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Existence of positive solution for some class of nonlinear fractional differential equations ☆

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Abstract

In this paper, we investigate the multiple and infinitely solvability of positive solutions for nonlinear fractional differential equation $Du(t) = t^{\nu} f(u)$, $0 < t < 1$, where $D = t^{-\beta\delta} D_{\beta}^{\gamma-\delta,\delta}$, $\beta > 0$, $\gamma \geqslant 0$, $0 < \delta < 1$, $\nu > -\beta(\gamma + 1)$. Our main work is to deal with limit case of $f(s)/s$ as $s \to 0$ or $s \to \infty$ and $\Phi(s)/s$, $\Psi(s)/s$ as $s \to 0$ or $s \to \infty$, where $\Phi(s)$, $\Psi(s)$ are functions connected with function f. In J. Math. Appl. 252 (2000) 804–812, we consider the existence of a positive solution for the particular case of Eq. (1.1), i.e., the Riemann–Liouville type ($\beta = 1$, $\gamma = 0$) nonlinear fractional differential equation, using the super-lower solutions method. Here, we devote to the existence of positive solution and multi-positive solutions for Eq. (1.1) by means of the fixed point theorems for the cone.
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1. Introduction

In this paper, we will consider the nonlinear generalized fractional differential equation

$$Du(t) = g(t, u), \quad 0 < t < 1, \tag{1.1}$$

where $D = t^{-\beta\delta} D_{\beta}^{\gamma-\delta,\delta}$, $\beta > 0$, $\gamma \geqslant 0$, $0 < \delta < 1$, is the generalized operator of fractional differentiation, corresponding to a generalized R–L (Riemann–Liouville) fractional integral $R = t^{\beta\delta} I_{\beta}^{\gamma,\delta}$, $\gamma \geqslant 0, 0 < \delta < 1$.

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Many papers and books on fractional calculus, fractional differential equations have appeared recently (see [1–16]). Most of them are devoted to the solvability of the linear fractional equation in terms of a special function (see [9,10]) and to problems of analyticity in the complex domain (see [16]). Moreover, some have been devoted to the solvability and the existence for positive solutions of Riemann–Liouville type nonlinear fractional differential equations (see [11–15]). No contribution exists, as far as we know, concerning the existence of positive solution and multi-positive solutions for the generalized nonlinear fractional differential equation

$$Du(t) = g(t, u), \quad 0 < t < 1.$$

2. Preliminaries

For convenience, we give the definitions of the generalized fractional integral and derivative.

Let α be a arbitrary real number, k a nonnegative integer.

Denote by $C_\alpha^{(k)}$ the linear space of functions

$$C_\alpha^{(k)} = \left\{ f(x) = x^p \tilde{f}(x);\ p > \alpha,\ \tilde{f} \in C[0,\infty) \right\}.$$

Definition 2.1 [3, Definition 1.1.1]. Let $m \geqslant 1$ be an integer, $\beta > 0$, $\gamma_1, \dots, \gamma_m$ and $\delta_1 > 0, \dots, \delta_m > 0$ be arbitrary real numbers. Consider the set $\gamma = (\gamma_1, \dots, \gamma_m)$ as a multiweight and $\delta = (\delta_1, \dots, \delta_m)$ as a positive multiorder of integration. For functions $f \in C_\alpha$, $\alpha \geqslant \max_k [-\beta(\gamma_k + 1)]$, we define the multiple Erdélyi–Kober (multi-E–K) operators in the following way:

$$I_{\beta,m}^{(\gamma_k),(\delta_k)} f(x) = \int_0^1 G_{m,m}^{m,0}\left[\sigma \,\middle|\, \begin{matrix} (\gamma_k + \delta_k)_1^m \\ (\delta_k)_1^m \end{matrix}\right] f(x\sigma^{1/\beta})\, d\sigma,$$

where $G_{m,m}^{m,0}$ is a special case of the Meijer G-functions.

Then, each operators of the form

$$Rf(x) = x^{\beta\delta_0} I_{\beta,m}^{(\gamma_k),(\delta_k)} f(x), \quad \text{with arbitrary } \delta_0 \geqslant 0,$$

is said to be a generalized operators of fractional integration of Riemann–Liouville type, or briefly, a generalized R–L fractional integral.

Remark 2.1. The multiple Erdélyi–Kober operators are well defined in the space C_α with $\alpha \geqslant \max_k [-\beta(\gamma_k + 1)]$ from Lemma 1.2.1 in [3].

