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Stable sets and polynomials

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Abstract

Several applications of methods from nonlinear algebra to the stable set problem in graphs are surveyed. The most recent work cited was cowritten by A. Schrijver and involves nonlinear inequalities. These yield a procedure to generate facets of the stable set polytope. If a class of graphs has the property that all facets of the stable set polytope can be generated this way in a bounded number of setps, then the stable set problem is polynomially solvable for these graphs. Perfect, t-perfect and h-perfect graphs have this property.

0. Introduction

The stable set problem is one of the simplest and most fundamental problems concerning graphs. As is to be expected, the problem may serve as a pilot case in the application of various methods to NP-hard problems. Indeed, the stable set problem was among the first to which polyhedral and linear algebraic methods have been successfully applied.

In this paper we survey some applications of nonlinear commutative algebra (polynomials) in the study of the stable set problem. While several of our considerations will have a similar flavor, they have not yet been unified in a single theory. Rather, I consider them as pieces of a jigsaw puzzle, where it is easy to feel that they belong to one bigger picture but their exact position is not found yet.

A *stable set* in a graph $G=(V,E)$ is a set of nodes, no two of which are adjacent. The maximum size $\alpha(G)$ of a stable set is called the *stability number* of G. It is well known that the computation of $\alpha(G)$ is NP-hard.

However, quite often we are not concerned with the stability number of an explicitly given graph but rather with the stability number of graphs derived from some other combinatorial structure. In fact, many results in extremal combinatorics and coding theory can be formulated in this way. In such cases, general algebraic methods for the determination of $\alpha(G)$ have a greater chance of success.

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For example, let S be an n-element set, $V=\binom{S}{k}$, the set of its k-element subsets, and define a graph K_n^k on V by connecting two k-subsets iff they are disjoint. These graphs are called *Kneser graphs.* The stability number of K_n^k is the maximum number of mutually intersecting k-subsets of an n-set. The Erdős–Ko–Rado theorem asserts that this number is $\binom{n-1}{k-1}$ if $n\geqslant 2k$. There is at least one general algebraic upper bound on the stability number from which this bound can be derived [17].

1. Turán's theorem and the Motzkin–Straus formula

One of the first results in extremal graph theory is Turán's theorem, which can be formulated as a result on the stability number.

Theorem 1.1. *Let G be a graph with n nodes, m edges and stability number α. Write $n=\alpha q+r$. Then*

$$m\geqslant(\alpha-r)\binom{q}{2}+r\binom{q+1}{2}.$$

This theorem is usually formulated for the complement. The best way to remember the bound is that the graph with a given number of nodes n, with given stability number α and with a minimum number of edges is the union of α node-disjoint cliques, as equal in size as possible. The complement of this graph (a complete multipartite graph) is called the *Turán graph* T_n^α.

Turán's theorem can be proved by double induction on n and α (and in many other ways). For our purposes the most interesting is an algebraic proof by Motzkin and Strauss [21], which in fact gives an explicit formula for $\alpha(G)$.

Theorem 1.2. *Let $G=(V,E)$ be a graph. Then*

$$1-\frac{1}{\alpha(G)}=\max\left\{2\sum_{ij\notin E}x_ix_j\colon \sum_{i\in V}x_i=1,\ x_i\geqslant 0\right\}.$$

Turán's theorem (at least for the case when $\alpha|n$) can be obtained by substituting $x_i=1/n$.

Proof. Consider a vector x maximizing the right-hand side and (if it is not unique) having a maximum number of 0 entries. We claim that the support of x is a stable set. Assume, to the contrary, that nodes 1 and 2 are adjacent nodes belonging to the support of x. If $-x_1\leqslant t\leqslant x_2$ then the vector $(x_1+t, x_2-t, x_3, \dots)$ also satisfies the constraints $x_i\geqslant 0$ and $\sum_i x_i=1$. The objective function is a linear function of t (since the term x_1x_2 does not occur in the objective function), and hence it can assume its maximum at 0 only if it is constant. But then for $t=x_2$, we have another optimizing vector with more 0 entries.

Now if the support of x is a stable set A, then high school algebra shows that the maximum is attained when $x_i = 1/|A|$ for all $i \in A$, and then the value of the objective function is exactly $1 - 1/|A|$. The maximum is attained when A is a maximum stable set. □

2. Polynomial ideals, stable sets and chromatic number

With every graph $G = (V, E)$ on $V = \{1, \ldots, n\}$, we associate the polynomial

$$p_G(x_1, \ldots, x_n) = \prod_{ij \in E} (x_i - x_j)$$

(we need a reference orientation of the edges for this, but this plays a small role). It is interesting to remark that this correspondence was one of the starting points of graph theory. Hilbert (1898) (see König [12]) considered the question of decomposing such a polynomial in which every variable has degree d into factors in which every variable has degree 1. This is of course equivalent to decomposing a regular graph in 1-factors and was one of the motivations for the work of Petersen. The words *factor* and *degree* come from this correspondence.

More recently, Li and Li [13] related this polynomial to the independence number of the graph G. Their starting point was the following observation.

