View metadata, citation and similar papers at core.ac.uk

brought to you by CORE

provided by Elsevier - Publisher Connector

JOURNAL OF COMBINATORIAL THEORY, Series B **50**, 245–253 (1990)

On Circuit Decomposition of Planar Eulerian Graphs*

HERBERT FLEISCHNER

Institut für Informationsverarbeitung, Domus Universitetis, Wien, Österreichische, Akademie der Wissenschaften

AND

ANDRÁS FRANK

Department of Computer Science, Eötvös Loránd University, Budapest

Communicated by the Managing Editors

July 27, 1988

We give a common generalization of P. Seymour's "Integer sum of circuits" theorem and the first author's theorem on decomposition of planar Eulerian graphs into circuits without forbidden transitions. © 1990 Academic Press, Inc.

1. INTRODUCTION

It is well known that a non-negative integer-valued circulation can always be expressed as a non-negative integer combination of (incidence vectors of) directed circuits. Thus Hoffman's circulation theorem (See, e.g., [2]) can be interpreted as one giving a necessary and sufficient condition for the existence of a list of directed circuits of a digraph so that the number of circuits from the list containing any edge is between two integer bounds given in advance.

P. Seymour [3] proved the undirected counterpart of Hoffman's result.

THEOREM 1.1. *Let $G=(V,E)$ be an undirected graph endowed with two functions $f, g: E \to R_+$ for which $f \leqslant g$. There are non-negative variables $x(C)$ assigned to the circuits C of G for which $f(e) \leqslant \Sigma(x(C)$: C a circuit and*

* Work supported by Sonderforschungsbereich 303 (DFG), Institut für Operations Research, Universität Bonn.

582b/50/2-8

0095-8956/90 $3.00

Copyright © 1990 by Academic Press, Inc.
All rights of reproduction in any form reserved.

$e \in C) \leqslant g(e)$ holds for every edge e if and only if $f(e) \leqslant g(B-e)$ for every cut B and edge $e \in B$.

(Where S is a finite set, $X \subseteq S$ and $h: S \to R$ is a function, we use the notation $h(X) := \Sigma_{x \in X} h(x)$.)

An important difference between the directed and undirected case is that the special case $f \equiv g$ is trivial for directed graphs while this is the crucial part in Seymour's proof of the undirected case. This is why we formulate here this special case and why we are concerned with it henceforth.

THEOREM 1.2 (Sum-of-Circuits Theorem [3]). *Let $G=(V, E)$ be an undirected graph endowed with a function $f: E \to R_+$. There are non-negative variables $x(C)$ assigned to the circuits C of G for which $f(e) = \Sigma(x(C): C$ a circuit and $e \in C)$ holds for every edge e if and only if*

$$2f(e) \leqslant f(B) \tag{1.1}$$

for every cut B and edge $e \in B$.

Another essential difference between the directed and undirected case is that in the directed case, if f and g are integer-valued, then x can be chosen integer-valued. This is not so in the undirected case as is shown by K_4 (complete graph on 4 nodes) with $f \equiv 1$.

In order to have hope to get an integer packing of circuits it is obviously necessary for f that $\Sigma(f(e): e$ incident to $v)$ is even for every $v \in V$. Such an f is called *Eulerian.*

Unfortunately (1.1) is not sufficient even if f is Eulerian: let G be the Petersen graph and let f be 2 on the edges of a specified perfect matching of G and 1 otherwise. However, for planar graphs the situation is much better:

THEOREM 1.3A (Integer Sum-of-Circuits Theorem [3]). *Let $G=(V, E)$ be a planar graph and $f: E \to Z_+$ Eulerian. There are non-negative integer variables $x(C)$ assigned to the circuits C of G for which $f(e) = \Sigma(x(C): C$ a circuit and $e \in C)$ holds for every edge e if and only if* (1.1) *holds for every cut B and edge $e \in B$.*

Let us formulate this theorem in an equivalent form.