Remark 2.2. The generalized fractional integral $I_{\beta,1}^{\gamma,\delta}$ coincides with the well-known Erdélyi–Kober fractional integral, i.e.,

$$I_\beta^{\gamma,\delta} f(x) = \frac{x^{-\beta(\gamma+\delta)}}{\Gamma(\delta)} \int_0^x (x^\beta - \tau^\beta)^{\delta-1} \tau^{\beta\gamma} f(\tau)\, d(\tau^\beta) = I_{\beta,1}^{\gamma,\delta} f(x).$$

Remark 2.3. The Riemann–Liouville fractional integral is a particular case of the generalized R–L (Riemann–Liouville) fractional integral, i.e.,

$$I^\delta f(x) = \frac{1}{\Gamma(\delta)} \int_0^x (x-\tau)^{\delta-1} f(\tau)\, d\tau = x^\delta I_{1,1}^{0,\delta} f(x) = x^\delta I_1^{0,\delta} f(x).$$

Example 2.1. The kernel functions of the generalized fractional integrals are the simple elementary function (see [3])

$$G_{1,1}^{1,0}\left[\sigma \,\middle|\, \begin{matrix} \gamma+\delta \\ \delta \end{matrix}\right] = \begin{cases} \frac{(1-\sigma)^{\delta-1}\sigma^\gamma}{\Gamma(\delta)}, & 0<\sigma<1, \\ 0, & \sigma>1. \end{cases}$$

Example 2.2.

$$I_\beta^{\gamma,\delta} x^p = \frac{\Gamma(\gamma+p/\beta+1)}{\Gamma(\gamma+\delta+p/\beta+1)} x^p, \quad p > -\beta(\gamma+1).$$

Definition 2.2 [3, Definition 1.5.3]. Let $\gamma_k, \delta_k \geqslant 0$, $k=1,\dots,m$, be arbitrary real numbers and

$$\eta_k = \begin{cases} [\delta_k]+1, & \text{for noninteger } \delta_k, \\ \delta_k, & \text{for integer } \delta_k,\ k=1,\dots,m. \end{cases}$$

The differential operator

$$D_{\beta,m}^{(\gamma_k),(\delta_k)} \stackrel{\text{def}}{=} D_\eta I_{\beta,m}^{(\gamma_k+\delta_k),(\eta_k-\delta_k)} = \left[\prod_{k=1}^m \prod_{j=1}^{\eta_k} \left(\frac{1}{\beta} x \frac{d}{dx} + \gamma_k + j\right)\right] I_{\beta,m}^{(\gamma_k+\delta_k),(\eta_k-\delta_k)},$$

defined for functions of $C_\alpha^{(\eta_1+\cdots+\eta_m)}$, is said to be a generalized Erdélyi–Kober fractional derivative.

More generally, the generalized operators of fractional differentiation, corresponding to the generalized fractional integrals $R = x^{\beta\delta_0} I_{\beta,m}^{(\gamma_k),(\delta_k)}$, $\delta_0 \geqslant 0$ are defined as

$$Df(x) = x^{-\beta\delta_0} D_{\beta,m}^{(\gamma_k-\delta_0),(\delta_k)} f(x) = D_{\beta,m}^{(\gamma_k),(\delta_k)} x^{-\beta\delta_0} f(x), \quad \delta_0 \geqslant 0.$$

We introduce the following definition which we will use in this paper.

Definition 2.3 [3, Definition 1.5.4]. By the Erdélyi–Kober fractional derivative, we mean the differential operator

$$D_{\beta,1}^{\gamma,\delta} f(x) = D_\eta I_{\beta,1}^{\gamma+\delta,\eta-\delta} f(x) = \left[\prod_{j=1}^{\eta} \left(\frac{1}{\beta} x \frac{d}{dx} + \gamma + j\right)\right] I_{\beta,1}^{\gamma+\delta,\eta-\delta} f(x),$$

defined in spaces $C_\alpha^{(\eta)}$ with $\alpha \geqslant -\beta(\gamma+1)$ and

$$\eta = \begin{cases} [\delta]+1, & \text{for noninteger } \delta, \\ \delta, & \text{for integer } \delta. \end{cases}$$

More generally, the generalized operators of fractional differentiation, corresponding to the generalized fractional integrals $R = x^{\beta\delta} I_{\beta,1}^{\gamma,\delta}$, $\delta \geqslant 0$, are defined as

$$\begin{aligned} Df(x) &= x^{-\beta\delta} D_{\beta,1}^{(\gamma-\delta),(\delta)} f(x) = x^{-\beta\delta} D_{\beta,1}^{(\gamma-\delta),(\delta)} f(x) \\ &= D_{\beta,1}^{\gamma,\delta} x^{-\beta\delta} f(x), \quad \delta \geqslant 0. \end{aligned}$$

Remark 2.4. The Erdélyi–Kober fractional derivative is also written as

$$D_\beta^{\gamma,\delta} = D_{\beta,1}^{\gamma,\delta}.$$

Remark 2.5. The Riemann–Liouville fractional derivative of order $0 < \delta < 1$ is a particular case of the generalized operator of fractional differentiation, i.e.,

$$D^\delta f(x) = \frac{1}{\Gamma(1-\delta)} \frac{d}{dx} \int_0^x (x-\tau)^{-\delta} f(\tau)\, d\tau = x^{-\delta} D_1^{-\delta,\delta} f(x).$$

Example 2.3.

$$D_\beta^{\gamma,\delta} x^p = \frac{\Gamma(\gamma+\delta+p/\beta+1)}{\Gamma(\gamma+p/\beta+1)} x^p, \quad p > -\beta(\gamma+1),$$

particularly, $D_\beta^{\gamma,\delta} x^{-\beta(\gamma+1)} = 0$.

