Let $R$ be a prime Krull ring (see Section 1 for the appropriate definitions) and $M$ a torsion-less right $R$-module of finite rank. The main aim of this paper is to prove a non-commutative version of Bourbaki's theorem:

**Theorem.** There exists a short exact sequence of $R$-modules,
$$0 \rightarrow F \rightarrow M \rightarrow I \rightarrow 0,$$
where $F$ is a free module and $I$ is a right ideal of $R$.

If one adds the assumptions that (i) $R$ is a domain and (ii) $K \dim_{R} = 1$ then this result has been proved by Chamarie [2] and our proof is a refinement of his. In Chamarie's proof, condition (ii) arises from the fact that he proves the result by passing to a quotient category of the category of $R$-modules. This condition can be avoided by proving the result back in the category of $R$-modules. To generalise the result from domains to prime rings is technically more involved, but the idea is easy enough: given a reflexive prime ideal $P$ of $R$, the local ring $R_{P}$ is a full matrix ring over a domain $D$ and so one should try to make use of this domain $D$. This is done in Section 2 and used in Section 3 to generalise Bourbaki's Theorem. Section 1 contains some elementary results about Krull rings.

In fact, we prove a slight generalisation of Bourbaki's Theorem, which is concerned with giving the appropriate short exact sequence for our module $M$ and a submodule $N$. In particular, this has the following result as a consequence.

**Corollary.** Let $R$ be a prime Krull ring with no non-zero reflexive ideals. Then any finitely generated torsion right $R$-module $T$ can be realised as $T \cong I/J$ for right ideals $J \subset I$ of $R$. 
1. Krull Rings

In this section, we give the basic definitions and some easy results that we need concerning Krull rings. Most of these results are variants of ones from [1] and [2], and the reader may like to refer to those papers for a more detailed discussion of Krull rings.

Let \( R \) be a prime Goldie ring with simple Artinian quotient ring \( Q(R) \). Given a right (or left) \( R \)-module \( M \), set \( M^* = \text{Hom}(M, R) \). There is a natural homomorphism from \( M \) to \( M^{**} \) and we call \( M \) torsion-less (respectively, reflexive) if this homomorphism is injective (respectively, an isomorphism). Note that \( M \) is torsion-less if and only if it can be embedded in a free module. If \( M \) is an essential right (left) ideal of \( R \), then we will identify \( M^* \) with \( \{ f \in Q(R): fM \subseteq R \} \), respectively, \( \{ f \in Q(R): Mf \subseteq R \} \). A right ideal \( I \) of \( R \) is called closed if \( I = \{ x \in R: xJ \subseteq I \text{ for some right ideal } J \text{ with } J^* = R \} \). Finally, \( R \) is called a prime Krull ring if \( R \) is a maximal order in \( Q(R) \) and has ACC both on closed right ideals and on closed left ideals.

It is easy to see that reflexive right ideals are closed and so, in particular, a Krull ring has ACC on reflexive right ideals. Given an ideal \( I \) of a ring \( R \) write \( \mathcal{C}(I) \) for the elements of \( R \) that are regular mod \( I \). Throughout this section \( R \) will denote a prime Krull ring.

**Lemma 1.1.** The lattice of reflexive right ideals of \( R \) has Krull dimension one. Indeed, if \( a \in \mathcal{C}(0) \), then the set of reflexive right ideals of \( R \) containing \( a \) satisfies DCC. The same result holds for reflexive left ideals.

**Proof.** This is immediate from the fact that reflexive submodules of \( Ra^{-1} \) satisfy ACC. See, for example, the proof of [2, Proposition 4.1.1].

It is not clear whether this result can be generalised to show that the lattice of closed right ideals has Krull dimension one—indeed, this is precisely the condition \( K \dim_{\mathcal{C}} R = 1 \) that was mentioned in the introduction. Thus (in contrast to Chamarie's proof) we must be careful to use reflexive as opposed to closed, right ideals.

**Lemma 1.2.** Let \( M \) be a torsion-less right \( R \)-module of finite uniform dimension and \( \mathcal{C} \) some Ore set of regular elements of \( R \). Then, for any \( \theta \in \text{Hom}(M_{\mathcal{C}}, R_{\mathcal{C}}) \), there exists \( c \in \mathcal{C} \) such that \( c\theta \in M^* \).

