma}{\partial \mathbf{E}} = \mathbf{D} \]

The 2D model is supposed to reproduce the behavior of a sandwich column with small lateral dimensions, so that the plane stress hypothesis is adopted. Thus, tensor \( \mathbf{D} \) must be replaced by the appropriate reduced tensor \( \mathbf{C} \) whose plane components will be used subsequently:

\[ C_{XXX} = C_{YYY} = \Lambda_c^* + 2\mu_c \]

\[ C_{XXY} = C_{XYX} = \Lambda_s^* \]

\[ C_{YYX} = \Lambda_s^* \]

\[ \left( \Lambda_c^* = \frac{2\nu_c\mu_c}{\Lambda_s^* + 2\mu_c} \right) \]

and their equivalents obtained by major or minor symmetries.

Eventually, the bifurcation Eq. (1) of the whole sandwich structure writes:

\[ \forall \mathbf{u}_a, \mathbf{u}_b, \mathbf{u}_c: \]

\[ \int_{A_a} \nabla^T \mathbf{u}_a \cdot \left( \mathbf{C}^* - \mathbf{P}_a \mathbf{e}_b \otimes \mathbf{e}_b \otimes \mathbf{e}_b \otimes \mathbf{e}_b \right) \cdot \nabla \mathbf{X}_b d\Omega_a \]

\[ + \int_{A_b} \nabla^T \mathbf{u}_b \cdot \left( \mathbf{C}^* - \mathbf{P}_b \mathbf{e}_b \otimes \mathbf{e}_b \otimes \mathbf{e}_b \otimes \mathbf{e}_b \right) \cdot \nabla \mathbf{X}_b d\Omega_b \]

\[ + \int_{A_c} \nabla^T \mathbf{u}_c \cdot \mathbf{C}^* \cdot \nabla \mathbf{X}_c d\Omega_c = 0 \]

The compressive stress (\( P > 0 \)), identical for the two skins, is related to the enforced displacement \( \hat{\mathbf{u}} > 0 \) (which will act as the bifurcation parameter) by the following relation:

\[ P = \frac{E_c \hat{\mathbf{u}}}{L} \]

and \( \mathbf{u}_a, \mathbf{u}_b, \mathbf{u}_c, \mathbf{X}_a, \mathbf{X}_b, \mathbf{X}_c \) and \( \mathbf{X} \) represent the displacement field and bifurcation mode components, respectively, relative to the two skins (with indices \( \mathbf{e}_b \) and \( \mathbf{e}_c \)) and the foam core (with indices \( \mathbf{e}_f \)).

Let us now consider the bending problem of the skins in the XY-plane. The Euler–Bernoulli beam theory is employed, as transverse shear effects may be negligible in practice. The Euler–Bernoulli kinematics is defined by two scalar displacement fields \( U(X) \) and \( V(X) \), respectively the axial and transverse displacements of the centroid axis of the beam. When the skins buckle from the straight position (the fundamental solution) to a bent shape, the expressions for the bifurcation modes \( \mathbf{X} \), and the displacement variations \( \delta \mathbf{U} \) (with \( i = a, b, c \) depending on the skin considered) are both chosen according to the Euler–Bernoulli kinematics:

\[ \mathbf{X} = \begin{bmatrix} U_i - YV_i \cr V_i \end{bmatrix} \quad \delta \mathbf{U} = \begin{bmatrix} \delta U_i - Y\delta V_i \cr \delta V_i \end{bmatrix} \]

(13)

On the other side, the bifurcation mode \( \mathbf{X} \) and the displacement variation \( \delta \mathbf{U} \), in the foam core are classically expressed in the orthornormal basis \( \{\mathbf{e}_f, \mathbf{e}_r, \mathbf{e}_z\} \):

\[ \mathbf{X} = \begin{bmatrix} U_c \cr V_c \end{bmatrix} \quad \delta \mathbf{U} = \begin{bmatrix} \delta U_c \cr \delta V_c \end{bmatrix} \]

(14)

The global bifurcation equation then writes:

\[ \forall \delta \mathbf{u}_a, \delta \mathbf{V}_a, \delta \mathbf{u}_b, \delta \mathbf{V}_b, \delta \mathbf{u}_c, \delta \mathbf{V}_c, \]

\[ \int_{A_a} \left[ E_i (U_{a,XX} - YV_{a,XY}) (\delta U_{a,XX} - Y\delta V_{a,XY}) \right] d\Omega_a \]

\[ + \int_{A_b} \left[ E_i (U_{b,XX} - YV_{b,XY}) (\delta U_{b,XX} - Y\delta V_{b,XY}) \right] d\Omega_b \]

\[ + \int_{A_c} \left[ C_{XXX} \delta U_{c,XX} + C_{YYX} \delta V_{c,XY} + C_{XXY} \delta U_{c,XY} + C_{YYX} \delta V_{c,XY} \right] d\Omega_c = 0 \]

(15)

where \( Y \) stands for the \( Y \)-coordinate of a current point relative to the centroid axis of the corresponding zone.

