Journal of Pure and Applied Algebra 213 (2009) 840–845

Contents lists available at ScienceDirect

Journal of Pure and Applied Algebra

journal homepage: www.elsevier.com/locate/jpaa

The equivariant K-theory of toric varieties

Suanne Au, Mu-wan Huang, Mark E. Walker *

Department of Mathematics, University of Nebraska - Lincoln, Lincoln, NE 68588, USA

ARTICLE INFO

Article history:
Received 26 March 2008
Received in revised form 2 September 2008
Available online 22 November 2008
Communicated by C.A. Weibel

MSC:
19D50
14M25

ABSTRACT

This paper contains two results concerning the equivariant K-theory of toric varieties. The first is a formula for the equivariant K-groups of an arbitrary affine toric variety, generalizing the known formula for smooth ones. In fact, this result is established in a more general context, involving the K-theory of graded projective modules. The second result is a new proof of a theorem due to Vezzosi and Vistoli concerning the equivariant K-theory of smooth (not necessarily affine) toric varieties.

1. Introduction

Let k be a field, suppose U_σ is the affine toric k-variety associated to a strongly convex rational polyhedral cone σ in Euclidean n-space, and let T be the n-dimensional torus that acts on U_σ. If U_σ is smooth, then there is an equivariant isomorphism $U_\sigma \cong T_\sigma \times \mathbb{A}^r$, where $r = \dim(\sigma)$ and T_σ is the unique orbit of minimal dimension (namely, dimension $n - r$). Using basic properties of equivariant K-theory of smooth varieties (see, for example, [6]), one obtains natural isomorphisms

$$K_q^T(U_\sigma) \cong K_q^T(T_\sigma) \cong K_q(k) \otimes_{\mathbb{Z}} \mathbb{Z}[M_\sigma] \tag{1}$$

where $M_\sigma \cong \mathbb{Z}^{n-r}$ is the group of characters of T_σ.

This paper consists of two main results related to the isomorphism (1). The first, Theorem 4, shows that this isomorphism holds for all affine toric varieties, not just smooth ones. In fact, this theorem establishes the more general isomorphism

$$K_q^T(U_\sigma \times_k \operatorname{Spec} R) \cong K_q(R) \otimes_{\mathbb{Z}} \mathbb{Z}[M_\sigma], \tag{2}$$

where R is any k-algebra and the action of T on $\operatorname{Spec} R$ is trivial. Theorem 4 is actually a consequence of our Theorem 1, concerning the K-theory of graded projective modules.

The second main result of this paper is a new proof of a theorem due to Vezzosi and Vistoli [11, Theorem 6.2] that calculates the equivariant K-theory of an arbitrary smooth toric variety. See our Theorem 6 for the precise statement. The proof due to Vezzosi and Vistoli uses a more general result, one that applies to arbitrary actions by diagonalizable groups schemes. However, in the important special case of toric varieties, we recover their result using only Eq. (1), the theory of sheaf cohomology for fans, and Thomason's foundational work on equivariant K-theory [9].

2. The K-theory of graded projective modules

The first main goal of this paper is to establish the isomorphism (2). The action of T on U_σ is given by a grading (by the group of characters of T) of the associated ring of regular functions for U_σ, and an equivariant bundle on U_σ is given by a

* Corresponding author.
E-mail addresses: xau@math.unl.edu (S. Au), s-mhuang2@math.unl.edu (M. Huang), mwalker5@math.unl.edu (M.E. Walker).

0022-4049/$ – see front matter
doi:10.1016/j.jpaa.2008.10.010

graded projective module over this ring. Thus, our first theorem is really about the K-theory of graded projective modules. In this section, we state and prove a general theorem of this form.

Let R be any commutative ring, M an abelian group (written additively), and $A \subset M$ a submonoid. We form the associated monoid-ring $R[A]$. As a matter of notation, an element $a \in A$ is written as χ^a in $R[A]$ so that $\chi^a \chi^b = \chi^{a+b}$ for $a, b \in A$. The commutative ring $R[A]$ is an M-graded R-algebra, with elements of R declared to be of degree zero and for any $a \in A$, $\deg(\chi^a) := a \in A \subset M$. Let $\mathcal{P}(R)$ denote the category of finitely generated projective R-modules and let $\mathcal{P}^M(R[A])$ denote the category consisting of finitely generated M-graded projective $R[A]$-modules and with morphisms given by M-graded $R[A]$-module homomorphisms. Let $K_*^M(R[A])$ denote the K-theory of the exact category $\mathcal{P}^M(R[A])$.