THEOREM 1.3. *The edge set of a planar Eulerian graph can be partitioned into circuits of length at least three if and only if there is no cut in which more than half of the edges are parallel edges connecting the same pair of nodes.*

This result is one of the starting points of our investigations. The other one is due to H. Fleischner [1]. Let $G=(V, E)$ be a planar Eulerian graph. For every node v we are given disjoint pairs of edges incident to v. Such a pair is called a *forbidden transition*. Let us call a circuit of G a *good circuit* if it includes no forbidden transition.

THEOREM 1.4 (H. Fleischner [1]). *The edge set of a planar Eulerian graph G can be partitioned into good circuits if and only if no forbidden transition forms a (2-element) cut.*

The main purpose of this paper is to show a common generalization of Theorems 1.3 and 1.4. Before doing that let us mention one more theorem belonging to this topic.

THEOREM 1.5 (P. Seymour [4]). *The edge set of a planar Eulerian graph G can be partitioned into circuits of even length if and only if every block of G contains an even number of edges.*

Remark. The proof of each of Theorems 1.3, 1.4, 1.5 consists of two parts. In the first part the problem is reduced to the special case when every degree is at most 4. In Theorem 1.4 Fleischner proves this special case by carrying out a complicated case analysis. His proof does not rely on any other results. In Theorem 1.3 Seymour invokes the Four-color theorem. This however can be avoided because in this special case Theorem 1.4 includes Theorem 1.3.

As far as Theorem 1.5 is concerned Seymour settles the "degree $\leqslant 4$" case by invoking Theorem 1.4 along with a rather complicated argument. Let us show here that this latter can also be avoided and the "degree $\leqslant 4$" case of Theorem 1.5 follows from Theorem 1.4 by an easy trick: let G be a 2-connected planar graph with an even number of edges such that the degree of each node is either 2 or 4. It is possible to color the edges with red and blue so that for each node the number of red and blue edges incident to that node is equal. Indeed, color alternatingly the edges red and blue along an Eulerian circuit of G. Since the number of edges is even this coloration will do. Define the forbidden transitions at every vertex of degree 4 to be the red–red and the blue–blue pairs. By Theorem 1.4 there is a decomposition of the edge set into good circuits. In this case a good circuit is an alternating red–blue circuit. Consequently, this decomposition consists of circuits of even length.

Notation. Let $G=(V, E)$ be an undirected graph. The degree of a node $v \in V$ is denoted by $d(v)$. For a set $X \subseteq V$ of nodes the set of edges with exacty one end in X is denoted by $\nabla(X)$ and is called a *cut*. X and $V-X$ are called the two *sides* of the cut. If $|X|=1$, $\nabla(X)$ is called a *star*.

A minimal cut is called a *bond.* (It is well known that in a connected graph a cut $\nabla(X)$ is a bond if and only if both X and $V-X$ induce a connected subgraph.) The set of edges induced by X is denoted by $E(X)$. In particular, $E(\{u, v\})$ is the set of parallel edges connecting u and v.

2. Common Generalization

Let $G=(V, E)$ be an Eulerian graph. At every node $v \in V$ a partition $\mathscr{P}(v)$ of the edges incident to v is specified. A member of $\mathscr{P}(v)$ is called a *forbidden part* and a subset of a forbidden part with at least two elements is called a *forbidden set.* Let $\mathscr{P} := \bigcup (\mathscr{P}(v) : v \in V)$ denote the set of forbidden parts.

A circuit of G is called *good* if it includes no forbidden sets. Let us call a cut S *critical at* $P \in \mathscr{P}$ (with respect to $\mathscr{P}$) if it contains precisely $|S|/2$ elements from P. If S contains more than $|S|/2$ elements from P, then S is called *bad* (with respect to $\mathscr{P}$). (Note that one-element forbidden parts do not play any role.) The main result of the paper is as follows.

Theorem 2.1. *The edge set of a planar Eulerian graph can be partitioned into good circuits if and only if there are no bad cuts.*

Remarks. This theorem immediately implies Theorem 1.4 when each forbidden part has at most two elements. Theorem 1.3 follows if the forbidden parts are the sets of parallel edges.