First, integrating over the cross-sections of the beams, then integrating by parts with respect to \( X \) and \( Y \), and eliminating negligible higher-order terms (presupposing that \( \lambda_c \ll 1 \)) yields six local partial differential equations for the components \( \mathbf{u}_a, \mathbf{V}_a, \mathbf{u}_b, \mathbf{V}_b, \mathbf{u}_c, \mathbf{V}_c \) of the eigenmode:

\[ 2E_i h_d U_{a,XX} + \mu_i (U_{a,XY} + V_{a,XY}) |_{Y=-h_c} = 0 \]

\[ 2E_i h_d U_{b,XX} + \mu_i (U_{b,XY} + V_{b,XY}) |_{Y=-h_c} = 0 \]

\[ 2E_i h_d U_{c,XX} + \mu_i (U_{c,XY} + V_{c,XY}) |_{Y=-h_c} = 0 \]

(16)

The last two Eqs. (16a) and (16f) identify with classical local equilibrium equations of the core region in a 2D framework. Similarly, the first four Eqs. (16a)–(16d) look like the classical buckling differential equations of beams, but include new quantities characterizing the influence of stresses at the interface between the skin considered and the foam during the buckling phenomenon. Let us mention that these additional terms are here naturally obtained through integrations by parts performed in the foam core.

At this stage, one has to specify the boundary conditions in order to solve the previous system. First, connecting conditions for
the displacement fields (bifurcation mode) must be satisfied at the interfaces between the foam core and the two facings, namely:

\[
\forall X \in ]0, L[,
\begin{align*}
U_0 - h_1 V_{aX} - U_1 |_{Y = -h_1} &= 0 \\
U_0 + h_1 V_{bX} - U_1 |_{Y = -h_1} &= 0 \\
\forall Y, V_0 - V_{cY} |_{X = 0} &= 0 \\
\forall Y, V_0 - V_{cY} |_{X = L} &= 0
\end{align*}
\]

(17)

Owing to the enforced displacement boundary conditions, the two ends of each face act as if they were guided, which leads to the following kinematical constraints:

\[
V_{aX}(0) = V_{aX}(L) = V_{bX}(0) = V_{bX}(L) = 0
\]

Taking into account \(\partial V_{aX}(0) = \partial V_{aX}(L) = \partial V_{bX}(0) = \partial V_{bX}(L) = 0\) in the bifurcation Eq. (15) leads one, after integration by parts, to the remaining stress boundary conditions at the ends \(X = 0\) and \(X = L\):

\[
\begin{align*}
2E_s h_1 U_{aX}(0) &= 0 \\
2E_s h_1 U_{aX}(L) &= 0 \\
2E_s h_1 U_{bX}(0) &= 0 \\
2E_s h_1 U_{bX}(L) &= 0
\end{align*}
\]

(18)

The last boundary conditions refer to the two remaining edges of the foam zone. The enforced displacement boundary conditions induce kinematical constraints for the axial component of the bifurcation mode at \(X = 0\) and \(X = L\), that is to say:

\[
\forall X \in ]-h_1, h_1[, 
U_{1X}|_{X = 0} = U_{1X}|_{X = L} = 0
\]

(19)

Since these two edges are free in the \(Y\)-direction, the last two equations consist in the following stress boundary conditions:

\[
\begin{align*}
\mu_c(U_{1Y} + V_{1X})|_{X = 0} &= 0 \\
\mu_c(U_{1Y} + V_{1X})|_{X = L} &= 0
\end{align*}
\]

(20)

2.3. Solution procedure

The bifurcation modes of a single Euler–Bernoulli beam under axial compression with the boundary conditions defined above take the following form:

\[
\begin{align*}
U_t &= 0 \\
V &= \cos \frac{\pi X}{L}
\end{align*}
\]

(21)

where \(n\) is the associated half-wave number.