Recall that if G is an M-graded $R[A]$-module and $m \in M$, then $G[m]$ denotes the same module but with the grading shifted so that $G[m]_w = G_{w-m}$ for all $w \in M$. In particular, $R[A][m]$ is graded-free of rank one generated by an element of degree m.

Write $U(A)$ for the subgroup of units (i.e., elements with additive inverses) in the monoid A. We fix, once and for all, a set $S(A) \subset M$ of coset representatives for the subgroup $U(A)$ of M.

Theorem 1. *For a commutative ring R, an abelian group M, and a submonoid A of M, we have an isomorphism*

$$K_q(R) \otimes_{\mathbb{Z}} \mathbb{Z}[M/U(A)] \cong K_q^M(R[A]), \quad \textit{for all } q.$$

Under the identification of $K_q(R) \otimes_{\mathbb{Z}} \mathbb{Z}[M/U(A)]$ with $\bigoplus_{S(A)} K_q(R)$, this isomorphism is induced by the collection of exact functors sending (P, s), with $P \in \mathcal{P}(R)$ and $s \in S(A)$, to $P \otimes_R R[A][s]$.

The proof of the theorem requires the following two lemmas. Throughout the rest of this section, let $U = U(A)$ and $S = S(A)$.

Lemma 2. *The exact functor*

$$\psi : \bigoplus_S \mathcal{P}(R) \to \mathcal{P}^M(R[U])$$

determined by

$$(P_s)_{s \in S} \mapsto \bigoplus_{s \in S} P_s \otimes_R R[U][s]$$

is an equivalence of categories.

Proof. For $P, P' \in \mathcal{P}(R)$ and $s, s' \in S$, we have an isomorphism

$$\mathrm{Hom}^M_{R[U]}(P \otimes_R R[U][s], P' \otimes_R R[U][s']) \cong \begin{cases} \mathrm{Hom}_R(P, P') & \text{if } s = s' \text{ and} \\ 0 & \text{otherwise,} \end{cases} \tag{3}$$

determined by sending a graded homomorphism from $P \otimes_R R[U][s]$ to $P' \otimes_R R[U][s']$ to the induced map on the degree s pieces. It follows that ψ is fully faithful.

Given $F \in \mathcal{P}^M(R[U])$, the M-grading on F gives a decomposition $F = \bigoplus_m F_m$. If $m, m' \in M$ belong to different cosets of U, then $(R[U] \cdot F_m) \cap F_{m'} = 0$. Thus we have an internal direct sum decomposition

$$F = \bigoplus_{s \in S} Q_s$$

as M-graded $R[U]$-modules, where $Q_s = \bigoplus_{m \in s+U} F_m$. Since F is finitely generated, $Q_s = 0$ for all but a finite number of s. For each $s \in S$, we have $F_s \cong Q_s \otimes_{R[U]} R$ (where $R[U] \to R$ is the augmentation map), and hence F_s is a finitely generated and projective R-module. If m_1, m_2 belong to the same coset of U in M, then $\chi^{m_2-m_1} : F_{m_1} \xrightarrow{\cong} F_{m_2}$ is an isomorphism of R-modules. Using this, we see that the map

$$F_s \otimes_R R[U][s] \to Q_s$$

determined by $p \otimes \chi^u \mapsto \chi^u \cdot p$ is a graded isomorphism of $R[U]$-modules. It follows that F is isomorphic to $\psi((F_s)_{s \in S})$, and hence ψ is an equivalence. □

If C, C' are M-graded rings, $\phi : C \to C'$ an M-graded ring homomorphism and F is an M-graded C-module, then the module obtained from F via extension of scalars along ϕ, namely $C' \otimes_C F$, acquires the structure of an M-graded C'-module having the property that if $c' \in C'_{m_1}$ and $f \in F_{m_2}$ then $c' \otimes f \in (C' \otimes_C F)_{m_1+m_2}$ (see [7, Section 2.4]). In particular, the module obtained from $C[m]$ by extension of scalars along ϕ is $C'[m]$.