We have

$$D_\beta^{\gamma,\delta} I_\beta^{\gamma,\delta} f \equiv f,$$

for every $f \in C_\alpha$, $\alpha \geqslant -\beta(\gamma+1)$ from the above Definition 2.3, Remark 2.2, and Theorem 1.5.5 in [3].

We have the following lemma from Definition 2.3, Remark 2.1, and Example 2.3.

Lemma 2.1. *Let $0 < \delta < 1$. If we assume $u \in C_\alpha$, $\alpha \geqslant -\beta(\gamma+1)$, then the fractional differential equation*

$$D_\beta^{\gamma,\delta} u = 0$$

has $u = cx^{-\beta(\gamma+1)}$, $c \in R^1$, as unique solutions.

From this lemma and Remark 2.1, we deduce the following law of composition.

Proposition 2.1. *Assume that f is in C_α, $\alpha \geqslant -\beta(\gamma+1)$, with a fractional derivative $D_\beta^{\gamma,\delta}$, $f \in C_\alpha$, $\alpha \geqslant -\beta(\gamma+1)$. Then*

$$I_\beta^{\gamma,\delta} D_\beta^{\gamma,\delta} f(x) = f(x) + cx^{-\beta(\gamma+1)},$$

for some $c \in R^1$. When then function f is in $C[0,\infty)$, then $c = 0$.

The following lemma plays major role in our analysis.

Lemma 2.2 (see [17]). *Let X be a Banach space, and let $P \subset X$ be a cone in X. Assume Ω_1, Ω_2 are open subsets of X with $0 \in \Omega_1 \subset \overline{\Omega}_1 \subset \Omega_2$, and let $S : P \to P$ be a completely continuous operator such that either*

(1) $\|Sw\| \leqslant \|w\|$, $w \in P \cap \partial\Omega_1$, *and* $\|Sw\| \geqslant \|w\|$, $w \in P \cap \partial\Omega_2$, *or*
(2) $\|Sw\| \geqslant \|w\|$, $w \in P \cap \partial\Omega_1$, *and* $\|Sw\| \leqslant \|w\|$, $w \in P \cap \partial\Omega_2$.

Then S has a fixed point in $P \cap (\overline{\Omega}_2 \setminus \Omega_1)$.

3. Main results

Let $X = C[0,1]$ be the Banach space endowed with the sup norm.
We will introduce a concept of μ-positive homogeneous function ($\mu \neq 0$).
A real function is called μ-positive homogeneous function if

$$f(\lambda s) = \lambda^{\mu} f(s), \quad \text{for all } \lambda > 0,\ s \in R.$$

Define the cone

$$K = \left\{ u \in X;\ u(\lambda t) = \lambda u(t),\ 0 < \lambda \leqslant 1,\ \min_{1/2 \leqslant t \leqslant 1} u(t) \geqslant \frac{1}{4}\|u\| \right\}.$$

The positive solution which we consider in this paper is such that $u(0) = 0$, $u(t) > 0$, $0 < t \leqslant 1$, $u \in X$.

Through this paper, we assume that

(1) $g(t,u) = t^{\nu} f(u)$, where $\nu > -\beta(\gamma+1)$, $f : [0,+\infty) \to [0,+\infty)$ is a continuous function;
(2) f is μ ($\mu \neq 0$) positive homogeneous function, i.e.,

$$f(\lambda u) = \lambda^{\mu} f(u), \quad \text{for all } \lambda > 0,\ u \geqslant 0,$$

hold, where μ satisfy $\mu + \nu + \beta\delta = 1$.