According to Eq. (21) and preliminary numerical observations, the following assumptions are made for the beam components of the bifurcation mode of the sandwich column:

\[
\begin{align*}
U_a &= \alpha \sin \frac{\pi X}{L} \\
U_b &= \beta \sin \frac{\pi X}{L} \\
V_a &= \pm \cos \frac{\pi X}{L} \\
V_b &= \cos \frac{\pi X}{L}
\end{align*}
\]

(22)

for both faces. However, two cases are considered, depending on the relative sign of the two fields \(V_a\) and \(V_b\). The bifurcation mode of the sandwich column may thus be asymmetric (\(V_a = V_b\)) or symmetric (\(V_a = -V_b\)). Conversely, the longitudinal components \(U_a\) and \(U_b\) are no longer zero as in the case of a single beam, due to the presence of the foam on one side only. Sinusoidal shapes are also retained for these components, which are consistent with the associated boundary conditions, together with unknown amplitudes \(\alpha\) and \(\beta\) to be determined.

Concerning the foam modal displacement field, a separation of variables is performed and the following forms are presupposed, according to Eq. (22):

\[
\begin{align*}
U_c &= \varphi(Y) \sin \frac{\pi Y}{L} \\
V_c &= \varphi(Y) \cos \frac{\pi Y}{L}
\end{align*}
\]

(23)

2.3.1. Antisymmetric case

Assuming \(V_a = V_b = \cos \frac{\pi X}{L}\) allows one to derive the so-called antisymmetric critical displacements and corresponding modes. Solving the two equilibrium equations in the foam core (16a) and (16b), together with the four connecting conditions (17), leads to the expression of \(\alpha\) and \(\beta\) as functions of \(\varphi\) and \(\beta\). Then, Eqs. (16a) and (16c) are solved in order to determine \(\alpha\) and \(\beta\). The following expressions are obtained:

\[
\begin{align*}
\alpha &= -\frac{E_s c_1}{2 L E_s L_s L_h h_s a_1} \left[ \text{sinh} \left( \frac{n \pi h_s}{L} \right) \text{sinh} \left( \frac{n \pi h_s}{L} \right) + \text{sinh} \left( \frac{n \pi h_s}{L} \right) \text{sinh} \left( \frac{n \pi h_s}{L} \right) \right] \\
\beta &= \alpha
\end{align*}
\]

(24)

\[
\begin{align*}
\varphi(Y) &= K_1 \cos \frac{n \pi Y}{L} + K_2 Y \sinh \frac{n \pi Y}{L}
\end{align*}
\]

(25)

The remaining boundary conditions are automatically verified. Finally, Eqs. (16b) or (16d) can be solved and lead to the same critical displacement:

\[
\lambda_c^0 = \left( 4 E_s E_t n \pi L^2 h_s \left[ 4 n_s^2 \pi^2 h_s^2 + 2 L^2 \right] \cosh \frac{n \pi h_s}{L} \right.
\]

\[
+ \left. \left[ 3 E_s^2 L^4 + 12 E_s c_1 n_s^2 \pi^2 L^2 h_s^3 (1 - v_s) + 4 E_s^2 n_s^4 \pi^4 L^4 h_s^5 (3 + 2 v_s - v_s^2) \right] \right)
\]

\[
\times \cosh \frac{n \pi h_s}{L} \sinh \frac{n \pi h_s}{L} - 3 E_s n \pi L^4 \left[ 4 E_s h_s + E_s h_c \right]
\]

\[
+ \left. 12 E_s n \pi L^2 h_s^2 h_c (1 + v_s) + 4 E_s^2 n_s^2 \pi^4 h_s^7 h_c (1 + v_s) \right)
\]

\[
\left. \left( 12 E_s n \pi L h_s \left[ E_s L_s L_s \cosh \frac{n \pi h_s}{L} + E_s n \pi L (3 + 2 v_s - v_s^2) \right] \right) \right)
\]

\[
\times \cosh \frac{n \pi h_s}{L} \sinh \frac{n \pi h_s}{L} + E_s n \pi L^3 h_s (1 + v_s) \right)
\]

(26)