**Proof.** Clearly \( M \) embeds in a free \( R \)-module of finite rank, say, \( M \subseteq R^{(n)} \). Since \( R \) has ACC on reflexive submodules, so does \( R^{(n)} \). Thus \( M^{**} = (\sum_i m_i R)^{**} \) for some \( m_i \in M \). Pick \( c \in \mathcal{C} \) and \( i \in R \) such that \( \theta(m_i) = c^{-1}i \), for each \( i \). Then \( c\theta(\sum_i m_i R) \subseteq R \), which forces \( c\theta(M^{**}) \subseteq R \), as required.
LEMMA 1.3. Let $P$ be a semiprime, reflexive ideal of $R$. Then $\mathcal{G}(P)$ is an Ore set in $R$ and $R_P$ is a semilocal, principal ideal ring (PIR). Furthermore, $R_P \simeq M_n(D)$ for some integer $n$ and domain $D$.

Proof. By [1, Propositions 1.7 and 2.5], $R_P$ exists and is a semilocal PIR. The remaining assertions follow from [3, p. 45] and [5, Remark 3.5].

LEMMA 1.4. Let $I = I^{**} \subseteq J$ be essential left ideals of $R$ and set $P = 1\text{-ann } J/I$. Then:

(i) $P^{**}J^{**} \subseteq I$ and so $P = P^{**}$.

(ii) If $J^{**}$ is minimal among reflexive left ideals that strictly contain $I$, then $P$ is a prime ideal.

Remark. Since $R$ is a maximal order, $\{f \in Q(R): fP \subseteq R\} = \{g \in Q(R): Pg \subseteq R\}$. Thus, $P^*$ is well defined in the sense that it is the same whether $P$ is viewed as a right or left ideal of $R$.

Proof. (i) Since $PJ(PJ)^* \subseteq R$, the above remark implies that $J(PJ)^*P \subseteq R$. Thus $(PJ)^*P \subseteq J^* = (J^{**})^*$ and $J^{**}(PJ)^*P \subseteq R$. Repeating these two steps gives $P^{**}(PJ)^* \subseteq R$ and $P^{**}J^{**}(PJ)^* \subseteq R$. Thus,

$$P^{**}J^{**} \subseteq (PJ)^* \subseteq J^{**} = I,$$

as required.

(ii) Suppose that $AB \subseteq P$ for two ideals $A$ and $B$. Then, by (i), $A(I + BJ)^{**} \subseteq I$. If $BJ \not\subseteq I$, then the minimality of $J^{**}$ ensures that $(I + B - J)^{**} = J^{**}$. Thus $AJ^{**} \subseteq I$ and $A \subseteq P$, as required.

LEMMA 1.5. Let $P$ be a non-zero, reflexive prime ideal of $R$ and $\alpha \in \mathcal{G}(P)$. Then $(P + R\alpha)^* = R$.

Proof. By replacing $\alpha$ by $\alpha + p$ for some $p \in P$, we may assume that $\alpha$ is regular [7, Proposition 2.4]. Let $q \in (P + R\alpha)^*$. Then, $q \in (R\alpha)^* = \alpha^* R$; say, $q = \alpha^{-1}g$ for some $g \in R$. Now $q \in P^*$ and so $qP \subseteq R$. Since $\alpha \in \mathcal{G}(P)$, $\alpha R \cap P = \alpha P$. Thus $gP \subseteq \alpha R \cap P = \alpha P$. Therefore, $qP = \alpha^{-1}gP \subseteq P$. Finally, since $R$ is a maximal order, this says that $q \in R$, as required.

2. Generating Regular Elements

Let $M$ be a module over a prime Goldie ring $S$, with quotient ring $Q(S)$. Define the Goldie rank of $M$ by $\operatorname{rk}_S(M) = \operatorname{udim} M \otimes Q(S)$. Equivalently, $\operatorname{rk}(M)$ is the uniform dimension of $M$ modulo its torsion submodule. The subscript will be dropped whenever there is no confusion. Given a prime
ideal \( P \) of \( S \), with \( S/P \) Goldie, set \( \text{rk}(M, P) = \text{rk}_{S/P}(M/MP) \). Finally, given a left ideal \( I \) of \( S \), we set \( h(I, P) = \text{rk}(I + P/P, P) \) or, equivalently, \( h(I, P) = \text{rk}(S/P) - \text{rk}(S/I, P) \).