Lemma 2.1. *The graph G has independence number at most α if and only if the identification of any $\alpha + 1$ variables in p_G yields the 0 polynomial.*

Let I_n^α denote the set of all polynomials $f \in \mathbb{R}[x_1, \ldots, x_n]$ such that $f(x_1, \ldots, x_n) = 0$ if at least $\alpha + 1$ variables are equal. Clearly I_n^α is an ideal. Now Li and Li prove the following characterization of this ideal.

Theorem 2.2. *The ideal I_n^α is generated by the polynomials P_H, where H is a graph isomorphic to the complement of the Turán graph T_n^α.*

Let $\hat{I}_n^\alpha$ denote the ideal generated by the polynomials p_H, $H \equiv \bar{T}_n^\alpha$. Trivially $\hat{I}_n^\alpha \subseteq I_n^\alpha$. It is easy to see that an n-tuple $(y_1, \ldots, y_n)$ is a zero of $\hat{I}_n^\alpha$ if and only if some $\alpha + 1$ of the y_i are equal. Hence, by the Nullstellensatz of Hilbert, every $f \in I_n^\alpha$ has some power f^p which belongs to $\hat{I}_n^\alpha$. The theorem says that we have $p = 1$.

One consequence of this result is Turán's theorem. In fact, if G is a graph with n nodes, m edges, and $\alpha(G) = \alpha$, then $p_G \in I_n^\alpha$. By Theorem 2.2, this ideal is generated by homogeneous polynomials of degree $m_0 = |E(\bar{T}_n^\alpha)|$, and hence $m = \deg(p_G) \geqslant m_0$.

The proof of Theorem 2.1 is quite involved. We shall state and prove an analogue for the chromatic number, due to Kleitman and Lovász (unpublished), whose proof is simpler. We start with the following observation.

Lemma 2.3. *The graph G has chromatic number at least χ if and only if every way of identifying variables in p_G, so that at most $\chi-1$ distinct variables remain, yields the 0 polynomial.*

Let J_n^χ denote the set of all polynomials $f\in\mathbb{R}[x_1,\ldots,x_n]$ such that $f(x_1,\ldots,x_n)=0$ if at most $\chi-1$ variables are distinct. Clearly J_n^χ is an ideal. This ideal can be characterized as follows.

Theorem 2.4. *The ideal J_n^χ is generated by the polynomials P_H, where H is a graph consisting of a clique of size χ and isolated nodes.*

The relation of this theorem to the Nullstellensatz is similar to that of Theorem 2.2.

Proof. Let $\hat{J}_n^\chi$ denote the ideal generated by the polynomials p_H, where H consists of a χ-clique and isolated nodes. Trivially $\hat{J}_n^\chi\subseteq J_n^\chi$. To show the converse inclusion, let $f\in J_n^\chi$. For $S\subseteq\{1,\ldots,n-1\}$, let f_S denote the polynomial obtained from f by substituting x_n for each x_i, $i\in S$.

Clearly $f_S\in J_n^\chi$, and so by induction on the number of variables we may assume that $f_S\in\hat{J}_n^\chi$ for every nonempty S.

Consider also the polynomial

$$g=\sum_S(-1)^{|S|}f_S.$$

If we substitute x_n for any x_i $(1\leqslant i\leqslant n-1)$ in g, we get the 0 polynomial; hence g is divisible by the product $(x_1-x_n)(x_2-x_n)\cdots(x_{n-1}-x_n)$. Write

$$g=(x_1-x_n)(x_2-x_n)\cdots(x_{n-1}-x_n)h.$$

It is clear that $g\in J_n^\chi$ and hence h vanishes whenever at most $\chi-2$ of the variables $x_1,\ldots,x_{n-1}$ are distinct. So if we expand h by the powers of x_n, the coefficient of every x_n^k will belong to $J_{n-1}^{\chi-1}$. By induction on n, we may assume that these coefficients belong to $\hat{J}_{n-1}^{\chi-1}$. This implies that $g\in\hat{J}_n^\chi$.

It follows that

$$f=g-\sum_{S\neq\emptyset}(-1)^{|S|}f_S\in\hat{J}_n^\chi,$$

which proves Theorem 2.4. $\square$

A slight modification of the proof gives the following results.

Corollary 2.5. (a) *A graph G has $\alpha(G)\leqslant k$ iff p_G has a representation of the form*

$$p_G=p_{H_1}+p_{H_2}+\cdots+p_{H_N},$$

where each H_i is a graph on $V(G)$ containing k cliques covering all nodes.

(b) *A graph G has $\chi(G)\geqslant k$ iff p_G has a representation of the form*

$$p_G = p_{H_1} + p_{H_2} + \cdots + p_{H_N},$$

where each H_i is a graph on $V(G)$ containing a k-clique.