2.3.2. Symmetric case

Assuming now \(V_a = -\cos \frac{\pi X}{L} = -V_b\) leads to somewhat different expressions relative to the so-called symmetric buckling solution. The same procedure is employed and gives rise to the following expressions:
\[
\begin{align*}
\alpha &= \frac{E_c\left[2\pi h n \sinh\frac{\pi h}{L} + (1-v)\gamma \cosh\frac{\pi h}{L}\sinh\frac{\pi h}{L} - n\pi h(1+v)\cosh\frac{\pi h}{L}\sinh\frac{\pi h}{L}\right]}{2\left[\pi h E_0\cosh\frac{\pi h}{L} + 2\pi h n \sinh\frac{\pi h}{L} - n\pi h(1+v)\cosh\frac{\pi h}{L}\sinh\frac{\pi h}{L}\right]}, \\
\beta &= \alpha, \\
\gamma(Y) &= \frac{-K_1 n\pi(1+v) - K_2 n\pi(1-v)Y \sinh\frac{\pi h}{L}}{\pi h(Y+1)}, \\
\zeta(Y) &= K_1 \sinh\frac{\pi h}{L} + K_2 Y \cosh\frac{\pi h}{L} \\
\end{align*}
\]

(27)

With:

\[
K_1 = \frac{L^2(1-v) + n\pi^2 h_nh(1+v) - n\pi h_nh(1+v)\cosh\frac{\pi h}{L}\sinh\frac{\pi h}{L} - n\pi h(1+v)}{L(1-v) - n\pi h_nh(1+v)} \\
K_2 = \frac{n\pi(1+v)\cosh\frac{\pi h}{L} - (n\pi h_n - 2L(1+v)\cosh\frac{\pi h}{L}\sinh\frac{\pi h}{L})}{L(1-v) - n\pi h_nh(1+v)} \\
\]

(28)

In this case, the critical displacement writes:

\[
\delta_{cr} = \left(4E_cE_nL^2h^2[4n^2\pi^2h^2 + 3L^2]\cosh\frac{\pi h}{L} + 3E_cE_nL^2h^2[1-v] + 4E_cE_n^2\pi^4h^4(3 + 2v - v^2)\right) \\
\times \cosh\frac{\pi h}{L} - n\pi h_nh(1+v) - 16E_cE_n^2\pi^3L^3h^3 \\
- 12E_cE_n^2\pi^2Lh^2[1+v] - 4E_cE_n^2\pi^2h^2h(1+v)^3]\left(12E_n^2\pi nLh(1+v)\cosh\frac{\pi h}{L} + 4E_n^2\pi L^2h^2\sinh\frac{\pi h}{L} - n\pi h_nh(1+v)\cosh\frac{\pi h}{L}\sinh\frac{\pi h}{L}\right) \\
\times \sinh\frac{\pi h}{L} - E_cL^2 - E_n^2\pi^2h^2h(1+v)^3 \middle/ \cosh\frac{\pi h}{L} \\
\]

(29)

3. Analytical modeling of the buckling behavior of sandwich beams under pure bending

Let us now consider the case of a beam under pure bending. The same general assumptions as introduced in the previous case of a compressed column are made. The left end section of the sandwich beam is still subjected to a zero displacement boundary condition in the longitudinal direction, whereas the right end section is now submitted to a linear distribution of longitudinal displacements, symmetric with respect to the neutral axis of the sandwich beam, which is supposed to be equivalent to a global bending moment as far as the rotation of the section remains small (see Fig. 3).

The same bifurcation equation is used as before, with the compressive displacement \( \lambda > 0 \) of the neutral axis of the upper skin at the right end section acting as the new bifurcation parameter. The pre-critical stresses in the foam core are still neglected. As for the skins, the upper one is supposed to be uniformly compressed, with a stress level \( P > 0 \) related to the enforced displacement \( \lambda > 0 \) by Eq. (12), and the lower one is subjected to the opposite uniform tensile stress.