Suppose that \( S \) is now a prime Krull ring, and \( M \) is a torsion-less right \( S \)-module of finite rank. The aim of this section is to find \( \alpha \in M \) such that the numbers \( \text{rk}(M/\alpha S, P) \), as \( P \) runs through the reflexive prime ideals of \( S \), are as small as possible. Note that, by Lemma 1.3, each \( P \) is a localisable ideal of \( S \) (and so \( S/P \) is Goldie). Thus in calculating \( \text{rk}(M, P) \), one can freely localise at \( P \); that is, \( \text{rk}(M, P) = \text{rk}(M, P) \). We will use this observation frequently, and usually without comment. In particular, since \( S_p \) is a matrix ring over a domain, \( \text{rk}(S/P)/\text{rk}(S) \) is an integer, which we will denote by \( t_p \). Finally, we write \( \mathcal{P} \) for the set of reflexive prime ideals of \( S \).

**Lemma 2.1.** Let \( Q_1, \ldots, Q_n \) be prime ideals of a ring \( S \) such that each \( S/Q_i \) is a prime Goldie ring. Let \( X \subset Y \) be torsion-less right \( S \)-modules, \( a_1, a_2 \in Y \) and \( b \in S \). Then there exist \( \lambda \in S \) such that, for \( 1 \leq i \leq n \),

\[
\text{rk}(X + (a_1 + a_2 \lambda) S, Q_i) \geq \min\{\text{rk}(X + a_1 S + a_2 S, Q_i), \text{rk}(X, Q_i) + \text{rk}(bS, Q_i)\}.
\]

**Remark.** This is a variant on a number of results from the literature. In particular, if \( b = 1 \) the result is well known.

**Proof.** We first observe that, by the argument used in [7, Proposition 2.4], it suffices to prove the result for just one prime, say, \( Q \). By passing to \( S/Q \) we may assume that \( Q = 0 \).

Pick \( \lambda \in S \) such that \( \text{rk}(X + (a_1 + a_2 \lambda) S) \) is as large as possible, and replace \( a_1 \) by \( a_1 + a_2 \lambda b \). We assume that

\[
\text{rk}(X + a_1 S) < \min\{\text{rk}(X + a_1 S + a_2 S), \text{rk}(X) + \text{rk}(bS)\},
\]

as otherwise the result is proven. Pick \( f \in S \) such that \( a_1, f \in X \) but \( \text{rk} fS \) is as large as possible. Thus, by (2),

\[
\text{rk}(fS) = \text{rk} S - \text{rk}(a_1 S + X/X) > \text{rk} S - \text{rk}(bS).
\]

In particular, \( hf \neq 0 \). Now by (2), again, \( X + a_1 S \) is not essential in \( X + a_1 S + a_2 S \). Thus, there exists a cyclic, uniform submodule, say, \( a_2 tS \) of \( a_2 S \), such that \( a_2 tS \cap (X + a_1 S) = 0 \). Since \( S \) is prime and \( Y \) torsion-free, there exists \( r \in S \) with \( a_2 trbf \neq 0 \). Now consider \( K = X + (a_1 + a_2 trb) S \). Then

\[
K \supseteq (a_1 + a_2 trb) fS + X = a_2 trbf S + X \supseteq a_2 trbf S \neq 0.
\]

Thus, \( K \) contains an essential submodule of \( a_2 tS \) and hence of
$X + a_1 S + a_2 t S$. This contradicts the maximality of $\text{rk}(X + a_1 S)$ and completes the proof.

In the next lemma we collect various facts about PIR's that will prove useful. Most of these are well known, but by the comments at the beginning of this section, they also give information about arbitrary prime Krull rings.

**Lemma 2.2.** Let $N \subseteq M$ be torsion-less right modules of finite rank over a semilocal, prime PIR $S$. Then:

(i) Up to isomorphism there exists a unique uniform projective $S$-module, say, $B = bS$, and $M \cong B^{(n)}$ for some integer $n$.

(ii) $\text{rk}(M, P) = t_\rho \text{rk}(M)$ for each prime ideal $P$ of $S$. Furthermore, $\text{rk}(N + MP/MP, P) < t_\rho \text{rk}(N)$ with equality for all $P$ if and only if $M/N$ is torsion-free.

(iii) There exists $\phi \in \text{Hom}(B, S)$ such that $\text{rk}(S\phi(b), P) = t_\rho$ for each prime ideal $P$ of $S$.

(iv) There exists $\alpha \in N$ with $\alpha S$ uniform and $M = \alpha S \oplus L$ for some submodule $L$ if and only if $\text{rk}(N + MP/MP, P) > t_\rho$ for each prime ideal $P$ of $S$.

**Proof.** (i) By Lemma 1.3, $S \cong M_m(D)$ for some integer $m$ and principal ideal domain $D$. Every torsion-less $D$-module is free and by Morita equivalence this translates to the present statement for $S$.