Lemma 3. *The exact functor*

$$\mathcal{P}^M(R[U]) \to \mathcal{P}^M(R[A])$$

defined by extension of scalars induces a bijection on isomorphism classes of objects. In particular, objects of $\mathcal{P}^M(R[A])$ are projective in the category of all M-graded $R[A]$-modules.

Proof. For a projective R-module P and an arbitrary M-graded $R[A]$-module G, we have

$$\mathrm{Hom}_{\mathcal{P}^M(R[A])}(P \otimes_R R[A][m], G) \cong \mathrm{Hom}_R(P, G_m). \tag{4}$$

Since $G \mapsto G_m$ is an exact functor, $P \otimes_R R[A]$ is a projective object in the category of all M-graded $R[A]$-modules. In particular, the second assertion of the Lemma follows from the first one, using Lemma 2.

The M-graded R-algebra map $R[U] \to R[A]$ is split by the M-graded R-algebra map $R[A] \to R[U]$ defined by

$$\chi^a \mapsto \begin{cases} \chi^a & \text{if } a \in U \text{ and} \\ 0 & \text{if } a \notin U. \end{cases}$$

Since the composition $R[U] \hookrightarrow R[A] \twoheadrightarrow R[U]$ is the identity, the functor $\mathcal{P}^M(R[U]) \to \mathcal{P}^M(R[A])$ is split injective on isomorphism classes of objects.

The proof of the surjectivity on isomorphism classes will use the graded version of Nakayama's Lemma. Let $I \subset R[A]$ denote the kernel of the split surjection $R[A] \twoheadrightarrow R[U]$ — it is generated as an R-module by $\{\chi^a \mid a \notin U\}$. Clearly I is M-graded and, moreover, every maximal M-graded ideal of $R[A]$ contains I. Indeed, if $\mathfrak{m}$ is a maximal M-graded ideal, then $R[A]/\mathfrak{m}$ is an M-graded ring such that every non-zero homogeneous element is a unit (and whose inverse is, necessarily, homogeneous). For $a \notin U$, if $\overline{\chi^a} \neq 0$ in $R[A]/\mathfrak{m}$, then we would have $\overline{\chi^a} \cdot \overline{r\chi^b} = 1$ for some $r \in R$ and $b \in A$. But then $a + b = 0$, contrary to $a \notin U$. Thus $\chi^a \in \mathfrak{m}$ for all $a \notin U$. Since I is contained in every maximal M-graded ideal, the graded version of Nakayama's Lemma (see, for example, [8, Theorem 3.6] for a proof) gives us: If G is a finitely generated M-graded $R[A]$-module such that $IG = G$, then $G = 0$.

Given $E \in \mathcal{P}^M(R[A])$, let $F = E \otimes_{R[A]} R[U] \in \mathcal{P}^M(R[U])$ (with the map $R[A] \to R[U]$ being the above split surjection) and let $\tilde{F} = F \otimes_{R[U]} R[A]$. We prove $E \cong \tilde{F}$ in $\mathcal{P}^M(R[A])$. As noted above, (4) and Lemma 2 show that $\tilde{F}$ is a projective object in the category of all M-graded $R[A]$-modules. Thus the canonical map $\tilde{F} \twoheadrightarrow F$ lifts along the surjection $E \twoheadrightarrow F$ to give a morphism $\theta : \tilde{F} \to E$ in $\mathcal{P}^M(R[A])$. The map θ induces an isomorphism upon modding out by I and hence, by Nakayama's Lemma, $\mathrm{coker}(\theta) = 0$. Since E is projective as an ungraded R-module, the exact sequence

$$0 \to \ker(\theta) \to \tilde{F} \to E \to 0$$

remains exact upon application of $- \otimes_{R[A]} R[U]$, and hence, using Nakayama's Lemma again, $\ker(\theta) = 0$. □

Proof of Theorem 1. By Lemma 2, we have

$$K_q^M(R[U]) \cong \bigoplus_S K_q(R) \cong K_q(R) \otimes_{\mathbb{Z}} \mathbb{Z}[M/U].$$

In order to prove the theorem, it therefore suffices to prove the exact functor

$$\mathcal{P}^M(R[U]) \to \mathcal{P}^M(R[A]), \tag{5}$$

induced by extension of scalars, induces a homotopy equivalence on K-theory spaces.