Eventually, the bifurcation equation of the whole sandwich structure writes here:

\[
\forall \delta u_i, \delta u_p, \delta u_c, \int_\Omega \nabla^i \delta u_i : \left( \epsilon^i + \lambda c, \epsilon^i + \lambda c, \epsilon^i + \lambda c \right) : \nabla c d\Omega = 0 \\
+ \int_\Omega \nabla^p \delta u_p : \left( \epsilon^p + \lambda c, \epsilon^p + \lambda c, \epsilon^p + \lambda c \right) : \nabla c d\Omega = 0 \\
+ \int_\Omega \nabla^c \delta u_c : \left( \epsilon^c + \lambda c, \epsilon^c + \lambda c, \epsilon^c + \lambda c \right) : \nabla c d\Omega = 0
\]

(30)

what leads to the new local partial differential equations for the components \( u_i, \ u_p, \ u_c \) of the eigenmode:

\[
2E_c h_iu_{xx} + \mu_c (u_{xy} + u_{yx})|_{y=-h_i} = 0 \\
2E_c h_pu_{xx} + \mu_c (u_{xy} + u_{yx})|_{y=-h_p} = 0 \\
2E_c h_cu_{xx} + \mu_c (u_{xy} + u_{yx})|_{y=-h_c} = 0
\]

(31)

The displacement and stress boundary conditions, together with the connecting conditions, are totally unchanged.

According to preliminary numerical observations, the following assumptions are made for the beam components of the bifurcation mode of the sandwich structure:

\[
\begin{align*}
\delta u_i &= \alpha \sin\frac{\pi h}{L} \\
\delta u_p &= \beta \sin\frac{\pi h}{L} \\
\delta u_c &= \gamma \cos\frac{\pi h}{L} \\
\gamma &= \delta \cos\frac{\pi h}{L} \left( \lambda > 0 \right)
\end{align*}
\]

(32)

introducing a new parameter \( \delta \) as the amplitude of the lower skin displacement mode, which is also sinusoidal under the influence of the upper skin sinusoidal buckling response. Concerning the foam modal displacement field, the same separation of variables is performed and the previous forms (Eq. (23)) are presupposed.

The solution procedure is very similar than the one described in the previous section. The only difference occurs at the end, when dealing with Eqs. (31b) and (31d). Contrary to the previous case, these two equations differ from each other, due to the different signs of the pre-critical stresses in the two facings. These equations are thus no more redundant and may be used in order to derive both values of the critical displacement \( \lambda_{cr} \) and the new amplitude \( \delta \). When solving Eq. (31b), for instance, one gets the expression of \( \lambda_{cr} \) as a function of \( \delta \). Then, solving Eq. (31d) leads to two positive real roots \( \delta_1 \) and \( \delta_2 \), inverse from each other. One of them, say \( \delta_1 \), is thus less than unity. This solution corresponds to the case where the upper skin undergoes a compressive stress \( (\lambda_{cr} > 0) \), and displays a normalized sinusoidal buckled shape. In this case, the lower skin also deforms with a sinusoidal shape, but with a smaller amplitude \( \delta_1 \). When considering \( \delta_2 \) instead of \( \delta_1 \), one switches roles between the upper and lower skins. The critical displacement cor-

![Fig. 3. Two-dimensional representation of the sandwich beam under pure bending.](image-url)
responding to $d_2$ is therefore opposite to the value obtained for $d_1$, due to symmetry of the sandwich beam.

Explicit forms of the critical displacement and associated bifurcation mode can also be determined in this loading case, but they are here too cumbersome to be presented as closed-form expressions.

4. Results and validation

4.1. Numerical computation of the critical displacements and bifurcation modes

Two-dimensional numerical finite element computations (linearized buckling analyses) have been performed, using Abaqus software, in order to validate the previous analytical solutions. A 2D geometry of the sandwich beam-column is retained, where the foam core but also the two skins are represented by a 2D continuous solid satisfying the plane stress assumption. The associated finite element mesh is made up of 8-noded quadrangular elements with reduced integration. The same mesh is employed for most of the calculations, displaying 100 elements along the length, 30 elements in the thickness of the foam core and 5 elements in the thickness of each skin. The appropriate displacement boundary conditions are enforced at both ends as described above in the theoretical developments and a supplementary transverse displacement is fixed in order to prevent the structure from rigid modes. In accordance with the analytical solutions, two types of buckling modes are observed for the compressed column, namely symmetric and antisymmetric modes, the latter including both global and local modes, depending on the wave number. Conversely, in the case of a beam under pure bending, only one mode shape is obtained. All these bifurcation modes are depicted in Figs. 4–6.

4.2. Validation and parametric analysis in the case of a compressed sandwich column

Subsequently, the analytical solutions obtained with the present approach are compared to other analytical solutions in the literature and validated by confrontation with our own numerical results. Let us first consider the case of a sandwich column under axial compression. First, the analytical expressions for the critical buckling load derived by Léotoing et al. (2002) in the same context

Table 1
Material and geometric parameters.