In the theorem every forbidden part consists of edges incident to a node. What if we drop this property and the forbidden parts are arbitrary? The cut condition (namely, that no cut includes a forbidden part bigger than its half) is necessary to have a partition into good circuits. If we have just one forbidden part then the cut condition is sufficient: this is a theorem of P. Seymour [5] on planar multicommodity flows. However, it can be shown that the cut condition is not sufficient, in general, if there are two forbidden parts.

Proof of the Theorem. The necessity of the condition is obvious. The proof of sufficiency goes along a similar line as Seymour's proof of Theorem 1.3. The main difference occurs in Claim 6 which is trivial in Seymour's proof and rather complicated here.

First observe that if there is a bad cut, there is a bad bond. Indeed, any cut C is a partition of bonds and if none of these bonds is bad, then neither is C.

Let $G=(V, E)$ be a counterexample with respect to $\mathscr{P}$ such that $|E| + |V|$ is minimal and the number of one-element members of $\mathscr{P}$ is maximal.

Claim 1. G is 2-connected.

Proof. Every block of an Eulerian graph is an Eulerian graph. Let us consider any block B of G along with the restriction of $\mathscr{P}$ to B. Clearly, there is no bad cut so, if $B \neq G$, B can be partitioned into good circuits. The circuit-partitions of the blocks of G form a circuit-partition of G, a contradiction. ▮

Claim 2. Every critical cut is either a star or contracting one of its two sides results in a non-planar graph.

Proof. Let $S = \nabla(X)$ be a critical cut which is not a star, that is, $1 < |X| < |V| - 1$ and let $T \subseteq S$ be a forbidden set for which $2\,|T| = |S|$. Assume that the node v incident to the elements of T is in X. Suppose furthermore that contracting either X or $V - X$ results in a planar graph. First contract X into one node and delete the resulting loops. The new graph is smaller than G. It is Eulerian, planar and includes no bad cuts so it has a good circuit-partition. The same is true if we contract $V - X$ instead of X.

A circuit in the circuit-partition of the contracted graph G/X corresponds to either a good circuit of G lying entirely in $V - X$ or a good path (i.e., a path not using forbidden parts) of G that lies in $V - X$ apart from its two (possibly not distinct) endpoints so that one of its end-edges is in T while the other one is in $S - T$. Therefore we have a set of $|T|$ good paths each of which has v as an endpoint. An analogous statement holds for the other contracted graph $G/(V - X)$ except that v is the last node of these paths preceding the endnode. By leaving out the last edge of these $|T|$ paths occurring in T we make v the endpoint.

One can easily see that the two sets of paths can be paired together so as to form $|T|$ good circuits of G which, along with the other circuits arising from the circuit-partitions of G/X and $G/(V - X)$, form a good circuit partition of G, a contradiction. (Observe that we have exploited at this point that each forbidden part is incident to a node.) ▮

Let us call a node v *trivial* if $\mathscr{P}(v)$ consists of one-element parts.

Let $e_i \in E(v, u_i)$ $(i = 1, 2)$ be two edges of G so that v, u_1, u_2 are distinct and lie on the same face of G. By *splitting off* e_1 and e_2 we mean the following operation. Replace e_1 and e_2 by a new edge $e = u_1 u_2$ and if $e_i \in P_i \in \mathscr{P}(u_i)$, then in P_i replace e_i by e $(i = 1, 2)$. Denote the new graph by G' and the new set of forbidden parts by $\mathscr{P}'$. Clearly G' is planar and Eulerian.

Claim 3. If v is trivial, there is a bond in G' which is bad with respect to $\mathscr{P}'$. Furthermore, every bad bond in G' is a star, namely, either $\nabla'(v)$, $\nabla'(u_1)$, or $\nabla'(u_2)$.

Proof. First we show that there is a bad cut in G'. For otherwise, since

G' is smaller than G, there is a good circuit-partition of G'. The circuits in this partition not using e are good circuits in G. Replacing e by e_1 and e_2 in the circuit C of the partition containing e we obtain either a good circuit of G (if C does not go through v) or two edge-disjoint circuits. Since by the assumption v is trivial these two circuits are good as well, contradicting the fact that G is a counter-example. We have already seen that if there is a bad cut, there is a bad bond as well.