(ii) $\text{rk}(M, P) = t_\rho \text{rk}(M)$ certainly holds for $M = S$ and, therefore, by (i), it holds in general. Let $L$ be the largest essential extension of $N$ in $M$. Then $M/L$ is torsion-free and so $M \cong L \oplus M/L$. Thus $L \cap MP = LP$ and

\[
\text{rk}(L + MP/MP, P) = \text{rk}(L/LP, P) = t_\rho \text{rk}(L) = t_\rho \text{rk}(N).
\]

If $L = N$ this gives $\text{rk}(N + MP/MP, P) = t_\rho \text{rk} N$, as required. Suppose that $L \neq N$. As $S$ is fully bounded, $\text{rk}(L/N, P) \neq 0$ for some prime ideal $P$ [1, Proposition 1.10]. Since $L \cap MP = LP$, this implies that $\text{rk}(L + MP/MP, P) > \text{rk}(N + MP/MP, P)$, as required.

(iii) Under the identification $S \cong M_m(D)$ we can find $\phi \in B^*$ such that $\phi(b) = e_{11}$. Thus $S\phi(b)$ is a direct summand of $S$ and the result follows from (ii).

(iv) This can be obtained from [4, Theorem 3.3], but a direct proof is just as quick. For each module of the (finitely many) prime ideals $P$ of $S$ there exists $\alpha_P \in N$ with $\text{rk}(\alpha_P S + MP/MP, P) \geq t_\rho$. Let $b$ be as in part (i). By induction and Lemma 2.1, there exist $\lambda_P \in S$ such that, if $\alpha = \sum \alpha_P \lambda_P b$, then

\[
\text{rk}(\alpha S + MP/MP, P) \geq t_\rho
\]
for each prime ideal \( P \). Note that, as \( r \)-ann \( a \supseteq r \)-ann \( b \), \( aS \) is actually uniform. By (ii) (with \( N \) replaced by \( aS \)) this implies that \( aS \) is a direct summand of \( M \). The other direction is trivial.

**Lemma 2.3.** Let \( I \) be a right ideal of a Krull ring \( R \). Then \( h(I, P) = t_p \rk(I) \) for all but finitely many reflexive prime ideals \( P \) of \( R \).

**Proof.** Pick a right ideal \( K \) of \( R \) such that \( I \cap K = 0 \) but \( I + K \) is essential. By [1, Proposition 1.8b], there exist prime ideals \( P_1, \ldots, P_n \) of \( R \) such that \( (I + K) \cap \mathfrak{m}(P) \neq \emptyset \) for each \( P \in \mathcal{P} \setminus \{P_1, \ldots, P_n\} \). Equivalently, \( h(I + K, P) = \rk(R/P) = t_p \rk(R) \). Now, Lemma 2.2(ii) says that \( h(I, P) \leq t_p \rk I \) and \( h(K, P) \leq t_p \rk K \). Combining these facts gives \( h(I, P) = t_p \rk I \) (and \( h(K, P) = t_p \rk K \)) for each reflexive prime ideal \( P \in \mathcal{P} \setminus \{P_1, \ldots, P_n\} \).

We are now ready to prove the main result of this section, for which we need the following notation. Let \( M \) be a module over a ring \( S \). For \( \alpha \in M \) write \( O_M(\alpha) = \{\theta(\alpha) : \theta \in \mathcal{M}^*\} \). Note that \( O_M(\alpha) \) is a left ideal of \( S \).

**Proposition 2.4.** Let \( M \) be a torsion-less right module of finite rank over a prime Krull ring \( R \). Pick an integer \( r \) with \( r \leq \rk M \). Then there exists \( \alpha \in M \) such that:

(i) \( \rk(O_M(\alpha)) = \min\{r, \rk R\} \);

(ii) \( h(O_M(\alpha), P) \geq \min\{(r - 1) t_p, \rk R/P\} \) for every \( P \in \mathcal{P} \). Furthermore, \( \alpha \) can be chosen from any submodule \( N \) of \( M \) that satisfies:

(iii) \( \rk(N) \geq r \);

(iv) \( \rk(N + MP/MP, P) \geq rt_p \) for every \( P \in \mathcal{P} \).

**Remark.** If \( M = N \) and \( R \) is a domain, then our hypotheses say that \( \rk M \geq 2 \rk R \), and the result is known; see, for example, [2]. However, in order to prove Bourbaki's Theorem for prime Krull rings, it is the case \( \rk R < \rk M \leq 2 \rk R \) that is important and this is rather more subtle. The case where \( N \neq M \) will have some amusing consequences, but is not needed for the proof of Bourbaki's Theorem.