<table>
<thead>
<tr>
<th>$E_s$ (MPa)</th>
<th>$E_c$ (MPa)</th>
<th>$\nu_s$</th>
<th>$L$ (mm)</th>
<th>$h_s$ (mm)</th>
<th>$h_c$ (mm)</th>
</tr>
</thead>
<tbody>
<tr>
<td>50,000</td>
<td>70</td>
<td>0.4</td>
<td>600</td>
<td>0.5</td>
<td>15–30</td>
</tr>
</tbody>
</table>
will be used for comparison purposes, considering the same first two examples (the material and geometric parameters are summarized in Table 1). Despite some differences in the boundary conditions between the two analyses (the one from Léotoing et al. (2002) and the present one), both analytical solutions are similar. The only difference is a phase difference in the sinusoidal response of the sandwich column, which affects neither the critical loading, nor the buckling mode type and the corresponding wave number. Then, parametric analyses will be performed to see the relative influence of geometric and material parameters on the buckling mode type. In every instance, both the critical displacement and the wave number of the associated bifurcation mode will be analyzed for the first mode(s) encountered (corresponding to the minimum critical value(s)).

4.2.1. Case 1: $h_c = 15$ mm

First, the analytical critical displacements (Eq. (26) for the antisymmetric case and Eq. (29) for the symmetric one) are plotted against the analytical solutions of Léotoing et al. (2002) and our numerical results for a large range of wave numbers (see Figs. 7(a) and (b), respectively).

The present analytical solution is completely in accordance with the numerical results. The order of appearance of the different modes (when considering increasing critical values) is the same for both approaches and the relative error between the critical displacements is around 1% at the maximum. The results from Léotoing et al. (2002) are somewhat similar to the other ones in the symmetric case (with less than 5% of error), but slightly differ from ours in the antisymmetric case (with a maximum error around 7%). It clearly shows that the present formulation leads to a better accu-

![Graph](attachment:image.png)

Fig. 7. Comparison between analytical and numerical critical values ($h_c = 15$ mm).
racy than a simplified model with an approximate displacement field in the core material, at least for the antisymmetric modes.

Fig. 8 compares our analytical solutions for both antisymmetric and symmetric modes. The critical values highly differ from each other for small wave numbers, but conversely they almost coincide for large wave numbers (what was not observed in Léotoing et al. (2002)). For this particular sandwich, the first modes encountered (with the minimum critical values) are antisymmetric and, moreover, the very first mode is the so-called global mode corresponding to the antisymmetric mode with \( n = 1 \). Then, at a sufficiently high level of critical displacements, antisymmetric and symmetric modes may alternate.

4.2.2. Case 2: \( h_c = 30 \) mm

The same analysis is performed with a similar sandwich column, except for the core thickness which is twice as before. Figs. 9(a) and (b) show the comparison between the analytical and numerical critical displacements in the antisymmetric and symmetric cases, respectively.

The present analytical solution is again in very good accordance with the numerical results, with a relative error between the critical displacements less than 1%. With this new geometry, the results from Léotoing et al. (2002) differ now significantly from the other ones, both in the symmetric and antisymmetric cases (with a maximum error around 20%). Their solutions always overestimate the numerical results in the symmetric case, and underestimate the numerical values in the antisymmetric case. With such discrepancies in the critical values, the order of appearance of the successive modes is very likely wrong compared to the numerical predictions. The benefits of the present formulation appear here more clearly, when considering a thicker core layer and especially for large wave numbers. In such conditions, even the higher-order beam model developed by Léotoing et al. (2002) is not capable of capturing accurately the displacement field observed in the core material during the buckling phenomenon.

Fig. 10 compares again the analytical solutions for both antisymmetric and symmetric modes. The same tendency is observed as in case 1, namely symmetric critical values much larger than antisymmetric ones for small wave numbers, and then coincident critical values from a certain threshold wave number. Contrary to the previous case, the very first modes are local (that is to say with large wave numbers) and therefore alternatively antisymmetric and symmetric.

4.2.3. Parametric analysis

The main objective of the parametric study is to determine the relative influence of some geometric and material parameters on the critical displacement value and especially on the buckling mode type (namely symmetric or antisymmetric, global or local).