For any finite subset $F \subset S$, let $\mathcal{P}_F^M(R[A])$ denote the full subcategory of those objects in $\mathcal{P}^M(R[A])$ isomorphic to one of the form

$$\bigoplus_{i=1}^{l} P_i \otimes_R R[A][s_i]$$

such that $s_i \in F$ for $i = 1, \ldots, l$. Define $\mathcal{P}_F^M(R[U])$ similarly. Note that $\mathcal{P}_F^M(R[U])$ and $\mathcal{P}_F^M(R[A])$ are closed under direct sum and hence are exact subcategories. Since $\mathcal{P}^M(R[A]) = \varinjlim_{F \subset S} \mathcal{P}_F^M(R[A])$ where F ranges over all finite subsets of S and since K-theory commutes with filtered colimits, it suffices to prove

$$\mathcal{P}_F^M(R[U]) \to \mathcal{P}_F^M(R[A])$$

induces an equivalence on K-theory for all finite $F \subset S$. We proceed by induction on $\#F$. If $\#F = 1$, then by (3) and Lemma 3, $\mathcal{P}_F^M(R[U]) \to \mathcal{P}_F^M(R[A])$ is an equivalence of categories.

Define a partial order $\leq$ on S by declaring $s \leq s'$ if and only if $s' - s \in A$. Then for projective R-modules P, P' and elements $s, s' \in S$, we have

$$\mathrm{Hom}_{\mathcal{P}^M(R[A])}(P \otimes_R R[A][s], P' \otimes_R R[A][s']) \cong \begin{cases} \mathrm{Hom}_R(P, P') & \text{if } s \leq s' \text{ and} \\ 0 & \text{otherwise.} \end{cases} \tag{6}$$

Now assume $\#F > 1$ and let $m \in F$ be a maximal element. Define $F' = F \setminus \{m\}$. We have a commutative diagram of exact functors

$$\begin{array}{ccc} \mathcal{P}^M_{F'}(R[U]) \oplus \mathcal{P}^M_{\{m\}}(R[U]) & \longrightarrow & \mathcal{P}^M_{F'}(R[A]) \oplus \mathcal{P}^M_{\{m\}}(R[A]) \\ \downarrow & & \downarrow \\ \mathcal{P}^M_F(R[U]) & \longrightarrow & \mathcal{P}^M_F(R[A]) \end{array}$$

in which the vertical maps are given by direct sum and the horizontal maps are extensions of scalars. The left-hand vertical map and the top horizontal map induce equivalences on K-theory using Lemma 2 and induction, respectively. It therefore suffices to prove that the right-hand vertical map induces an equivalence on K-theory. This follows from Waldhausen's generalization of the Quillen Additivity Theorem, as we now explain.

Let $\mathcal{E}$ denote the exact category consisting of short exact sequences of objects of $\mathcal{P}^M_F(R[A])$ of the form

$$0 \to B \to P \to C \to 0 \tag{7}$$

with $B \in \mathcal{P}^M_{\{m\}}(R[A])$ and $C \in \mathcal{P}^M_{F'}(R[A])$. By Lemma 3, for any such short exact sequence, we have that P is isomorphic to $B \oplus C$. This exact sequence is isomorphic to

$$0 \to B \xrightarrow{\begin{pmatrix}1\\0\end{pmatrix}} B \oplus C \xrightarrow{(0,1)} C \to 0.$$

Thus $\mathcal{E}$ is equivalent to the full subcategory consisting of such "trivial" exact sequences. Moreover, by (6) there are no non-trivial maps from B to C, and hence a morphism from one such exact sequence to another is completely determined by the map on middle objects. That is, the functor $\mathcal{E} \to \mathcal{P}^M_F(R[A])$ sending the exact sequence (7) to P is an equivalence of categories. On the other hand, Waldhausen's Additivity Theorem [13] shows that the functor

$$\mathcal{E} \to \mathcal{P}^M_{\{m\}}(R[A]) \oplus \mathcal{P}^M_{F'}(R[A])$$

sending (7) to (B, C) induces an equivalence on K-theory. □

3. The equivariant K-theory of affine toric varieties

In this section we provide an interpretation of Theorem 1 for toric varieties.