According to Proposition 2.1, Eq. (1.1) is equivalent to the integral equation

$$u(t) = t^{\beta\delta} I_{\beta}^{\gamma,\delta} g\big(t, u(t)\big) = t^{\beta\delta} \frac{t^{-\beta(\gamma+\delta)}}{\Gamma(\delta)} \int_0^t (t^{\beta} - \tau^{\beta})^{\delta-1} \tau^{\beta\gamma} g\big(\tau, u(\tau)\big)\, d(\tau^{\beta})$$

$$= t^{\beta\delta} \frac{t^{-\beta(\gamma+\delta)}}{\Gamma(\delta)} \int_0^t (t^{\beta} - \tau^{\beta})^{\delta-1} \tau^{\beta\gamma} \tau^{\nu} f\big(u(\tau)\big)\, d(\tau^{\beta}). \tag{3.1}$$

Let $T : K \to K$ be the operator defined by

$$Tu(t) = t^{\beta\delta} \frac{t^{-\beta(\gamma+\delta)}}{\Gamma(\delta)} \int_0^t (t^{\beta} - \tau^{\beta})^{\delta-1} \tau^{\beta\gamma} \tau^{\nu} f\big(u(\tau)\big)\, d(\tau^{\beta}). \tag{3.2}$$

We have the following lemma.

Lemma 3.1. *Assume the assumptions* (1) *and* (2) *hold. Then the operator* $T : K \to K$ *is completely continuous.*

Proof. We first prove that $T : K \to K$.

For $u \in K$, we have

$$\begin{aligned}
Tu(t) &= t^{\beta\delta}\frac{t^{-\beta(\gamma+\delta)}}{\Gamma(\delta)}\int_0^t (t^\beta-\tau^\beta)^{\delta-1}\tau^{\beta\gamma}\tau^\nu f\bigl(u(\tau)\bigr)\,d(\tau^\beta)\\
&= t^{\beta\delta}\frac{t^{-\beta(\gamma+\delta)}}{\Gamma(\delta)}t^{\beta\gamma+\mu+\beta\delta+\nu}\int_0^1 (1-s^\beta)^{\delta-1}s^{\beta\gamma}s^\nu f\bigl(u(s)\bigr)\,d(s^\beta)\\
&= \frac{t}{\Gamma(\delta)}\int_0^1 (1-s^\beta)^{\delta-1}s^{\beta\gamma}s^\nu f\bigl(u(s)\bigr)\,d(s^\beta).
\end{aligned}$$

Thus, for $0 < \lambda \leqslant 1$, $0 \leqslant t \leqslant 1$,

$$Tu(\lambda t) = \frac{\lambda t}{\Gamma(\delta)}\int_0^1 (1-s^\beta)^{\delta-1}s^{\beta\gamma}s^\nu f\bigl(u(s)\bigr)\,d(s^\beta) = \lambda Tu(t).$$

On the other hand, we have

$$\begin{aligned}
\|Tu\| &= \max_{0\leqslant t\leqslant 1}\left|t^{\beta\delta}\frac{t^{-\beta(\gamma+\delta)}}{\Gamma(\delta)}\int_0^t (t^\beta-\tau^\beta)^{\delta-1}\tau^{\beta\gamma}\tau^\nu f\bigl(u(\tau)\bigr)\,d(\tau^\beta)\right|\\
&= \max_{0\leqslant t\leqslant 1}\left|\frac{t}{\Gamma(\delta)}\int_0^1 (1-s^\beta)^{\delta-1}s^{\beta\gamma}s^\nu f\bigl(u(s)\bigr)\,d(s^\beta)\right|\\
&= \frac{1}{\Gamma(\delta)}\int_0^1 (1-s^\beta)^{\delta-1}s^{\beta\gamma}s^\nu f\bigl(u(s)\bigr)\,d(s^\beta),
\end{aligned}$$

$$\begin{aligned}
\min_{1/2\leqslant t\leqslant 1} Tu(t) &= \min_{1/2\leqslant t\leqslant 1}\frac{t}{\Gamma(\delta)}\int_0^1 (1-s^\beta)^{\delta-1}s^{\beta\gamma}s^\nu f\bigl(u(s)\bigr)\,d(s^\beta)\\
&= \frac{1}{2}\frac{1}{\Gamma(\delta)}\int_0^1 (1-s^\beta)^{\delta-1}s^{\beta\gamma}s^\nu f\bigl(u(s)\bigr)\,d(s^\beta)\\
&\geqslant \frac{1}{4}\frac{1}{\Gamma(\delta)}\int_0^1 (1-s^\beta)^{\delta-1}s^{\beta\gamma}s^\nu f\bigl(u(s)\bigr)\,d(s^\beta)
\end{aligned}$$

$$= \frac{1}{4} \max_{0 \leqslant t \leqslant 1} \left| \frac{t}{\Gamma(\delta)} \int_0^1 (1 - s^\beta)^{\delta-1} s^{\beta\gamma} s^\nu f\big(u(s)\big)\, d(s^\beta) \right|$$

$$= \frac{1}{4} \max_{0 \leqslant t \leqslant 1} \left| t^{\beta\delta} \frac{t^{-\beta(\gamma+\delta)}}{\Gamma(\delta)} \int_0^t (t^\beta - \tau^\beta)^{\delta-1} \tau^{\beta\gamma} \tau^\nu f\big(u(\tau)\big)\, d(\tau^\beta) \right| = \frac{1}{4} \|Tu\|.$$

So $T: K \to K$.