In the two examples above, it has been noticed that the antisymmetric and symmetric critical displacements tend towards the same value when the wave number tends to infinity. This phenomenon is always observed, but the two critical values are shown to coincide for varying wave numbers. Therefore, for a relatively thin core layer, the first (local) modes are purely antisymmetric, and the first symmetric mode occurs much later for a greater critical value (see Fig. 8). Conversely, for a thicker shell, the minimum critical value corresponds to both a symmetric and an antisymmetric mode, which means that the first mode is in some way a mixed mode. Let us compare the first symmetric and antisymmetric local modes for various ratios of thickness \( h_c/h_s \) and modulus \( E_s/E_c \). The same geometric and material parameters are used subsequently (see Table 1), apart from the core thickness \( h_c \) and Young’s modulus \( E_c \) which vary.

Fig. 11 shows that, for a given length, the minimum local antisymmetric and symmetric critical displacements coincide for a thickness ratio \( h_c/h_s \) greater than a unique transition value (say about 40), regardless of the modulus ratio \( E_s/E_c \). Moreover, the common critical value for thicknesses greater than this transition value does only depend on the material moduli but not on the thicknesses. The so-called transition value of the thickness ratio represents the limit case above which the two facings buckle in an uncoupled way, as if the core layer was infinitely thick. The relative sign of the buckling mode component of each face (determining the antisymmetric or symmetric nature of the overall buckling mode) has thus no influence on the critical displacement.

If interest is now focusing on antisymmetric modes, it could be of significant practical relevance to determine whether the first mode is global or local, depending on the geometric and/or material properties. To this end, the minimum (antisymmetric) critical
displacements, either global (corresponding to $n = 1$) or local (with $n > 1$), are plotted in Fig. 12 for the same ratios of thickness ($h_c/h_s$) and modulus ($E_s/E_c$) as before. It allows us to identify two regions, separated by the dotted line. The global or local nature of the first mode highly depends on the thickness ratio $h_c/h_s$. The first mode is likely to be global if the core layer is particularly thin, whereas it will be local for greater thickness ratios $h_c/h_s$. The material parameters also influence the buckling response of the sandwich column. The greater the modulus ratio $E_s/E_c$, the smaller the transition value of the thickness ratio $h_c/h_s$ at which both a global and local mode can occur for the same minimum critical displacement.

From a general point of view, this new transitional value seems to be greater than the previous transition corresponding to the coincidence of symmetric and antisymmetric critical values. It means that, in practice, the buckling phenomenon will first occur either in a global way ($n = 1$) or in a local way. In the latter case, the geometry is always such that the first antisymmetric and symmetric critical displacements coincide and thus expressions (26) and (29) can equally be used for the determination of the minimum critical value.

All these observations only concern a unique length. It is obvious that the length-to-thickness ratio of the sandwich column has a great influence on the critical values and the buckling mode type (wave number), but a brief parametric analysis allows us to draw similar conclusions for other lengths in a realistic range.

4.3. Validation in the case of a sandwich beam under pure bending

Next, the present analytical solution for the buckling response of a sandwich beam under pure bending will be compared to the recent analytical results from Jasion et al. (2012) and our numerical results obtained using the finite element model presented above (the mesh may display 200 or 400 elements along the length, depending on the wave number of the first buckling mode, in order
to properly describe the sinusoidal displacement field). The same examples as in Jasion et al. (2012) are considered, corresponding to the geometric and material parameters listed in Table 2.

For each value of $E_c$ considered, both the critical displacement (which corresponds to the displacement at the end of the neutral axis of the compressed face at the critical time) and the wave number of the associated bifurcation mode are evaluated for the first mode (with the minimum critical value). The results are summarized in Table 3 and plotted in Fig. 13.

The analytical solution developed in this paper shows a very good agreement with the numerical results obtained with a 2D finite element model. The half-wave numbers of the first buckling modes are almost the same and the critical values differ by no more than 2% between the two approaches. For comparison purposes, it can be noticed that the results from Jasion et al. (2012) are less likely than ours. A few differences between half-wave numbers can be mentioned. However, the most important thing is that their analytical solutions overestimate the reference numerical values (what makes them non-conservative) with a relative error up to 13% in the most practical case of a soft core layer.