Fig. 1 shows a representation of P_{C_5} illustrating both parts of the theorem. This shows that there is a certain 'graph calculus' involved here. We can write $\sum_i G_i = 0$ instead of $\sum_i p_{G_i} = 0$; here the G_i are directed graphs but we identify two orientations differing on an even number of edges. Let us call a relation

$$\sum_{i=1}^{N} G_i = 0$$

a *graph identity*. It might be interesting to study the structure of graph identities. For example, it is not difficult to show that (after cancellation of edges common to all terms) there are two graph identities with 3 terms, illustrated in Fig. 2. Is the number of graph identities with a fixed number of terms finite? Can one verify a graph identity in polynomial time? (It can be verified in randomized polynomial time by substituting random values for the variables.) One would expect that the number of terms N in the representations in Corollary 2.5 is exponentially large in the worst case (else, NP would be equal to randomized co-NP). Can one prove this about some particular series of graphs? How does one find representations for graphs for which a lower bound for $\alpha(G)$, or an upper bound for $\chi(G)$, has been proved by other means (e.g. for the Kneser graphs)?

Recently Alon and Tarsi [1] obtained another related result. Let $f(x)$ be any polynomial in one variable with k distinct roots. Consider the polynomial ideal Q_n^k in

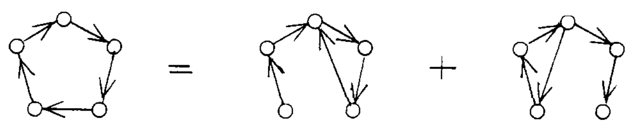

Fig. 1.

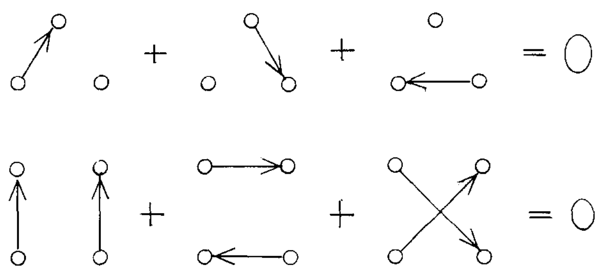

Fig. 2.

$\mathbb{R}[x_1,\dots,x_n]$ generated by the polynomials $f(x_1)$, $f(x_2),\dots,f(x_n)$; an n-tuple $(u_1,\dots,u_n)$ is a zero of this ideal iff every entry u_i is a root of f. It is easy to see that G has chromatic number greater than k iff it vanishes on all zeros of Q_n^k. It is not difficult to prove again the stronger 'Nullstellensatz', asserting that this is equivalent to saying that $p_G \in Q_n^k$.

Alon and Tarsi use this to prove the following sufficient condition on colorability.

Theorem 2.6. *Assume that the expansion of p_G contains a monomial with nonzero coefficient in which every exponent is less than k. Then G is k-colorable.*

It should be remarked that every expansion term $x_1^{a_1}x_2^{a_2}\dots x_n^{a_n}$ of p_G corresponds to an orientation of G in which node i has outdegree a_i. However, different orientations may give expansion terms with different sign, which therefore may cancel. So the condition in Theorem 2.6 requires more than just the existence of an orientation with outdegrees less than k (which would not suffice anyway, as the complete graphs show).

An advantage of this method is that it extends to restricted colorations in which every node has a set of at most k colors assigned and we want a legal coloration where the color of each node is one of the colors assigned to it.

3. α-Critical graphs

A graph G is called *α-critical* if it has no isolated nodes and, for every edge e, $\alpha(G-e)>\alpha(G)$. This is equivalent to saying that $\tau(G')<\tau(G)$ for every proper subgraph G' of G, and so these graphs can also be called τ-critical. Having a good description of α-critical graphs would yield a good description of α; this means that we probably cannot have a complete list of α-critical graphs (in particular, the class of α-critical graphs is probably neither in NP nor in co-NP).

Cliques and odd circuits are α-critical, and so are many other graphs; we just mention the icosahedron graph as an interesting example. Several monographs treat α-critical graphs [5, 18], so here we only sketch some of the most important results and discuss one proof in detail which relates to our main topic. For the following paragraphs, let G be an α-critical graph with n nodes and m edges, and set $\alpha=\alpha(G)$, $\tau=\tau(G)$.

The first result on α-critical graphs was obtained by Erdős and Gallai [7].

Theorem 3.1. *For every α-critical graph,*

$$\alpha \leqslant n/2$$

(equivalently, $\tau \geqslant n/2$). Equality holds iff G is a matching.

The quantity $\delta=\delta(G)=n-2\alpha=2\tau-n=\tau-\alpha$ turns out to be an important measure of the complexity of the α-critical graph, as we shall see below. This was first suggested by Gallai, and we call $\delta(G)$ the *Gallai class number* of the α-critical graph G.

The result of Erdős and Gallai can be viewed as a Helly-type result: it is equivalent to saying that *if the edges of every subgraph of a graph G with at most 2k nodes can be blocked by k nodes then all edges of G can be blocked by k nodes.* There is an analogous result on the number of edges, proved by Erdős et al. [8].

Theorem 3.2. *For every α-critical graph,*

$$m \leqslant \binom{\tau+1}{2}.$$

Hajnal [11] proved two extensions of the Erdős–Gallai theorem. For every set $A \subseteq V$, let $\Gamma(A)$ denote the set of nodes in G adjacent to some node in A.