We adopt the notational conventions for toric varieties found in Fulton's book [4]. An affine toric variety is defined from a strongly convex rational polyhedral cone σ in $N \otimes_{\mathbb{Z}} \mathbb{R}$ where $N \cong \mathbb{Z}^n$ is an n-dimensional lattice. Let $M := \mathrm{Hom}_{\mathbb{Z}}(N, \mathbb{Z}) \cong \mathbb{Z}^n$ be the dual lattice and define the dual cone of σ by

$$\sigma^\vee := \{u \in M \otimes_{\mathbb{Z}} \mathbb{R} | u(v) \geq 0 \text{ for all } v \in \sigma\}.$$

We have that $\sigma^\vee \cap M$ is a finitely generated abelian monoid, by Gordan's Lemma, and hence, for any commutative ring R, the corresponding monoid ring $R[\sigma^\vee \cap M]$ is a finitely generated R-algebra. We let

$$U_{\sigma,\mathbb{Z}} = \mathrm{Spec}\, \mathbb{Z}[\sigma^\vee \cap M],$$

the *affine toric scheme* over $\mathbb{Z}$ associated to σ.

Note that for any commutative ring R, we have

$$U_{\sigma,R} := U_{\sigma,\mathbb{Z}} \times \mathrm{Spec}\, R = \mathrm{Spec}\, R[\sigma^\vee \cap M].$$

In particular, for a field k, the affine k-variety $U_{\sigma,k} = \mathrm{Spec}\, k[\sigma^\vee \cap M]$ is the classical toric k-variety associated to σ.

For any commutative ring R, the R-algebra $R[\sigma^\vee \cap M]$ is an M-graded R-algebra, and this grading amounts to an action of the n-dimensional torus scheme $T := \mathrm{Spec}\, \mathbb{Z}[M]$ on $U_{\sigma,R}$. Viewing $U_{\sigma,R}$ as $U_{\sigma,\mathbb{Z}} \times \mathrm{Spec}\, R$, the action of T is given by the usual action on $U_{\sigma,\mathbb{Z}}$ and the trivial action $\mathrm{Spec}\, R$. An equivariant vector bundle over $U_{\sigma,R}$ is identified as a projective module over $R[\sigma^\vee \cap M]$ that is M-graded. We therefore obtain

$$K^M_*(R[\sigma^\vee \cap M]) \cong K^T_*(U_{\sigma,R}).$$

Finally, observe that $U(\sigma^\vee \cap M) = \sigma^\perp \cap M$, and we define $M_\sigma := M/(\sigma^\perp \cap M)$. The following is thus an immediate consequence of Theorem 1.

Theorem 4. *For any commutative ring R and strongly convex rational cone σ, there is a natural isomorphism*

$$K^T_q(U_{\sigma,R}) \cong K_q(R) \otimes_{\mathbb{Z}} \mathbb{Z}[M_\sigma].$$

In particular, we see that Eq. (1) holds for any affine toric variety, not only the smooth ones. Observe that M_σ, as just defined, coincides with the group of characters on the minimal orbit of U_σ.

Remark 5. The isomorphism of Theorem 1 is natural in R in the obvious sense and is natural in A in the following sense: If $A \subset A' \subset M$ is an inclusion of submonoids of M, then

$$\begin{array}{ccc} K_q(R) \otimes_{\mathbb{Z}} \mathbb{Z}[M/U(A)] & \xrightarrow{\cong} & K_q^M(R[A]) \\ \downarrow & & \downarrow \\ K_q(R) \otimes_{\mathbb{Z}} \mathbb{Z}[M/U(A')]) & \xrightarrow{\cong} & K_q^M(R[A']) \end{array}$$

commutes, where the left-hand map is the canonical quotient map and the right-hand map is induced by extension of scalars.