5. Conclusions

In this study, we investigated the buckling behavior of sandwich beam-columns under various loadings. Exact analytical solutions were obtained for the critical loading and bifurcation mode, deriving from a 3D bifurcation analysis, in both cases of a sandwich column under axial compression and a sandwich beam under pure bending. A 2D model was defined, for simplicity purposes, where the skins were assumed to behave like Euler-Bernoulli beams while the core material was modeled as a 2D continuous solid, without considering any simplified deformation field. In the first case of compression, two buckling mode types were identified (namely antisymmetric and symmetric), leading to two different
expressions for the critical loading, both depending on a wave number. Furthermore, it has been proved that the so-called “global” and “local” modes can be represented by the same expressions (the global modes can be viewed as antisymmetric modes with small wave numbers). In the second case of pure bending, just one buckling mode type was observed, since only one skin is supposed to buckle under such loading conditions.

The present analytical solutions were compared to previous ones in the literature and 2D finite element results performed for validation purposes. The comparison focused on the order of appearance of the different modes and predominantly on the first mode encountered in practice displaying the minimum critical value. In all cases, our analytical predictions are in very good agreement with the reference numerical results (with less than 2% of error). Moreover, significant discrepancies have been noticed between our solutions and the ones from previous studies (with relative differences up to 10–20% at the maximum and possibly a

<table>
<thead>
<tr>
<th>$E_i$ (MPa)</th>
<th>$E_c$ (MPa)</th>
<th>$v_i$</th>
<th>$L$ (mm)</th>
<th>$h_s$ (mm)</th>
<th>$h_c$ (mm)</th>
</tr>
</thead>
<tbody>
<tr>
<td>65,600</td>
<td>10–50–100–400–800–1200</td>
<td>0.3</td>
<td>800</td>
<td>0.5</td>
<td>9</td>
</tr>
</tbody>
</table>

Table 3
Half-wave numbers and critical displacements of the first buckling mode in pure bending.

<table>
<thead>
<tr>
<th>$E_i$ (MPa)</th>
<th>Analytical</th>
<th>Numerical (FE)</th>
<th>Jasion et al.</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$n$</td>
<td>$\lambda_{cr}$ (mm)</td>
<td>$n$</td>
</tr>
<tr>
<td>10</td>
<td>23</td>
<td>1.47</td>
<td>23</td>
</tr>
<tr>
<td>50</td>
<td>37</td>
<td>4.03</td>
<td>37</td>
</tr>
<tr>
<td>100</td>
<td>45</td>
<td>6.35</td>
<td>45</td>
</tr>
<tr>
<td>400</td>
<td>70</td>
<td>16.29</td>
<td>70</td>
</tr>
<tr>
<td>800</td>
<td>87</td>
<td>26.39</td>
<td>88</td>
</tr>
<tr>
<td>1200</td>
<td>99</td>
<td>35.09</td>
<td>100</td>
</tr>
</tbody>
</table>

Fig. 12. Minimum global or local antisymmetric critical values for various thickness and modulus ratios.

Fig. 13. Comparison between analytical and numerical minimum critical values in pure bending.
disarrangement of the successive modes). In view of these results, one can state that it is important to describe the real deformation behavior of the foam core with a 2D continuous model, instead of using a simplified beam-like model, in order to better identify the first buckling mode and estimate the sought critical value more accurately. Likewise, it is essential to take into account the deformation state of the two skins, even the one which is subjected to tensile pre-stresses, in the case of pure bending.

In the case of compression, a parametric analysis has finally been performed with respect to geometric and material parameters in order to see the influence of the thickness and modulus ratios between the core material and the skins on the type of buckling mode encountered in the first place (global/local, antisymmetric/symmetric). It has been shown that the first mode is likely to be global if the core layer is particularly thin, whereas it will be local for a relatively thick foam core. The symmetric critical values are always greater than the antisymmetric ones. Besides, with the length considered in this paper, whenever the first mode is local, the core layer is found to be sufficiently thick so that the two skins buckle in an uncoupled way. Thus, the antisymmetric and symmetric critical displacements coincide and do not depend anymore on the thickness ratio, as soon as it remains above the transition value where a local mode and a global mode may simultaneously occur. Therefore, only the expression of the critical displacement in the antisymmetric case is useful in practice, since it may represent both global and local buckling behaviors and give critical values below or most often equal to the ones provided by the expression in the symmetric case.

Since the methodology developed here is general, the present results may be adapted to the case of sandwich plates under biaxial compression (tension). Otherwise, these analytical results may be used as a starting point for the development of a 1D finite element model which would rely on the exact deformation mode of each layer in order to capture local effects (such as the wrinkling of the faces) during the buckling analysis of the sandwich, following the example of Hu et al. (2009).

References