Theorem 3.3. *If A is stable then* $|\Gamma(A)| \geqslant |A|$.

The Erdős–Gallai theorem corresponds to the case when A is a maximum stable set. Theorem 3.3 implies that G has a perfect 2-matching, i.e. a collection of node-disjoint circuits and edges covering all nodes.

The other theorem of Hajnal asserts the following.

Theorem 3.4. *The degree of any node in an α-critical graph G is bounded by* $\delta(G)+1$.

Besides the Erdős–Gallai theorem, this also implies the theorem of Erdős et al. [8] by summing over all elements of a minimum blocking set. Surányi (1975) obtained the following common generalization of all the results mentioned so far.

Theorem 3.5. *If A is a stable set and* $a \in A$ *then the degree of a is at most* $|\Gamma(A)|-|A|+1$.

The proof of this theorem is elegant but elementary (see [18, Theorem 12.1.13]). The following generalization of the theorem of Erdős et al. [8], however, is not known to follow by elementary methods [15].

Theorem 3.6. *A blocking set B in an α-critical graph G spans at most* $\binom{|B|-\alpha+1}{2}$ *edges.*

The Erdős–Hajnal–Moon theorem is obtained when $B=V(G)$. Another interesting case is when $|B|=\tau$: *a minimum blocking set in an α-critical graph spans at most* $\binom{\delta+1}{2}$ *edges.*

Since the proof of Theorem 3.6 fits in our discussions, we give it here. The key is the following lemma.

Lemma 3.7. *Let H be any graph with n nodes and m edges, and let, to each* $i \in V(H)$, *a vector* $v_i \in \mathbb{R}^k$ *be assigned. Assume that these* v_i *span* $\mathbb{R}^k$, *and*

(i) *every stable set corresponds to linearly independent vectors;*

(ii) *deleting any edge this will not hold any more.*

Then $m \leqslant \binom{n-k+1}{2}$.

Each α-critical graph provides an example when the conditions of the lemma hold: choose vectors $v_i \in \mathbb{R}^\alpha$ in general position. More interesting examples will come up in the proof of Theorem 3.6.

Proof of Lemma 3.7. Let us assign a variable x_i to each node i, and consider the following equations:

$$x_i x_j = 0 \quad (ij \in E(H)),$$

$$\sum_{i \in V(H)} x_i v_i = 0.$$

Note that the first set of equations is equivalent to saying that the support of x is a stable set in H; the second set implies that the support of x corresponds to linearly dependent vectors. So by assumption (i) in the lemma, this system has no nontrivial solution; moreover, (ii) implies that dropping any equation from the first set, the remaining system will have a nontrivial solution.

The solutions of $\sum_i x_i v_i = 0$ form a linear space with dimension $n-k$; over this space, the first set of equations gives m quadratic equations. By the above remarks, these quadratic equations must be linearly independent. Since the linear space of all quadratic equations over an $(n-k)$-dimensional space as dimension $\binom{n-k+1}{2}$, the lemma follows.

Proof of Theorem 3.6. For simplicity, we describe the proof in the case when B is a minimum blocking set in G. Let $A = V(H) \setminus B$ and let H be the subgraph of G induced by B. Let the numbers ξ_{ij} ($i \in B$, $j \in A$) be algebraically independent transcendentals, and let, for each $i \in B$, v_i be the vector in $\mathbb{R}^A$ whose jth entry is ξ_{ij} if $ij \in E(G)$ and 0 otherwise. The reader familiar with matroid theory will recognize that this is a representation of the transversal matroid induced on B by the edges connecting A to B. In particular, a set $S \subseteq B$ corresponds to linearly independent vectors iff every $S' \subseteq S$ has at least $|S'|$ neighbors in A.

We claim that H and the set of vectors $\{v_i\}$ satisfy the conditions of the lemma. The fact that $\{v_i\}$'s span the space $\mathbb{R}^A$ follows e.g. from Theorem 3.3. To verify (i), let S be a stable set in G and assume that the vectors $\{v_i : i \in S\}$ are not linearly independent. Then there exists a set $S' \subseteq S$ having fewer than $|S'|$ neighbors in A. But then $A \setminus \Gamma(S') \cup S'$ is a stable set in G larger than A, a contradiction. The proof that (ii) holds is similar.

Thus Lemma 3.7 implies that H has at most $\binom{|B|-\alpha+1}{2} = \binom{\delta+1}{2}$ edges.

Corollary 3.6 plays an important role in the proof of the following theorem.

Theorem 3.8. *For every $\delta \geq 0$, the number of α-critical graphs with Gallai class number δ and with all degrees at least 3 is finite.*

For $\delta = 1$, this follows trivially from Theorem 3.4. It was proved by Andrásfai [2] that every connected α-critical graph with $\delta = 2$ arises from K_4 by subdividing every edge by an even number of nodes; this implies Theorem 3.8 in this case. Case $\delta=3$ was settled by Surányi, and the general case by Lovász [16].

It should be mentioned that those α-critical graphs with Gallai class number δ and having nodes with degree 1 or 2 can be generated from this finite 'basis' in a rather simple way: one may add node-disjoint copies of K_2, or split a node into two and connect these two to a new node.