Thus, we can prove this lemma by the Arzela–Ascoli theorem as proving Lemma 2.1 in [15]. □

We let

$$\frac{B}{e} < A \leqslant \frac{\Gamma(1+\gamma+\delta+\nu/\beta)}{\Gamma(1+\gamma+\nu/\beta)} < \frac{4e\Gamma(1+\gamma+\delta+\nu/\beta)}{\Gamma(1+\gamma+\nu/\beta)} \leqslant B,$$

where $e \geqslant 1$ is a suitable constant,

$$\Phi(s) = \max\big\{ f(\tau) \mid 0 \leqslant \tau \leqslant s \big\},$$
$$\Psi(s) = \min\left\{ f(\tau) \;\middle|\; \frac{1}{4}s \leqslant \tau \leqslant s \right\}.$$

Obviously, $\Phi(s)$ and $\Psi(s)$ are also continuous functions by the assumption (1).

We also let

$$f_0 = \lim_{s \to 0} \frac{f(s)}{s}, \qquad f_\infty = \lim_{s \to \infty} \frac{f(s)}{s}.$$

We obtain the following lemma.

Lemma 3.2. *Assume the conditions* (1) *and* (2) *hold. If there exists two positive constants* a, b, $a \neq b$ *such that* $\Phi(a) \leqslant aA$, $\Psi(b) \geqslant bB$, *then Eq.* (1.1) *has one positive solution* $u \in K$, *and* $\min\{a, b\} \leqslant \|u\| \leqslant \max\{a, b\}$.

Proof. We only need to prove the operator T has one fixed point in K by the equivalence between Eq. (1.1) and the integral equation (3.1). Operator $T: K \to K$ is completely continuous by Lemma 3.1. $\forall u \in K$, $\|u\| = a$, thus $0 \leqslant u(t) \leqslant a$, $t \in [0, 1]$, and

$$\max\big\{ f(s) \mid 0 \leqslant s \leqslant a \big\} = \Phi(a) \leqslant aA.$$

Then we have the following equalities by Example 2.2:

$$Tu(t) = t^{\beta\delta} I_\beta^{\gamma,\delta} g\big(t, u(t)\big) = t^{\beta\delta} \frac{t^{-\beta(\gamma+\delta)}}{\Gamma(\delta)} \left(\int_0^t (t^\beta - \tau^\beta)^{\delta-1} \tau^{\beta\gamma} g\big(\tau, u(\tau)\big)\, d(\tau^\beta) \right)$$

$$= t^{\beta\delta} \frac{t^{-\beta(\gamma+\delta)}}{\Gamma(\delta)} \left(\int_0^t (t^\beta - \tau^\beta)^{\delta-1} \tau^{\beta\gamma} \tau^\nu f\big(u(\tau)\big)\, d(\tau^\beta) \right)$$

$$\leqslant t^{\beta\delta}\frac{aAt^{-\beta(\gamma+\delta)}}{\Gamma(\delta)}\left(\int_0^t (t^\beta-\tau^\beta)^{\delta-1}\tau^{\beta\gamma}\tau^\nu\,d(\tau^\beta)\right)$$
$$= aA\frac{\Gamma(\gamma+\nu/\beta+1)}{\Gamma(\gamma+\delta+\nu/\beta+1)}t^{\beta\delta+\nu}\leqslant a=\|u\|.$$

$\forall u\in K$, $\|u\|=b$, thus $(1/4)b\leqslant u(t)\leqslant b$, $t\in[1/2,1]$, and

$$\min\left\{f(\tau)\;\middle|\;\frac{1}{4}b\leqslant\tau\leqslant b\right\}=\Psi(b)\geqslant bB,$$

$$Tu(1)=t^{\beta\delta}I_\beta^{\gamma,\delta}g\bigl(1,u(1)\bigr)=\frac{1}{\Gamma(\delta)}\left(\int_0^1(1-\tau^\beta)^{\delta-1}\tau^{\beta\gamma}\tau^\nu f\bigl(u(\tau)\bigr)\,d(\tau^\beta)\right)$$
$$\geqslant bB\frac{1}{\Gamma(\delta)}\left(\int_0^1(1-\tau^\beta)^{\delta-1}\tau^{\beta\gamma}\tau^\nu\,d(\tau^\beta)\right)$$
$$= bB\frac{\Gamma(\gamma+\nu/\beta+1)}{\Gamma(\gamma+\delta+\nu/\beta+1)}\geqslant b=\|u\|.$$

By Lemma 2.2, operator T has a fixed point $u\in K$ and $\min\{a,b\}\leqslant\|u\|\leqslant\max\{a,b\}$. So Eq. (1.1) has a positive solution. □

Theorem 3.1. *Assume the conditions* (1) *and* (2) *hold, and* $f_0=\infty$, $f_\infty=0$. *Then Eq.* (1.1) *has one positive solution* $u\in K$.