**Proof.** The proof is essentially by induction on \( r \). Formally, let \( 1 \leq s \leq r - 1 \) be an integer. Assume that there exists \( \alpha \in N \) and \( \theta_1, \ldots, \theta_s \in \mathcal{M}^* \) such that, if \( V = \sum R\theta_i(\alpha) \), then:

(a) \( \rk V = \min\{s, \rk R\} \);

(b) \( h(V, P) \geq \min\{(s - 1) t_p, \rk R/P\} \) for each \( P \in \mathcal{P} \).

(c) If \( K = \bigcap \ker \theta_i \), then \( \rk K \geq \rk M - s \).
(We note that the case \( s = 1 \) does hold. Let \( \alpha \) be any non-zero element of \( N \) and \( \theta \) any element of \( M^* \) such that \( \theta(\alpha) \neq 0 \) but \( \text{rk}(\text{Ker} \ \theta) = \text{rk} \ \text{M} - 1 \). It is readily checked that these elements satisfy (a), (b), (c). Alternatively, one could begin the induction at \( s = 0 \).)

In order to prove the proposition, we suppose that \( \alpha \in N \) satisfies (a), (b) and (c), and show how to modify \( \alpha \) so that it satisfies these conditions with \( s \) replaced by \( s + 1 \). We begin by noting some consequences of these conditions. First, \( M/K \) is torsion-free. For, if \( m \in M \) and \( c \in \mathfrak{C}(O) \) satisfy \( mc \in K \), then \( \theta_i(m) c = 0 \) for each \( i \), whence \( \theta_i(m) = 0 \) and \( m \in K \). Thus, by Lemma 2.2(ii), \( \text{rk}(K + MP/MP, P) = t_p(\text{rk} \ K) \) for each \( P \in \mathcal{P} \). Thus condition (c) implies

\[
(\text{d}) \quad \text{rk}(N \cap K) \geq \text{rk} N - s \geq r - s.
\]

\[
(\text{e}) \quad \text{rk}(N \cap K + MP/MP, P) \geq (r - s) t_p \quad \text{for all} \quad P \in \mathcal{P}.
\]

Secondly, by Lemma 2.3, \( V \) also satisfies:

\[(\text{f}) \quad \text{There exist reflexive, non-zero prime ideals} \quad P_1, \ldots, P_m \quad \text{of} \quad R \quad \text{such that} \quad \mathfrak{h}(V, P) = \min \{ \text{rk} R/P, st_p \} \quad \text{for all} \quad P \in \mathcal{P} \setminus \{ P_1, \ldots, P_m \}.\]

Let \( T = P_1 \cap \cdots \cap P_m \). By Lemma 1.2, \( T \) is localisable and \( R_T \) is a PIR. Note that the non-zero prime ideals of \( R_T \) are just the \( P_R \). By condition (e), \( \text{rk}(K \cap N + MP_i/MP_i) \geq t_p \) for \( 1 \leq i \leq m \). Thus, by Lemma 2.2(iv), there exists \( \beta \in (K \cap N) \) such that \( \beta R_T \) is a uniform, direct summand of \( M_T \); say, \( M_T = \beta R_T \oplus L \). By replacing \( \beta \) by \( \beta c \) for some \( c \in \mathfrak{C}(T) \), we may suppose that \( \beta \in N \cap K \). By Lemma 2.2(iii), there exists \( \phi \in (M_T)^* \) such that \( \phi(L) = 0 \) but \( \text{rk}(R_T \phi(\beta), P_i R_T) = t_p \) for \( 1 \leq i \leq m \). By Lemma 1.2, there exists \( c \in \mathfrak{C}(T) \) such that \( c \phi \in M^* \) and, again, we can, without loss of generality, replace \( \phi \) by \( c \phi \). Note that this means that we can reinterpret the above observations down in \( M \). In particular, \( \text{rk}(R \phi(\beta), P_i) = t_p \), for \( 1 \leq i \leq m \), and \( \text{rk} \ R \phi(\beta) = \text{rk} \ R \beta R = 1 \).