Let us remark that there are many unsolved problems concerning α-critical graphs. The known bounds on the maximum number of nodes of an α-critical graph G with $\delta(G)=\delta$ and all degrees at least 3 are very poor: the proof of Theorem 3.8 gives an upper bound of 2^{δ^2}, while a construction of Surányi (1978) gives a lower bound of δ^2. Is it true that if G' is an α-critical subgraph of an α-critical graph G then $\delta(G')<\delta(G)$? The following conjecture of Chvátal is also unsettled: *any k edges of an α-critical graph G adjacent to the same node are contained in an α-critical subgraph G' with $\delta(G')=k-1$*. (For $k=2$ this is a result of Berge.) Recently, Sewell [22] proved that *every α-critical graph G with $\delta(G)>2$ contains an even subdivision of K_4* (i.e. a connected α-critical subgraph H with $\delta(H)=2$).

4. The stable set polytope

At this point we have to recall some results on the stable set polytope. A detailed account can be found e.g. in Grötschel et al. [10].

Let $G=(V,E)$ be a graph. For every subset $S\subseteq V$, let $\chi^S\in\mathbb{R}^V$ denote its incidence vector, i.e. the vector defined by

$$\chi_i^S=\begin{cases}1 & \text{if } i\in S,\\ 0 & \text{otherwise.}\end{cases}$$

The *stable set polytope* STAB(G) of G is the convex hull of incidence vectors of all stable sets.

Since the stable set polytope is defined in terms of its vertices, it is natural to ask for a description of its facets. There is no hope of getting a complete description, but many interesting classes of facets, and more generally of inequalities valid for STAB(G), are known. We mention some of them. To exclude some trivial complications, we assume that G has no isolated nodes.

There are two classes of trivial inequalities: the *nonnegativity constraints*

$$x_i\geqslant 0 \quad (i\in V) \tag{4.1}$$

and the *edge constraints*

$$x_i+x_j\leqslant 1 \quad (ij\in E). \tag{4.2}$$

These inequalities define a polytope FRAC(G) which is in general larger than STAB(G). In fact, one has STAB$(G)=$FRAC(G) iff G is bipartite. An important property of FRAC(G) is that its vertices are half-integral.

A natural extension of (4.2) is the class of *clique constraints*:

$$\sum_{i \in B} x_i \leqslant 1, \quad \text{where } B \text{ is a clique.} \tag{4.3}$$

Inequalities (4.1) and (4.3) suffice to describe STAB(G) iff G is perfect.

The next important class of inequalities valid for STAB(G) is the class of *odd hole constraints*:

$$\sum_{i \in B} x_i \leqslant \frac{|C|-1}{2}, \quad \text{where } B \text{ induces a chordless odd cycle.} \tag{4.4}$$

A graph is called *t-perfect* if (4.1), (4.2) and (4.4) suffice to describe STAB(G), and *h-perfect* if (4.1), (4.3) and (4.4) suffice to describe STAB(G).

A further, perhaps less well studied, class of inequalities is the class of *odd antihole constraints*:

$$\sum_{i \in B} x_i \leqslant 2, \quad \text{where } B \text{ induces the complement of a chordless odd cycle.} \tag{4.5}$$

All these inequalities are facets at least if $B = V$. If there are nodes not occurring in the inequality then they may sometimes be added to the constraint with nonzero coefficient; this is called *lifting* (we do not discuss this procedure here).

A wider class of facets was found by Chvátal [6].

Theorem 4.1. *Let $G=(V,E)$ be an α-critical graph. Then the inequality $sum_{i\in V} x_i \leqslant \alpha(G)$ defines a facet of* STAB(G).

This theorem indicates how complex the facets of STAB(G) can be. Another important consequence of it is that we may in a sense consider facets of STAB(G) as generalizations of α-critical graphs and extend the theory of α-critical graphs to facets. This idea is rather unexplored; most of the results are due to Sewell [22]. We only state one result (found also by Lovász and Schrijver [19]) which will be needed later on.

Let $\sum_i a_i x_i \leqslant b$ be an inequality defining a facet of STAB(G). We define its *Gallai class number* as $\sum_i a_i - 2b$. The following lemma implies that this value is nonnegative, and in fact positive except for the edge constraints.

Lemma 4.2. *Let $\sum_i a_i x_i \leqslant b$ be a facet of* STAB(G). *Then*

$$\max\{\sum_i a_i x_i \colon x \in \mathrm{FRAC}(G)\} = \tfrac{1}{2} \sum_i a_i.$$

In other words, the left-hand side of any facet of STAB(G) is maximized over FRAC(G) by the vector $(1/2, \ldots, 1/2)^{\mathrm{T}}$. It can also be shown that if $a_i > 0$ for all i, and the facet is different from the edge constraints, then this is the unique vector maximizing the left-hand side.