Proof. We only need to prove the operator T has one fixed point in K by the equivalence between Eq. (1.1) and the integral equation (3.1). And operator $T:K\to K$ is completely continuous by Lemma 3.1.

By $f_0=\infty$, for arbitrary $M>0$, there exists one positive constant a such that

$$f(s)\geqslant sM,\quad \text{for all } 0\leqslant s\leqslant a.$$

Similarly, by $f_\infty=0$ for $\forall\varepsilon>0$, there exists one positive constant d such that

$$f(s)\leqslant\varepsilon s,\quad \text{for all } s\geqslant d.$$

Let

$$c=\max_{0\leqslant s\leqslant d}f(s)+1.$$

Taking

$$b>\left\{a,2c,\frac{2\Gamma(\gamma+\nu/\beta+1)\varepsilon}{\Gamma(\gamma+\delta+\nu/\beta+1)}\right\},$$

then by means of Lemma 2.2, operator T has a fixed point $u\in K$ and $a\leqslant\|u\|\leqslant b$. So Eq. (1.1) has a positive solution. □

Lemma 3.3. (1) *If $f_0 < A$, then there exists $d > 0$ such that*

$$\Phi(a) \leqslant aA, \quad \textit{for all } 0 \leqslant d.$$

(2) *If $f_\infty > (1/e)B$, where $e \geqslant 1$ is a suitable constant, then there exists $d_1 > 0$ such that*

$$\Psi(b) \geqslant \frac{1}{4e} bB, \quad \textit{for all } b \geqslant d_1.$$

Proof. (1) By $f_0 < A$, then there exists $d > 0$, such that

$$f(s) \leqslant sA, \quad 0 \leqslant s \leqslant d.$$

So for all $0 \leqslant a \leqslant d$,

$$\Phi(a) = \max_{0 \leqslant s \leqslant a} f(s) \leqslant \max_{0 \leqslant s \leqslant a} sA = aA.$$

(2) By $f_\infty > (1/e)B$, then there exists $d_1 > 0$, such that

$$f(s) \geqslant \frac{1}{e} sB, \quad s \geqslant d_1.$$

So for all $b \geqslant d_1$,

$$\Psi(b) = \min_{(1/4)b \leqslant s \leqslant b} f(s) \geqslant \min_{(1/4)b \leqslant s \leqslant b} \frac{1}{e} sB = \frac{1}{4e} bB. \qquad \square$$

We obtain the following existence result.

Theorem 3.2. *Assume the conditions* (1) *and* (2) *hold. If* $\sup_{s>0}(\Phi(s)/s) < A$ *and* $\inf_{s>0}(\Psi(s)/s) > B$, *then Eq.* (1.1) *has one positive solution u.*

Proof. We only need to prove the operator T has one fixed point in K by the equivalence between Eq. (1.1) and the integral equation (3.1). Operator $T : K \to K$ is completely continuous by Lemma 3.1.

By the assumptions of the theorem, there exist two positive constant $a, b > 0$, such that

$$\Phi(a) \leqslant aA, \qquad \Psi(b) \geqslant bB.$$

Noticing that $0 < A < B$, then $a \neq b$, so by Lemma 3.2, operator T has a fixed point $u \in K$, and $\min\{a, b\} \leqslant \|u\| \leqslant \max\{a, b\}$, which completes the proof. $\square$

Theorem 3.3. *Assume the conditions* (1) *and* (2) *hold, and $f_0 < A$. If there exists $b > 0$ such that $\Psi(b) \geqslant bB$, then Eq.* (1.1) *has one positive solution u.*

Proof. We only need to prove the operator T has one fixed point in K by the equivalence between Eq. (1.1) and the integral equation (3.1). Operator $T : K \to K$ is completely continuous by Lemma 3.1.

By (1) of Lemma 3.3, there exists $d > 0$ such that

$$\Phi(a) \leqslant aA, \quad \text{for all } 0 < a \leqslant d.$$

Thus, operator T has a fixed point $u \in K$ satisfying $\min\{a,b\} \leqslant \|u\| \leqslant \min\{a,b\}$ via Lemma 3.2. □

Theorem 3.4. *Assume the conditions* (1) *and* (2) *hold, and* $f_\infty > (1/e)B$, *where* $e \geqslant 1$ *is a suitable constant. If there exists* $a > 0$ *such that* $\Phi(a) \leqslant aA$, *then Eq.* (1.1) *has one positive solution* u.

Proof. We only need to prove the operator T has one fixed point in K by the equivalence between Eq. (1.1) and the integral equation (3.1). Operator $T: K \to K$ is completely continuous by Lemma 3.1.