Now apply the left-hand version of Lemma 2.1, with \( X = V \subseteq Y = R \), \( a_i = \phi(\alpha_i) \), \( a_i = 1 \), \( b = \phi(\beta) \) and \( \{ Q_1, \ldots, Q_n \} = \{ P_1, \ldots, P_m, 0 \} \). This provides \( \lambda \in R \) such that for \( 1 \leq i \leq n \),

\[
\text{rk}(V + R \phi(\alpha + \beta \lambda), Q_i) = \min \{ \text{rk} R/Q_i, \text{rk}(V, Q_i) + \text{rk}(R \phi(\beta), Q_i) \}.
\]

Now \( \beta \in \text{Ker} \ \theta_i \) for \( 1 \leq i \leq s \) and so \( \sum R \theta_i(\alpha + \beta \lambda) = \sum R \theta_i(\alpha) - V \). Set \( \theta_{i+s} = \phi \) and \( W = \sum_{i=1}^{s+1} R \theta_i(\alpha + \beta \lambda) \). Then (3), combined with (a), (b) and (f), respectively, gives

\[
(\text{a'}) \quad \text{rk}(W) = \min \{ \text{rk} R, s + 1 \},
\]

\[
(\text{b'}) \quad h(W, P_i) = \min \{ \text{rk} R/P_i, st_p \} \quad \text{for all} \quad 1 \leq i \leq m \quad \text{and}
\]

\[
(\text{b''}) \quad h(W, P) \geq h(V, P) = \min \{ st_p, \text{rk} R/P \} \quad \text{for all} \quad P \in \mathcal{P} \setminus \{ P_1, \ldots, P_m \}.
\]
Finally, since $\phi(L \cap M) = 0$, $rk(\Ker \phi) = rk M - 1$ and so $rk(\bigcap_{i=1}^{s+1} \Ker \theta_i) \geq rk M - s - 1$ by (c). This therefore completes the inductive step and the proof is complete.

3. BOURBAKI'S THEOREM AND OTHER APPLICATIONS

The main aim of this section is to prove the theorem of the introduction. This will be an easy corollary of the following result.

**Theorem 3.1.** Let $R$ be a Krull ring and $M$ a torsion-less right $R$-module of finite rank, but with $rk M > rk R$. Then there exists $\alpha \in M$ such that

$$O_M(\alpha) = R.$$  

Furthermore, $\alpha$ may be chosen from any submodule $N$ of $M$ which satisfies

(i) $rk N \geq rk R + 1$ and (ii) $rk(N + MP/MP, P) \geq rk R/P + t_P$ for all $P \in \mathcal{P}$. 

**Remark.** If one is just interested in Bourbaki's Theorem, then the submodule $N$ can be ignored.

**Proof.** Throughout the proof $O(x)$, for $x \in M$, will denote $O_M(x)$. By Proposition 2.4, there exists $\alpha \in N$ such that $h(O(x), P) = rk R/P$, for all $P \in \mathcal{P}$ (including $P = 0$). In particular, there exists $\theta \in M^*$ such that $\theta(\alpha) \in \mathcal{G}(0)$. By Lemma 2.3, pick non-zero, reflexive prime ideals $P_1, \ldots, P_m$ such that $\theta(\alpha) \notin \mathcal{G}(P)$ for all $P \in \mathcal{P} \setminus \{P_1, \ldots, P_m\}$. Set $T = P_1 \cap \cdots \cap P_m$. Note that, as $rk N > rk R$, $N \cap \Ker \theta \neq 0$. Pick $\beta \in (N \cap \Ker \theta) T$ with $\beta \neq 0$. Set $K = R\theta(\alpha)$. Since $K$ is an essential left ideal, Lemma 1.1 implies that there exists a saturated chain of reflexive left ideals

$$K_0 = R \supset K_1 \supset \cdots \supset K_n = K.$$  

(In other words, for each $i$, $K_i$ is minimal among reflexive left ideals that strictly contain $K_{i+1}$.) The proof of the theorem will be by induction on this chain:

**Sublemma 3.2.** Let $0 \leq r \leq n$ be an integer. Then there exists $\lambda \in R$ such that $\{K_r + O(\alpha + \beta \lambda)\}^* = R$.

Observe that the sublemma proves the theorem. For, take $r = n$. Then $\{K_n + O(\alpha + \beta \lambda)\}^* = R$ for some $\lambda \in R$. However, as $\beta \in \Ker \theta$, $R\theta(\alpha + \beta \lambda)^* = K_n$. Thus $O(\alpha + \beta \lambda)^* = R$, as required.