This lemma also generalizes Hajnal's theorem (3.3). It also follows that the Gallai class number is a kind of 'integrality gap':

$$\sum_i a_i - 2b = 2\max\left\{\sum_i a_i x_i \colon x \in \mathrm{FRAC}(G)\right\} - 2\max\left\{\sum_i a_i x_i \colon x \in \mathrm{STAB}(G)\right\}.$$

Certain extensions of Theorems 3.5 and 3.6 were given by Sewell. It would be interesting to find extensions of Theorem 3.8 to facets.

We introduce one further class of inequalities valid for STAB(G). An *orthonormal representation* of a graph G is an assignment of a vector $v_i \in \mathbb{R}^k$ to each node i (for some $k \geqslant 1$) so that $|v_i| = 1$ and $v_i^{\mathrm{T}} v_j = 0$ for every pair i, j of nonadjacent nodes. Consider any vector c with $|c| = 1$, and the inequality

$$\sum_i (c^{\mathrm{T}} v_i)^2 x_i \leqslant 1.$$

If x is the incidence vector of a stable set A then the left-hand side is $sum_{i \in A}(c^{\mathrm{T}} v_i)^2$, and the inequality holds by Parseval's formula since the vectors v_i, $i \in A$, are mutually orthogonal. So this inequality holds true for every vector $x \in \mathrm{STAB}(G)$. These inequalities are called *orthogonality constraints*. The set of vectors satisfying all orthogonality constraints is denoted by TH(G).

The definition of TH(G) is rather complicated. In particular, there are infinitely many orthogonality constraints for a fixed graph and, accordingly, TH(G) is in general not a polytope. But TH(G) has some surprisingly nice properties (see [9, 10]). We mention a few. If $\bar{G}$ denotes the complement of the graph G, then TH$(\bar{G})$ is the antiblocker of TH(G). TH(G) is polyhedral iff G is perfect, and in this case it is equal to the stable set polytope of G. Perhaps most important is the following fact: every linear objective function can be optimized over TH(G) in polynomial time.

5. Projection representations

If we project a polytope to a subspace then the number of vertices can, of course, not increase, but we have little control over the number of facets, which may increase substantially. This seemingly negative fact can be turned around: a polytope having an inconveniently large number of facets may be represented as the projection of a polytope with a much smaller number of facets. This technique in polyhedral optimization is relatively new and promising [3, 4, 14]; at the same time, many of the basic questions are unsettled.

As an example, consider the stable set polytope of a comparability graph G, obtained by taking a partially ordered set $(V, \leqslant)$ and connecting two elements of V iff they are comparable in this partial order. This graph is perfect and so its stable set polytope is described by the nonnegativity and clique constraints:

$$x_i \geqslant 0 \quad \text{for every } i \in V,$$

$$\sum_{i \in B} x_i \leqslant 1 \quad \text{for every chain } B \subseteq V.$$

This latter family of constraints is typically exponentially large. However, STAB(G) can be represented as the projection of a polytope with $\mathrm{O}(|V|^2)$ facets.

Lemma 5.1. *Let $(V, \leqslant)$ be a poset. Assign two variables x_i, y_i to each $i \in V$ and consider the polytope P defined by the inequalities*

$$0 \leqslant x_i \leqslant y_i \leqslant 1 \quad \textit{for all } i \in V,$$

$$y_i + x_j \leqslant y_j \quad \textit{for all pairs } i < j.$$

Then projecting P on the x-coordinates, we obtain STAB(G).

It is a very interesting question whether STAB(G) in general can be represented as the projection of a polytope with a polynomial number of facets. The answer is in the affirmative for comparability graphs and their complements, chordal graphs and their complements, and several other classes of perfect graphs, but is not known for all perfect graphs. The answer is positive for t-perfect graphs (cf. the next section). One suspects that the answer will be negative for a general graph. However, the only negative result to date is that of Yannakakis (1988), who proves that if we also require that the automorphism group of G 'lifts up' to isometries of the polytope then the stable set polytope of the line-graph of K_n (i.e. the matching polytope of K_n) cannot be represented as the projection of a polytope with a polynomial number of facets.

Yannakakis also formulated the following combinatorial necessary condition. Let $\{U_1, W_1\}, \{U_2, W_2\}, \ldots, \{U_N, W_N\}$ be partitions of $V(G)$ into two classes. We say that this familiy *separates cliques and stable sets* if for every stable set A and clique B such that $A \cap B = \emptyset$ there exists an i with $A \subseteq U_i$ and $B \subseteq W_i$.

Lemma 5.2. *If* STAB(G) *can be represented as the projection of a polytope with N facets then there exist N partitions of $V(G)$ separating cliques from stable sets.*

It is not known whether in every graph (or even in every perfect graph) cliques can be separated from stable sets by a polynomial number of partitions. It follows from the theory of communication complexity that $\mathrm{O}(n^{\log n})$ partitions suffice in every graph. This was improved by A. Hajnal (unpublished) to $\mathrm{O}(n^{(1/2)\log n})$.

6. Quadratic inequalities

The polyhedral theory of stable sets can be viewed as a theory of linear inequalities valid for the incidence vectors of stable sets. In view of the results discussed in

Sections 1 and 3, it is natural to ask for quadratic inequalities (and, of course, higher degree inequalities) valid for the incidence vectors of stable sets.