By (2) of Lemma 3.3, there exists $d_1 > 0$ such that

$$\Psi(b) \geqslant \frac{1}{4e} bB, \quad \text{for all } b \geqslant d_1.$$

Thus, operator T has a fixed point $u \in K$ satisfying $\min\{a,b\} \leqslant \|u\| \leqslant \max\{a,b\}$ via Lemma 3.2. □

Theorem 3.5. *Assume the conditions* (1) *and* (2) *hold. If* $f_0 < A$, $f_\infty > (1/e)B$, *where* $e \geqslant 1$ *is a suitable constant, then Eq.* (1.1) *has one positive solution* u.

Proof. We only need to prove the operator T has one fixed point in K by the equivalence between Eq. (1.1) and the integral equation (3.1). Operator $T: K \to K$ is completely continuous by Lemma 3.1.

By $f_0 < A$, $f_\infty > (1/e)B$ and Lemma 3.3, there exist two positive number a, b such that

$$\Phi(a) \leqslant aA, \qquad \Psi(b) \geqslant \frac{1}{4e} bB.$$

We may assume that $a < b$. Thus, operator T has a fixed point $u \in K$ satisfying $a \leqslant \|u\| \leqslant b$ via Lemma 3.2. □

The following are existence results of multiple solutions.

Theorem 3.6. *Assume the conditions* (1) *and* (2) *hold, and* $f_\infty > (1/e)B$, *where* $e \geqslant 1$ *is a suitable constant. If there exist*

$$0 < b_{m+1} < a_m < b_m < \cdots < a_2 < b_2 < a_1 < +\infty,$$

such that $\Phi(a_i) \leqslant a_i A$ $(i = 1, 2, \ldots, m)$, $\Psi(b_i) \geqslant b_i B$ $(i = 2, \ldots, m+1)$, *then Eq.* (1.1) *has* $2m$ *positive solutions.*

Theorem 3.7. *Assume the conditions* $(1), (2)$ *and* $f_0 < A$ *hold. If there exist*

$$0 < b_1 < a_2 < b_2 < \cdots < a_m < b_m < a_{m+1} < +\infty,$$

such that $\Phi(a_i) \leqslant a_i A$ $(i = 2, \ldots, m+1)$, $\Psi(b_i) \geqslant b_i B$ $(i = 1, 2, \ldots, m)$, *then Eq.* (1.1) *has* $2m$ *positive solutions.*

Since the proofs of Theorems 3.6 and 3.7 are analogous, we only prove Theorem 3.6.

Proof of Theorem 3.6. We only need to prove the operator T has fixed point in K by the equivalence between Eq. (1.1) and the integral equation (3.1). Operator $T: K \to K$ is completely continuous by Lemma 3.1.

By the continuity of Φ and Ψ, it is easy to know that there exist

$$a'_k > a_k > a''_k > b'_{k+1} > b_{k+1} > b''_{k+1} > a'_{k+1} > a_{k+1} > a''_{k+1},$$

such that

$$\Phi\left(a'_k\right) \leqslant a'_k A, \quad \Phi\left(a''_k\right) \leqslant a''_k A \quad (k = 1, 2, \ldots, m),$$
$$\Psi\left(b'_k\right) \geqslant b'_k B, \quad \Psi\left(b''_k\right) \geqslant b''_k B \quad (k = 2, \ldots, m+1).$$

On the other hand, by $f_\infty > (1/e)B$ and Lemma 3.3, we may take $b''_1 > a'_1$, $b'_1 > a''_1$ such that

$$\Psi\left(b''_1\right) \geqslant b''_1 B, \qquad \Psi\left(b'_1\right) \geqslant b'_1 B.$$

Now, for every $\{a'_k, b''_k\}$, $\{a''_k, b'_k\}$ $(k = 1, 2, \ldots, m)$, there exist $u_k, u_i \in K$ $(k, i = 1, 2, \ldots, m)$ such that

$$Tu_k = u_k, \qquad Tu_i = u_i$$

and

$$a'_k \leqslant \|u_k\| \leqslant b''_k, \qquad a''_i \leqslant \|u_i\| \leqslant b'_i.$$

So we can draw our conclusion. $\square$

We have the following results of infinitely solvability for Eq. (1.1).

Theorem 3.8. *Assume the conditions* (1) *and* (2) *hold. If* $\lim_{s\to 0} \Phi(s)/s < A$ *and* $\lim_{s\to 0} \Psi(s)/s > B$*, then Eq.* (1.1) *has a sequence of positive solutions* $\{u_k\}_{k=1}^\infty$.

Theorem 3.9. *Assume the conditions* (1) *and* (2) *hold. If* $\lim_{s\to \infty} \Phi(s)/s < A$ *and* $\lim_{s\to \infty} \Psi(s)/s > B$*, then Eq.* (1.1) *has a sequence of positive solutions* $\{u_k\}_{k=1}^\infty$.