**Proof of Sublemma 3.2.** The sublemma obviously holds for $r = 0$, so suppose that it holds for $r = s - 1$. Pick $\lambda \in R$ such that $\{K_{s-1} + O(\alpha + \beta \lambda)\}^* = R$. Now, since $\beta \in \Ker \theta$, $O(\alpha + \beta \lambda) \cap \mathcal{G}(P) \neq \emptyset$ for
each \( P \in \mathcal{P} \), except possibly for \( P = P_i, 1 \leq i \leq m \). However, for \( 1 \leq i \leq m \) there does exist \( \phi_i \in M^* \) such that \( \phi_i(\alpha) \in \mathcal{C}(P_i) \), by the initial choice of \( \alpha \). Since \( \beta \in M \), \( \phi_i(\beta) \in P_i \) and \( \phi_i(\alpha + \beta \lambda) \in \mathcal{C}(P_i) \). Thus \( O(\alpha + \beta \lambda) \cap \mathcal{C}(P) \neq \emptyset \) does hold for all \( P \in \mathcal{P} \). Let \( P = \text{l-ann } K_{s-1}/K_s \). By Lemma 1.4(ii), \( P \) is a reflexive prime ideal of \( R \).

Suppose, first, that \( P \neq 0 \). Certainly \( P(K_{s-1} + O(\alpha + \beta \lambda)) \subseteq (K_s + O(\alpha + \beta \lambda))^* \). Thus, by Lemma 1.4, again,

\[
P = PR = P(K_{s-1} + O(\alpha + \beta \lambda))^* \subseteq (K_s + O(\alpha + \beta \lambda))^*.
\]

But, \( O(\alpha + \beta \lambda) \cap \mathcal{C}(P) \neq \emptyset \). Thus, by Lemma 1.5, \( (K_s + O(\alpha + \beta \lambda))^* = R \), as required.

This leaves the case \( P = 0 \). Pick \( \psi \in M^* \) such that \( \psi(\beta) \neq 0 \) and \( f \in \mathcal{C}(0) \) such that \( f\psi(\alpha + \beta \lambda) \in K_s \). Now \( Rf\psi(\beta) R \) is a two-sided ideal of \( R \) and so \( Rf\psi(\beta) K_{s-1} \nsubseteq K_s \). Thus there exists \( g \in K_{s-1} \) such that \( f\psi(\beta) g \in K_{s-1} \setminus K_s \). Now

\[
K_s + O(\alpha + \beta \lambda + \beta g) \supseteq K_s + Rf\psi(\alpha + \beta \lambda + \beta g)
\]

\[
= K_s + Rf\psi(\beta) g \supseteq K_s.
\]

By construction, \( Rf\psi(\beta) g \in K_{s-1} \) which is minimal among reflexive left ideals containing \( K_s \). Thus (4) implies that \( (K_s + O(\alpha + \beta \lambda + \beta g))^* \supseteq K_{s-1} \). Therefore, by induction and the fact that \( g \in K_{s-1} \),

\[
(K_s + O(\alpha + \beta \lambda + \beta g))^* \supseteq (K_{s-1} + O(\alpha + \beta \lambda + \beta g))^*
\]

\[
= (K_{s-1} + O(\alpha + \beta \lambda))^* = R,
\]

as required.

Chamarie in [2] shows that Bourbaki's theorem follows easily from his version of Theorem 3.1. Using the same proof, we obtain:

**Theorem 3.3.** Let \( N \subset M \) be torsion-less right modules of finite rank over a prime Krull ring \( R \). Suppose that \( \text{rk}(N + MP/MP, P) = \text{t}_P \text{rk}(N) \) for all \( P \in \mathcal{P} \). Then there exists a free submodule \( F \) of \( N \) such that \( N/F \) is isomorphic to a right ideal of \( R \) and \( M/F \) is torsion-less.

**Proof.** Clearly, we may suppose that \( \text{rk}(N) > \text{rk}(R) \). Thus, by Theorem 3.1, there exists \( \alpha \in N \) such that \( O_{M}(\alpha)^* = R \). We claim that \( M/\alpha R \) is torsion-free. For, suppose that \( yc = zd \) for some \( y \in M, c \in \mathcal{C}(O) \) and \( d \in R \). Then, for any \( \theta \in M^* \),

\[
\theta(\alpha) dc^{-1} = \theta(y) \in R.
\]

Thus, \( dc^{-1} \in O(\alpha)^* = R \) and \( y \in \alpha R \), as required. Hence \( M/\alpha R \), and
therefore \( N/\alpha R \), are torsion-free. Thus these modules are torsion-less (use the argument of [2, Proposition 4.3.5]). Set \( \overline{N} = N/\alpha R \subset \overline{M} = M/\alpha R \). For every \( P \in \mathcal{P} \),

\[
\text{rk}(\overline{N} + \overline{MP}/\overline{MP}, P) \geq \text{rk}(N + MP/MP, P) - \text{rk}(\alpha R, P)
\]

\[
= t_P \text{rk}(N) - t_P \text{rk} R = \text{rk}(\overline{N}),
\]

where the final inequality comes from Lemma 2.2(ii). So we may imply induction to find a free submodule \( F \) of \( \overline{N} \) such that \( \overline{N}/F \) is isomorphic to a right ideal of \( R \), and \( \overline{M}/F \) is torsion-less. Finally, as \( \alpha R \simeq R \), the inverse image \( F \) of \( F \) in \( N \) is still free. And the proof is complete.