At first sight it seems that we are getting too much too easily. Let $G=(V,E)$ be a graph and consider the following system of equations:

$$x_i^2=x_i \quad \text{for every node } i\in V, \tag{6.1}$$

$$x_ix_j=0 \quad \text{for every edge } ij\in E. \tag{6.2}$$

Trivially, the solutions of (6.1) are precisely the 0–1 vectors, and so the solutions of (6.1) and (6.2) are precisely the incidence vectors of stable sets. Unfortunately, little is known about the solutions of systems of quadratic equations. In fact, what this shows is that even the solvability of such a simple system of quadratic equations (together with a linear equation $\sum_i x_i=\alpha$) is NP-hard.

However, we can use this system to derive some other constraints. Equation (6.1) implies that for every node i,

$$x_i=x_i^2\geqslant 0, \quad 1-x_i=(1-x_i)^2\geqslant 0, \tag{6.3}$$

and using this (6.2) implies that for every edge ij,

$$1-x_i-x_j=1-x_i-x_j+x_ix_j=(1-x_i)(1-x_j)\geqslant 0. \tag{6.4}$$

So we can derive the edge constraints from (6.1) and (6.2) formally. We can go on and use these to derive the odd hole constraints. Consider e.g. a pentagon $(1,2,3,4,5)$. Then we have

$$\begin{aligned}1-x_1-x_2-x_3+x_1x_3&=1-x_1-x_2-x_3+x_1x_2+x_1x_3\\&=(1-x_1)(1-x_2-x_3)\geqslant 0,\end{aligned}$$

and similarly

$$1-x_1-x_4-x_5+x_1x_4\geqslant 0.$$

Furthermore,

$$x_1-x_1x_3-x_1x_4=x_1(1-x_3x_4)\geqslant 0.$$

Summing these inequalities, we get the odd hole constraint

$$2-x_1-x_2-x_3-x_4-x_5\geqslant 0. \tag{6.5}$$

We can also derive the clique constraints. Assume that nodes $1,2,3,4,5$ induce a complete 5-graph. We start with the trivial inequality

$$(1-x_1-x_2-x_3-x_4-x_5)^2\geqslant 0.$$

Expanding, we get

$$1+\sum_{i=1}^{5} x_i^2-2\sum_{i=1}^{5} x_i+2\sum_{i\neq j} x_ix_j\geqslant 0.$$

Here the first sum is just $\sum_i x_i$ by (6.1) and the third sum is 0 by (6.2), so we get

$$1-x_1-x_2-x_3-x_4-x_5 \geqslant 0. \tag{6.6}$$

We can in fact derive any orthogonality constraint. Let $\{v_i: i\in V\}$ be any orthonormal representation of G and let c be a unit vector in the same space. Then we have the trivial inequality

$$\left(c-\sum_i (c^{\mathrm{T}} v_i)\, x_i v_i\right)^2 \geqslant 0.$$

Expanding, we get

$$c^2+\sum_i (c^{\mathrm{T}} v_i)^2 x_i^2 v_i^2 - 2\sum_i (c^{\mathrm{T}} v_i)^2 x_i + 2\sum_{i\neq j} (c^{\mathrm{T}} v_i)(c^{\mathrm{T}} v_j) x_i x_j v_i^{\mathrm{T}} v_j \geqslant 0.$$

Here $x_i^2 v_i^2 = x_i$ by (6.1) and by $|v_i|=1$. Moreover, $x_i x_j v_i^{\mathrm{T}} v_j = 0$; this follows from (6.2) if i and j are adjacent, and from the definition of orthonormal representations if i and j are nonadjacent. Using also that $|c|=1$ we get

$$1-\sum_i (c^{\mathrm{T}} v_i)^2 x_i \geqslant 0, \tag{6.7}$$

which is just an orthogonality constraint.

We can formalize these procedures: if we have a family of linear inequalities valid for STAB(G), then by multiplying pairs of them we obtain quadratic inequalities. Other sources of quadratic inequalities are (6.1) and (6.2) and the fact that the square of a linear form is nonnegative. By taking nonnegative linear combinations of such quadratic inequalities, we may be able to get rid of all quadratic terms and obtain a linear inequality. The above examples show that quite complicated linear inequalities can be derived in this way.

This way of deriving inequalities was introduced and studied by Lovász and Schrijver [19, 20] and we do not repeat the details here. But we have to address the question: what is the point in deriving inequalities algebraically, which can be proved true anyway by trivial combinatorial considerations?

Let F be a family of linear inequalities valid for STAB(G); we shall assume that F contains the family F_0 of inequalities (6.3). As a minimal set of operations, allow the following: multiply each member of F by x_i and by $1-x_i$ to get a family of quadratic inequalities; take nonnegative linear combinations of these inequalities; use (6.1) and (6.2) to get rid of certain quadratic terms. Let $T(F)$ be the family of linear inequalities obtained in this way. Let $T_+(F)$ be the set of linear inequalities obtained if also squares of linear functions can be taken at the start. Clearly $F\subseteq T(F)\subseteq T_+(F)$. We denote by $T^k(F)$ the family of inequalities obtained by repeating the T operator k times.