Suppose now that a bond $S' = \nabla'(X)$ is bad with respect to $\mathscr{P}'$. Obviously $\nabla(X)$ is critical (in G) with respect to $\mathscr{P}$ and $e_i \in \nabla(X)$, $i = 1, 2$.

Since u_1 and u_2 lie on one face of G, $G + e$ is a planar graph. Since S' is a bond of G', both X and $V - X$ induce a connected subgraph of $G + e$. Therefore contracting either X or $V - X$ into one node we obtain a planar graph from G. By Claim 2 $\nabla(X)$ is a star. Clearly the only stars that may become bad while splitting off e_1 and e_2 are the stars of v, u_1, u_2, as required. ∎

(Note that in the proof $V - X$ does not necessarily induce a connected subgraph of G. This is why Claim 2 is stated in the present form.)

Call $\nabla(u_1)$ *dangerous* at $P \in \mathscr{P}(u_2)$ if $\nabla(u_1)$ is critical in G at P and $P \cap E(v, u_2) \neq \varnothing$. Obviously, $\nabla'(u_1)$ is bad at P' if and only if $\nabla(u_1)$ is dangerous at P and $e_2 \in P \cap E(v, u_2)$.

Claim 4. If $\nabla(u_1)$ is dangerous at $P \in \mathscr{P}(u_2)$, then $d(u_1) < d(u_2)$.

Proof. We have $d(u_1) = 2\,|P \cap \nabla(u_1)| = 2\,|P \cap E(u_1, u_2)| < 2\,|P \cap \nabla(u_2)| \leqslant d(u_2)$. ∎

The following claim is obvious.

Claim 5. If, for some node x, $\nabla(x)$ is critical both at $P \in \mathscr{P}(s)$ and $T \in \mathscr{P}(t)$ (where $P \neq T$, $s \neq x \neq t$), then $s \neq t$ and x has s and t as the only neighbours.

Claim 6. There is no trivial node.

Proof. We are going to show that if v is trivial, then splitting off two appropriate edges at v yields no bad star and this will contradict Claim 3.

Case 1. v has exactly two neighbours.

Let u_1 and u_2 be the two neighbours of v and $d(u_1) \geqslant d(u_2)$. By Claim 4 $\nabla(u_1)$ is not dangerous at any member of $\mathscr{P}(u_2)$.

Suppose that $\nabla(u_2)$ is critical at $P \in \mathscr{P}(u_1)$. Then $E(v, u_1) - P \neq \varnothing$, for otherwise, $\nabla\{(u_2, v\})$ would be a bad cut at P. Furthermore, $\nabla(v)$ cannot be critical at P since otherwise $|\nabla(\{u_2, v\})| < d(v) + d(u_2) = 2\,|P \cap E(v, u_1)| + 2\,|P \cap E(u_2, u_1)| = 2\,|P \cap \{u_2, v\}|$; i.e., $\nabla(\{u_2, v\})$ would be bad again.

Therefore, irrespective of $\nabla(u_2)$ being critical or not, the following choice of e_1 and e_2 is possible.

Let us choose $e_i \in E(v, u_i)$ so that $e_i \in P_i$ if v is critical at a certain $P_i \in \mathscr{P}(u_i)$ $(i=1, 2)$ and, moreover, $e_1 \notin P$ if $\nabla(u_2)$ is critical at $P \in \mathscr{P}(u_1)$. Now splitting off e_1 and e_2 cannot make either of the stars of v, u_1, u_2 bad, contradicting Claim 3.

Case 2. v has at least three distinct neighbours.

By Claim 5 (applied to $x=v$) $\nabla(v)$ cannot be critical at more than one forbidden part. If $\nabla(v)$ is critical at a certain $T \in \mathscr{P}(t)$, let us choose $u_2 := t$. If $\nabla(v)$ is not critical, let u_2 be a neighbour of v of minimum degree. Let u_1 and u_1' be two other distinct neighbours of v such that both $\{v, u_1, u_2\}$ are on one face of G and $\{v, u_1', u_2\}$ are on one face of G. Applying Claim 5 to $x=u_2$ we see that $\nabla(u_2)$ cannot be critical both at a member of $\mathscr{P}(u_1)$ and at a member of $\mathscr{P}(u_1')$. By symmetry let us assume that $\nabla(u_2)$ is not critical at any member of $\mathscr{P}(u_1)$.