Consequently, the isomorphism of Theorem 4 is natural in R and with respect to the inclusion of a face τ into σ. In the latter case, the map

$$K_q^T(U_{\sigma,R}) \to K_q^T(U_{\tau,R})$$

is induced by pullback along the equivariant open immersion $U_{\tau,R} \subset U_{\sigma,R}$ and the map

$$K_q(R) \otimes_{\mathbb{Z}} \mathbb{Z}[M_\sigma] \to K_q(R) \otimes_{\mathbb{Z}} \mathbb{Z}[M_\tau]$$

is the map induced by the canonical surjection $M_\sigma \twoheadrightarrow M_\tau$.

4. The Vezzosi–Vistoli theorem

In this section, we use (1) from the introduction, the theory of sheaves on fans and the foundational results of Thomason [9] concerning equivariant K-theory to recover a result due to Vezzosi and Vistoli [11,12]: For a field k and a smooth toric k-variety $X = X(\Delta)$ defined by a fan Δ, the sequence

$$0 \longrightarrow K_q^T(X) \longrightarrow \bigoplus_{\sigma \in \mathrm{Max}(\Delta)} K_q^T(U_\sigma) \xrightarrow{\partial} \bigoplus_{\delta,\tau \in \mathrm{Max}(\Delta), \delta < \tau} K_q^T(U_{\delta\cap\tau})$$

is exact. Here, $\mathrm{Max}(\Delta)$ is the set of maximal cones in Δ and we choose, arbitrarily, a total ordering for this set. The map ∂ is given as follows. For $f = (f_\sigma)_{\sigma\in\mathrm{Max}(\Delta)}$ in $\bigoplus_{\sigma\in\mathrm{Max}(\Delta)} K_q^T(U_\sigma)$, the $(\delta < \tau)$-component of its image is $f_\tau|_{U_{\delta\cap\tau}} - f_\delta|_{U_{\delta\cap\tau}} \in K_q^T(U_{\delta\cap\tau})$.

In fact, we prove that the sequence

$$0 \to K_q^T(X) \to \bigoplus_{\sigma} K_q^T(U_\sigma) \to \bigoplus_{\delta<\tau} K_q^T(U_{\delta\cap\tau}) \to \bigoplus_{\delta<\tau<\epsilon} K_q^T(U_{\delta\cap\tau\cap\epsilon}) \to \cdots \tag{8}$$

is exact, where $\bigoplus_\sigma K_q^T(U_\sigma) \to \bigoplus_{\delta<\tau} K_q^T(U_{\delta\cap\tau}) \to \cdots$ is the Čech complex of the presheaf K_q^T for the equivariant open cover $\mathcal{V} = \{U_\sigma \mid \sigma \text{ is a maximal cone in } \Delta\}$. Using Eq. (1) (or our Theorem 4), the exactness of this sequence is equivalent to the existence of an exact sequence of the form

$$0 \to K_q^T(X) \to \bigoplus_{\sigma} K_q(k) \otimes_{\mathbb{Z}} \mathbb{Z}[M_\sigma] \to \bigoplus_{\delta<\tau} K_q(k) \otimes_{\mathbb{Z}} \mathbb{Z}[M_{\delta\cap\tau}] \to \cdots . \tag{9}$$

We define a topology on the finite set of cones comprising the fan Δ by declaring the open subsets to be the subfans of Δ; see [2] or [3]. In other words, we view Δ as a poset via face containment, $\prec$, and we give Δ the "poset topology", in which an open subset Λ is a subset satisfying the condition what whenever $x \prec y$ and $y \in \Lambda$, we have $x \in \Lambda$. For a cone $\sigma \in \Delta$, let $\langle\sigma\rangle$ denote the fan consisting of σ and all its faces (i.e., the smallest open subset of Δ containing σ). Observe that for a sheaf $\mathcal{F}$ on Δ, we have $\mathcal{F}(\langle\sigma\rangle) = \mathcal{F}_\sigma$, the stalk of $\mathcal{F}$ at the point σ.