The following facts are the key to the algorithmic applications of these methods.

Theorem 6.1. *Every linear inequality valid for* STAB(G) *occurs in* $T^k(F_0)$ *for some* $k \leqslant n$.

Let K be the solution set of F; we denote by $N(K)$ $[N_+(K)]$ the solution set $T(F)$ $[T_+(K)]$ (it is easy to see that this does indeed depend on K only). Let K^0 denote the convex hull of 0–1 solutions of F. Then it follows from these considerations that

$$K^0 \subseteq N_+(K) \subseteq N(K) \subseteq K,$$

and that iterating the N operator n times we get down to K^0.

We define the N-index of a linear inequality valid for STAB(G) as the least k for which it belongs to T^k ((6.3) and (6.4)). The N_+-index is defined analogously. It follows that every inequality has an N-index at most n, but in fact the N-index of an inequality is usually much smaller.

Theorem 6.2. *The N-index of an inequality is at most its Gallai class number. The N-index of an inequality is* 1 *iff it is a nonnegative combination of odd hole constraints. The N_+-index of an odd antihole constraint is* 2. *The N-index of a clique constraint involving k nodes is $k-2$.*

Corollary 6.3. *A graph is t-perfect if and only if $T(F)$ defines its stable set polytope, where F consists of the edge constraints.*

The T_+ procedure generates inequalities even faster.

Theorem 6.4. *Clique, orthogonality, odd hole, and odd antihole constraints have N_+-index* 1.

It follows that if G is perfect then STAB(G)$=N_+$(FRAC(G)); if G is t-perfect then STAB(G)$=N$(FRAC(G)).

It is also worth noting that $N(K)$ is the projection of a convex set in the $\binom{n}{2}+n$ dimensional space, obtained as the solution set of the following inequalities: take all quadratic inequalities generated from F by multiplying by x_i or $1-x_i$, and replace the products $x_i x_j$ by a new variable y_{ij}. So if $|F|=m$ then $N(K)$ is the projection of a polytope defined by at most $2\,mn$ inequalities. Corollary 6.5 follows.

Corollary 6.5. *If G is t-perfect then* STAB(G) *can be obtained as the projection of a polytope with* O(mn) *facets.*

It is also interesting to remark that the convex (but generally nonpolyhedral) set TH(G) is just the solution set of all linear inequalities derived from (6.1), (6.2) and all inequalities $l^2 \geqslant 0$, where l is a linear form.

An important feature of the N- and N_+-operators is that they preserve algorithmic 'niceness'. Stating this somewhat imprecisely, if every linear objective function can be optimized over K in polynomial time, then the same is true for $N(K)$ and $N_+(K)$ (with a different polynomial in the time bound). In particular, the following holds.

Theorem 6.6. *Let $c>0$ be any constant. Then for the class of graphs G such that* STAB(G) *can be described by inequalities with N_+-index at most c, the value $\alpha(G)$ is polynomial-time-computable.*

The theorem applies in particular to perfect, t-perfect and h-perfect graphs. It also applies to graphs whose stable set polytope can be defined by constraints with bounded Gallai class number. We remark that the class of graphs whose stable set polytope can be defined by rank constraints with bounded Gallai class number was studied in detail by Sewell [22].

Let $G=(V,E)$ be a graph and let $I(G)$ denote the polynomial ideal generated by the polynomials $x_i^2-x_i$ $(i\in V)$ and by the polynomials x_ix_j $(ij\in E)$. Consider the quotient ring $R=\mathbb{R}[x_1,\dots,x_n]/I(G)$ (this is related to the so-called *Reisner–Stanley ring* of the graph; cf. [24]). We can introduce an order of polynomials by writing $f\geqslant 0 \pmod{I(G)}$ iff $f(x)\geqslant 0$ for every root of the ideal $I(G)$; clearly, this induces an order on the quotient ring. It is not difficult to prove the following lemma.

Lemma 6.7. *For any polynomial f, we have $f\geqslant 0 \pmod{I(G)}$ iff there exist polynomials $g_1,\dots,g_N$ such that $f\equiv g_1^2+\cdots+g_N^2 \pmod{I(G)}$.*

In fact, this holds for every ideal with a finite number of zeros. But the following theorem shows an interesting connection between this property and the perfectness of a graph.

Theorem 6.8. *A graph G is perfect if and only if the following holds:*

$(*)$ For any linear polynomial f, we have $f\geqslant 0 \pmod{I(G)}$ iff there exist linear polynomials $g_1,\dots,g_N$ such that $f\equiv g_1^2+\cdots+g_N^2 \pmod{I(G)}$.

The proof of this theorem follows from the characterization of perfectness in terms of the body TH(G). One could formulate analogous characterizations of t-perfect graphs using Corollary 6.10. It would be interesting to know which polynomial ideals in general have property $(*)$.

The method sketched here is not restricted to the stable set problem; in fact, it can be applied to any 0–1 optimization problem. Moreover, it can be extended from quadratic to higher-order inequalities. For these extensions, see [19, 20, 23].

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