Assume first that $\nabla(v)$ is not critical. Then the choice of u_2 and Claim 4 show that $\nabla(u_1)$ is not dangerous at any member of $\mathscr{P}(u_2)$. Let us choose $e_i \in E(v, u_i)$ $(i=1, 2)$. Now splitting off e_1 and e_2 cannot make either of the stars of v, u_1, u_2 bad, contradicting Claim 3.

Second, assume that $\nabla(v)$ is critical at $T \in \mathscr{P}(u_2)$. Let $e_1 \in E(v, u_1)$ and $e_2 \in T \cap E(v, u_2)$. By the choice of u_1, e_1, and e_2, splitting off e_1 and e_2 does not make the stars of v and u_2 bad. We claim that the star of u_1 cannot become bad either. Indeed, if $\nabla'(u_1)$ is bad (in G'), then (since $e_2 \in T$) it must be bad at T', so $\nabla(u_1)$ is critical at T (in G). But then $|\nabla(\{u_1, v\})| < d(v) + d(u_1) = 2\,|T \cap E(v, u_2)| + 2\,|T \cap E(u_1, u_2)| = 2\,|T \cap \{u_1, v\}|$; i.e., $\nabla(\{u_1, v\})$ would be bad at T (in G) and the claim follows. ∎

Since G is a counterexample there is a forbidden part P with $|P| \geqslant 2$. Let P have a minimum number of elements and assume that $P \in \mathscr{P}(r)$ $(r \in V)$. Let $e \in P$. Modify $\mathscr{P}(r)$ in such a way that $\mathscr{P}''(r) := \mathscr{P}(r) - \{P\} \cup \{P-e\} \cup \{e\}$, that is we replace $\{P\}$ by $\{P-e\}$ and $\{e\}$. Since G is a counterexample, where the number of one-element forbidden parts is maximal, G is not a counterexample with respect to $\mathscr{P}''$. Therefore there is a partition $\mathscr{C}$ of the edge set of G into circuits which are good with respect to $\mathscr{P}''$.

$\mathscr{C}$ is almost good with respect to $\mathscr{P}$ except that circuit C_0 containing e contains another element f of $P-e$. (C_0 may consist only of e and f.)

Let us construct an auxiliary digraph $D=(V, A)$ as follows. A directed edge uv belongs to A if there is a circuit $C \in \mathscr{C} - \{C_0\}$ going through u and v so that either C does not use P or else if it contains an element $rs \in P$, then r, s, u, v follows each other on C in this order. (Note that C cannot use more than one edge of P since C is good with respect to $\mathscr{P}$.)

Claim 7. In D there is a directed path from r to the set $V(C_0)-r$.

Proof. It there is no such path, let X denote the set of nodes in D reachable from r along a directed path. The only node of C_0 belonging to X is r.

We are going to show that $\nabla(X)$ is bad at P (with respect to $\mathscr{P}$). Indeed, let $g=uv\in\nabla(X)-P$ where $u\in V-X$, $v\in X$ and let C be the circuit in $\mathscr{C}$ going through g. Since no edge of D leaves X, C must contain an edge $h=rs\in P$, where $s\notin X$. Furthermore the segment of C between s and u is entirely in $V-X$ while the segment of C between r and v is in X.

In other words, to every edge $g\in\nabla(X)-P$ an edge h from $P\cap\nabla(X)$ can be uniquely assigned. Furthermore, $e, f\in P$ also belong to $\nabla(X)$, and hence $\nabla(X)$ is bad at P. ∎

Let S be a shortest path in D from r to $V(C_0)-r$. Let $r=v_0, v_1, ..., v_k$ ($v_k\in V(C_0)-r$) be the nodes of S (in this order). Suppose that an edge $v_{i-1}v_i$ of S belongs to D because of circuit $C_i\in\mathscr{C}-\{C_0\}$ ($i=1, 2, ..., k$).