Since the proofs of Theorems 3.8 and 3.9 are analogous, we only prove Theorem 3.9.

Proof of Theorem 3.9. We only need to prove the operator T has fixed point in K by the equivalence between Eq. (1.1) and the integral equation (3.1). Operator $T: K \to K$ is completely continuous by Lemma 3.1.

By the assumptions of the theorem, there exist sequence of positive constants $a_k > 0$, $b_k > 0$ $(k = 1, 2, \ldots)$ such that

$$\Phi(a_k) \leqslant a_k A, \qquad \Psi(b_k) \geqslant b_k B.$$

Without loss of the generality, we may assume that

$$a_1 > b_1 > a_2 > b_2 > \cdots > a_k > b_k > \cdots.$$

For every $\{b_k, a_k\}$, $k = 1, 2, \ldots$, operator T has fixed point $u_k \in K$, and $b_k \leqslant \|u_k\| \leqslant a_k$ by Lemma 3.2.

So we can draw our conclusion. □

Example 3.1. Consider the following fractional differential equation:

$$Du(t) = \frac{\Gamma(\gamma + 1/(2\beta) + 1)}{4\Gamma(\gamma - \delta + 1/(2\beta) + 1)} t^{1/2-\beta\delta} u^{1/2}, \quad 0 < t < 1,$$

where $\beta(\gamma - \delta + 1) + 1/2 > 0$.

We easily see that

$$f_0 = \infty, \qquad f_\infty = 0.$$

Then by Theorem 3.1, this equation has one positive solution $u \in K$.

Example 3.2. Consider the following fractional differential equation:

$$Du(t) = \frac{5\Gamma(1 + \gamma + \delta)}{\Gamma(1 + \gamma)} t^{-\beta\delta} u, \quad 0 < t < 1.$$

We take $e = 5$,

$$A = \frac{6\Gamma(1 + \gamma + \delta)}{\Gamma(1 + \gamma)}, \qquad B = \frac{20\Gamma(1 + \gamma + \delta)}{\Gamma(1 + \gamma)}.$$

Then

$$\frac{4\Gamma(1 + \gamma + \delta)}{\Gamma(1 + \gamma)} = \frac{B}{e} < A = \frac{6\Gamma(1 + \gamma + \delta)}{\Gamma(1 + \gamma)} < 4e\frac{\Gamma(1 + \gamma + \delta)}{\Gamma(1 + \gamma)} = B.$$

We see that

$$f_0 = \frac{5\Gamma(1 + \gamma + \delta)}{\Gamma(1 + \gamma)} < A = \frac{6\Gamma(1 + \gamma + \delta)}{\Gamma(1 + \gamma)},$$

$$f_\infty = \frac{5\Gamma(1 + \gamma + \delta)}{\Gamma(1 + \gamma)} > \frac{B}{e} = \frac{4\Gamma(1 + \gamma + \delta)}{\Gamma(1 + \gamma)}.$$

So by Theorem 3.5, this equation has one positive solution $u \in K$.

Remark 3.1. The generalized R–L (Riemann–Liouville) fractional calculus (fractional integration and fractional derivative) used in this paper may be extend to half-axis; it is apparent from their definitions. And the (Riemann–Liouville) fractional calculus (fractional integration and fractional derivative) used in [14,15] is a case of generalized R–L (Riemann–Liouville) fractional calculus, so it can also be extend to half-axis; for the detailed case, see [1–3]. The fractional calculus can be applied in various fields of science, it covers the widely known classical fields, such as Abel integral equation and viscoelasticity, and also less-known fields, including analysis of feedback amplifiers, capacitor theory, fractances, generalized voltage dividers, and other.

Remark 3.2. There are another kinds of fractional calculus: Riemann–Liouville integral and derivative of fractional order on the finite interval $[a, b]$, where $-\infty < a < b < \infty$, which are called Riemann–Liouville left-sided and right-sided fractional calculus. In this case, a, b must be finite numbers; when a, b are infinite numbers, there is another definition which differs from the definition for the function in the finite interval; for the detailed case, please see [2]. From [4], we see that the notions of left and right fractional derivatives can be considered from the physical and the mathematical viewpoints. Sometimes the following physical interpretation of the left and right derivative can be helpful.

Suppose that t is time and function $f(t)$ describes a certain dynamical process developing in time. If we take $\tau < t$, where t is the present moment, then the state $f(\tau)$ of the process belongs to the past of this process; if we take $\tau > t$, then $f(\tau)$ belongs to the future of the process f. From such a point of view, the left derivative is an operation performed on the past states of the process f and the right derivative is an operation performed on the future states of the process f. On the other hand, from the viewpoint of mathematics the right derivatives remind us of the operators conjugate to the operators of the left differentiation.

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