**Corollary 3.4** (Bourbaki's Theorem). Let \( M \) be a torsionless right module of finite rank over a prime Krull ring \( R \). Then there exists a short exact sequence

\[
0 \to F \to M \to I \to 0,
\]

where \( F \) is a finitely generated free right \( R \)-module and \( I \) is a right ideal of \( R \).

**Proof.** Apply Theorem 3.3 with \( N = M \). (Note that \( \text{rk}(M + MP/MP, P) = t_P \text{rk} M \) does hold by Lemma 2.2(ii).)

The next corollary shows why we have been interested in the submodule \( N \) in the earlier results.

**Corollary 3.5.** Let \( R \) be a prime Krull ring with no reflexive ideals (apart from zero), and let \( T \) be a finitely generated, torsion right \( R \)-module. Then \( T \simeq J/I \) where \( I \subset J \) are uniform right ideals of \( R \).

**Proof.** Write \( T = A/B \) for some finitely generated, free right \( R \)-module \( A \) and some \( B \subset A \). Let \( K \) be a uniform right ideal of \( R \). Then \( T \simeq A \oplus K/B \oplus K \). Clearly, \( \text{rk}(A \oplus K) = \text{rk}(B \oplus K) \). Thus, by Theorem 3.3, there exists a free submodule \( F \) of \( B \oplus K \) such that \( (B \oplus K)/F \simeq I \) is a right ideal of \( R \) and \( J = (A \oplus K)/F \) is torsion-less. A dimension count shows that \( I \)--and therefore \( J \)--is uniform, as required.

It is clear that Corollary 3.5 fails when \( R \) has reflexive prime ideals. For, if \( P \) is a reflexive, nonzero, prime ideal of \( R \), then \( R/P \oplus R/P \) cannot be realised as a subfactor of \( R \) (use Lemma 1.3). However, this is really the only counterexample, since Corollary 3.5 easily generalises to give:

**Corollary 3.6.** Let \( R \) be a prime Krull ring and \( T \) a finitely generated torsion right \( R \)-module, such that \( \text{rk}(T, P) = 0 \) for all \( P \in \mathcal{P} \). (For example, take \( T = (R/Q)^{(n)} \) for any non-reflexive, prime ideal \( Q \) of \( R \).) Then \( T \simeq J/I \) for some reflexive right ideals \( I \subset J \) of \( R \).
The proof of Corollary 3.6 is left to the reader, but we note the following amusing consequence.

**Corollary 3.7.** Let $R$ be a prime Krull ring. Suppose that there exists a uniform bound on the number of generators of one-sided ideals of $R$. Then $R$ is a (Noetherian) Asano order. The converse holds if $R$ has finite (Rentschler–Gabriel) Krull dimension.

**Proof.** Suppose that such a bound exists. By Corollary 3.6, $R$ is a Noetherian Krull ring in which every prime ideal is reflexive. Thus, every prime ideal is maximal. If $P$ is a prime ideal of $R$, it follows that $P^*P \supseteq P$ (use, for example, Lemma 1.3) and therefore that $P^*P = R$. So every prime ideal is invertible. An easy induction shows that every non-zero ideal is invertible, which is equivalent to $R$ being an Asano order [5, Proposition 2.1]. The converse is [6, Theorem 5.4].

There are several further results in [2], which are proved under the restriction $K \dim_R R = 1$, that can be proved in general using the methods of this paper. For example:

**Proposition 3.8.** Let $I$ be a right ideal of a Krull ring $R$ and $J$ an essential submodule of $I$. Then there exists $y \in I$ such that $I^{**} = (J + yR)^{**}$.

The proof of Proposition 3.8 is left to the reader, since it is a case of redoing the proof of [2, Corollaire 4.2.5] inside the category of $R$-modules, and is similar to the proof of Theorem 3.1.

**Corollary 3.9.** Let $J \subseteq I$ and $K$ be essential, reflexive right ideals of a prime Krull ring $R$. Then there exists $x \in Q(R)$ such that $J = I \cap xK$.

**Proof.** Use Proposition 3.8 in the proof of [2, Corollaire 4.2.6].

**References**