Let G' denote the subgraph formed by the union of circuits C_i ($i=0, 1, ..., k$). Let $\mathscr{P}'$ be the restriction of $\mathscr{P}$ onto G'.

Claim 8. In G' there is no bad cut with respect to $\mathscr{P}'$.

Proof. Since G' is the union of circuits all of which are good except C_0, a cut can be bad only at P' (where P' denotes the restriction of P). Since C_1 cannot contain an element of P, C_0 contains exactly two of them and $C_2, ..., C_k$ each contains at most one element of P, no cut can be bad at P'. ∎

Claim 9. $G'=G$.

Proof. If G' is a proper subgraph of G, then G' is not a counterexample with respect to $\mathscr{P}'$. By Claim 8 the edge set of G' partitions into circuits that are good with respect to $\mathscr{P}'$. This partition along with the circuits in $\mathscr{C}-\bigcup(\{C_i\}: i=0, 1, ..., k)$ would form a good circuit-partition of G (with respect to $\mathscr{P}$). ∎

Let $L_1=\{C\in\mathscr{C}: E(C)\cap P=\varnothing\}$ and $L_2=\{C\in\mathscr{C}: |E(C)\cap P|=1\}$. Because of the minimal choice of S every node of S belongs to at most two members of L_1 while r is in exactly one member of L_1. By Claim 9 $G'=G$ so we have $d(r)=2\,|L_2|+4=2\,|P|$ and $d(v_i)\leqslant 2\,|L_2|+4$ $(=2\,|P|)$ ($i=1, ..., k$). By Claim 6 and by the minimal choice of P, for every i ($i=1, ..., k$) there is a forbidden part $Q_i\in\mathscr{P}(v_i)$ for which $|Q_i|\geqslant|P|$. Since there is no bad cut, $d(v_i)\geqslant 2\,|Q_i|$ and so $d(v_i)\geqslant 2\,|Q_i|\geqslant 2\,|P|$.

Consequently, the inequalities above are equalities and every node v_i ($i=1, 2, ..., k-1$) occurs in exactly two circuits from L_1. Both v_0 and v_k occur in one circuit of L_1 and each is contained in C_0. C_1 belongs to L_1.

Since v_i ($i = 1, ..., k-1$) is contained in two circuits of L_1 the minimality of S shows that the circuit C_{i+1} defining edge $v_i v_{i+1}$ of D cannot belong to L_2. Thus L_2 must be empty.

Therefore we have proved that no node of G is contained in more than two circuits of $\mathscr{C}$. Consequently, the degree of every node is at most four. By Theorem 1.4 such a graph cannot be a counterexample, a contradiction. ∎

Remarks. If the forbidden parts at every node v partitions the edge set incident to v in such a way that each part is continuous in the cyclic order, then the main theorem can be easily reduced to Theorem 1.3A by applying a natural node-splitting technique. In the general case however this technique gives rise to a non-planar graph so Theorem 1.3A does not apply and this is why we needed the above refinement of Seymour's proof of Theorem 3.1A.

Finally let us call attention to a recent paper of B. Alspach, L. Goddyn, and Cun-Quan Zhang in which they prove an integer sum-of-circuit theorem for graphs not containing a subdivision of the Petersen graph.

References

1. H. Fleischner, Eulersche Linien und Kreisuberdeckungen die vorgegebene Durchgange in den Kanten vermeiden, *J. Combin. Theory Ser. B* **29** (1980), 145–167.
2. L. R. Ford and D. R. Fulkerson, "Flows in Networks," Princeton Univ. Press, Princeton, NJ, 1962.
3. P. D. Seymour, Sums of circuits, *in* "Graph Theory and Related Topics" (J. A. Bondy and U. S. R. Murty, Eds.), pp. 341–350, Academic Press, New York 1979.
4. P. D. Seymour, Even circuits in planar graphs, *J. Combin. Theory Ser. B* **31** (1981), 327–338.
5. P. D. Seymour, On odd cuts and plane multicommodity flows, *Proc. London Math. Soc.* **42** (1981), 178–192.
6. B. Alspach, L. Goddyn and Cun-Quan Zhang, Graphs with the circuit cover property, submitted for publication.