For this topology, sheaves are uniquely determined by their stalks and the maps between their stalks arising from comparable elements of the poset (see [1, Section 4.1]). That is, there is an equivalence between the category of contravariant functors from the poset Δ to the category of abelian groups and the category of sheaves of abelian groups on the topological space Δ. (Recall that a poset may be viewed as a special type of category.) Given a sheaf $\mathcal{F}$ on the space Δ, the associated functor on the poset Δ sends $\sigma \in \Delta$ to $\mathcal{F}_\sigma = \mathcal{F}(\langle\sigma\rangle)$ and sends a face inclusion $\tau \prec \sigma$ to the map induced by $\langle\tau\rangle \subset \langle\sigma\rangle$. Given a contravariant functor F on the poset Δ, the value of associated sheaf $\mathcal{F}$ on an open subset Λ of Δ is given by

$$\mathcal{F}(\Lambda) = \varprojlim_{\sigma\in\Lambda} F(\sigma).$$

Theorem 6. *Assume that $X = X(\Delta)$ is a smooth toric variety defined over an arbitrary field k. Then the presheaf $\Lambda \mapsto K_q^T(X(\Lambda))$ defined on Δ is a flasque sheaf. Moreover, there is an isomorphism*

$$K_q^T(X) \cong K_q(k) \otimes K_0^T(X).$$

and the sequences (8) and (9) are exact.

Proof. Let $\mathcal{A}_q$ be the sheaf on Δ associated to the functor sending a cone σ to $K_q(k) \otimes \mathbb{Z}[M_\sigma]$ and a face inclusion $\tau \prec \sigma$ to the map induced by the canonical quotient $M_\sigma \twoheadrightarrow M_\tau$.

The sheaf $\mathcal{A}_0$ is flasque by [1]. Since $\mathcal{A}_0$ is a flasque sheaf of torsion free abelian groups, the presheaf $K_q(k) \otimes_{\mathbb{Z}} \mathcal{A}_0$ is actually a sheaf. Indeed, for any open subset U and open covering $U = \cup_i V_i$ of it, the map from $\mathcal{A}_0(U)$ to the associated Čech complex is a quasi-isomorphism by [5, III.4.3], and since $\mathcal{A}_0$ is torsion free, this map remains a quasi-isomorphism upon tensoring by any abelian group. It now follows from the correspondence between functors and sheaves that $\mathcal{A}_q \cong K_q(k) \otimes \mathcal{A}_0$. In particular, $\mathcal{A}_q$ is also flasque.

For a subfan Λ of Δ, let $\mathcal{V}$ be the Zariski open covering $\{U_\sigma \mid \sigma \text{ is a maximal cone in } \Lambda\}$ of $X(\Lambda)$ and let $\mathcal{U}$ be the open covering $\{\langle\sigma\rangle \mid \sigma \in \mathrm{Max}(\Lambda)\}$ of Λ. By Eq. (1) (or Theorem 4), the Čech cohomology complex of the presheaf $\mathcal{K}_q^T(-)$ on $X(\Lambda)$ for the open covering $\mathcal{V}$ coincides with the Čech cohomology complex of the sheaf $\mathcal{A}_q$ for the open covering $\mathcal{U}$. Since the higher Čech cohomology of flasque sheaves vanishes [5, III.4.3], we have

$$\check{H}^p\left(\mathcal{V}, K_q^T\right) = \check{H}^p\left(\mathcal{U}, \mathcal{A}_q\right) = 0, \quad \text{for all } p > 0. \tag{10}$$

Thomason [9] has proven that $\mathcal{K}^T$ coincides with equivariant G-theory (defined from equivariant coherent sheaves) and that the latter satisfies the usual localization property relating X, an equivariant closed subscheme, and its open complement. From this one deduces that if $X(\Lambda) = U \cup V$ is covering by equivariant open subschemes, then

$$\begin{array}{ccc} \mathcal{K}^T(X(\Lambda)) & \longrightarrow & \mathcal{K}^T(U) \\ \downarrow & & \downarrow \\ \mathcal{K}^T(V) & \longrightarrow & \mathcal{K}^T(U \cap V) \end{array}$$

is a homotopy cartesian square. Arguing just as in [10, Section 8], one obtains a convergent spectral sequence

$$\check{H}^p\left(\mathcal{V}, K_q^T\right) \Longrightarrow K_{q-p}^T(X(\Lambda)).$$

Using (10), this spectral sequence collapses to give

$$\check{H}^0\left(\mathcal{V}, K_q^T\right) \cong K_q^T(X(\Lambda)), \quad \text{for all } q. \tag{11}$$

Combining (11) and (10) gives that the complexes

$$0 \to K_q^T(X(\Lambda)) \to \bigoplus_\sigma K_q^T(U_\sigma) \to \bigoplus_{\delta<\tau} K_q^T(U_{\delta\cap\tau}) \to \cdots$$

and

$$0 \to \mathcal{A}_q(\Lambda) \to \bigoplus_\sigma K_q(k) \otimes_{\mathbb{Z}} \mathbb{Z}[M_\sigma] \to \bigoplus_{\delta<\tau} K_q(k) \otimes_{\mathbb{Z}} \mathbb{Z}[M_{\delta\cap\tau}] \to \cdots$$

are exact and isomorphic to each other. In particular, $\Lambda \mapsto \mathcal{K}_q^T(X(\Lambda))$ is isomorphic to the flasque sheaf $\mathcal{A}_q$.

The remaining assertions of the theorem follow immediately. □

Acknowledgements

We are very grateful to Dan Grayson for suggesting the use of the Additivity Theorem in the proof of Theorem 1. This allowed us to simplify our original proof and to make the result more general. We are also indebted to the anonymous referee for many thoughtful suggestions which greatly improved the exposition of this paper. Walker's research was partially supported by NSF grant DMS-0601666.

References

[1] Silvano Baggio, Equivariant K-theory of smooth toric varieties, Tohoku Math. J. (2) 59 (2) (2007) 203–231.
[2] Gottfried Barthel, Jean-Paul Brasselet, Karl-Heinz Fieseler, Ludger Kaup, Equivariant intersection cohomology of toric varieties, in: Algebraic Geometry: Hirzebruch 70 (Warsaw, 1998), in: Contemp. Math., vol. 241, Amer. Math. Soc, Providence, RI, 1999, pp. 45–68.
[3] Paul Bressler, Valery A. Lunts, Intersection cohomology on nonrational polytopes, Compos. Math. 135 (3) (2003) 245–278.
[4] William Fulton, Introduction to Toric Varieties, in: Annals of Mathematics Studies, vol. 131, Princeton University Press, Princeton, NJ, 1993, The William H. Roever Lectures in Geometry.
[5] Robin Hartshorne, Algebraic Geometry, Springer-Verlag, New York, 1977, Graduate Texts in Mathematics, No. 52.
[6] Alexander S. Merkurjev, Equivariant K-theory, in: Handbook of K-Theory, vol. 1, 2, Springer, Berlin, 2005, pp. 925–954.
[7] Constantin Năstăsescu, Freddy Van Oystaeyen, Methods of Graded Rings, in: Lecture Notes in Mathematics, vol. 1836, Springer-Verlag, Berlin, 2004.
[8] Markus Perling, Graded rings and equivariant sheaves on toric varieties, Math. Nachr. 263–264 (2004) 181–197.
[9] R.W. Thomason, Algebraic K-theory of group scheme actions, in: Algebraic Topology and Algebraic K-Theory (Princeton, N.J., 1983), in: Ann. of Math. Stud., vol. 113, Princeton Univ. Press, Princeton, NJ, 1987, pp. 539–563.
[10] R.W. Thomason, Thomas Trobaugh, Higher algebraic K-theory of schemes and of derived categories, in: The Grothendieck Festschrift, Vol. III, in: Progr. Math., vol. 88, Birkhäuser, Boston, Boston, MA, 1990, pp. 247–435.
[11] Gabriele Vezzosi, Angelo Vistoli, Higher algebraic K-theory for actions of diagonalizable groups, Invent. Math. 153 (1) (2003) 1–44.
[12] Gabriele Vezzosi, Angelo Vistoli, Erratum: "Higher algebraic K-theory for actions of diagonalizable groups", Invent. Math. 153 (1) (2003) 1–44; Invent. Math. 161 (1) (2005) 219–224.
[13] Friedhelm Waldhausen, Algebraic K-theory of spaces, in: Algebraic and Geometric Topology (New Brunswick, N.J., 1983), in: Lecture Notes in Math., vol. 1126, Springer, Berlin, 1985, pp. 318–419.