On solving vague systems of linear equations with pattern-shaped columns

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Abstract

In [J. Nedoma, Ann. Oper. Res. 47 (1993) 483–496], a method for optimizing a linear function on the solution set of a vague system of linear equations was proposed. In this paper, we deal with the so-called simultaneous version of this method. We show that it works provided the system matrix is pattern-shaped. In this case, moreover, the iteration process is extremely simple, not requiring any matrix operation after being started. The application of this method to the mentioned optimization problem is straightforward provided a list of the space orthants covering the solution set is given. A way of finding this list is outlined briefly. © 2001 Elsevier Science Inc. All rights reserved.

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1. Introduction

The goal of the sign-accord algorithm (SAA), proposed by Rohn [10], is to find a specific vertex of the solution set of a system of interval linear equations by using a way of sequential improving a given representative of the interval system matrix. The algorithm is finite provided the interval system matrix is regular, not containing any singular matrix. Using SAA for optimizing a linear function on the solution set, we need at most $2^n$ of its applications, and a similar assertion holds for finding
the interval hull. In [10–12], several methods for reducing the necessary number of applications were developed. One of the simplest cases occurs if the solution components are sign-stable. Then we obtain the interval hull after $2^n$ applications of the algorithm.

In [4,7,8], a method for optimizing a linear function on the solution set of a more general vague linear system is described. It is based on sequential optimizing an auxiliary linear function on the column sets of the vague system matrix. Let us refer to it as to the column optimization method (COM). The specification of COM to an interval problem is identical with a modification of SAA. Both of these methods exist in two versions.

- **Lexicographic version:** In each step, we modify one column only using a systematic rule for its choice. Namely, we can choose the least index of the set of “unsatisfying column indices”. This version works provided the interval (vague) matrix is regular [4,8,10].

- **Simultaneous version:** All the column representatives are modified in each step. The correctness of this version is proved provided the interval matrix is $\sigma$-regular, i.e., there exists a right-hand side vector which provides sign-stable solutions [4,7]. This assumption, however, is not necessary. In fact, there has not yet been found any regular interval matrix such that the simultaneous version fails. If the vague matrix is regular, local convergence of the simultaneous version is guaranteed [8]. Of course, regularity of the system matrix cannot be verified before starting COM. As was shown in [9], testing regularity of an interval matrix is NP-hard. If the simultaneous version of COM does not converge, there is a good reason to suppose that the system matrix is singular. To verify this suspicion, we can switch to the lexicographic version. On the other hand, a successful application of COM does not exclude possible singularity.

The main purpose of this paper is to show that the simultaneous version of COM converges provided the vague matrix is regular and pattern-shaped. Consequently, this method is finite for any regular interval matrix which can be brought to cubic form by scaling. Moreover, the respective specification of COM is extremely simple, not involving any matrix operation in the current steps. By a systematic application of this algorithm, the interval hull of the solution set of the respective system of linear equations can be found. The method represents a powerful instrument for evaluation of the errors of the solutions of problems with inexact data.

2. Square vague system of linear equations

Let $\mathcal{A}^j \subset \mathbb{R}^n$, $j \in J = \{1, \ldots, n\}$, be nonvoid compact convex sets. The set of matrices

$$A^V = \{A \mid A = (a^1, \ldots, a^n), \; a^j \in \mathcal{A}^j, \; j = 1, \ldots, n\}$$
is called an \((n \times n)\)-vague matrix (V-matrix). A vague vector \(b^V\) is a nonvoid compact convex subset of \(\mathbb{R}^n\). The elementary algebraic operations with vague matrices are defined by using the natural way: the operation is applied to all their representatives.

Let us denote

\[ A^V x = b^V \leftrightarrow \exists(A \in A^V, \ b \in b^V): Ax = b, \]

\[ X(A^V, b^V) = \{x \mid A^V x = b^V, \ x \in \mathbb{R}^n\}, \]

\[ \sigma = (\sigma_1, \ldots, \sigma_n) \in \{ \pm 1 \}^n, \ T_\sigma = \text{diag} \, \sigma, \]

\[ X_\sigma(A^V, b^V) = X(A^V, b^V) \cap R^n_\sigma, \ R^n_\sigma = \{x \mid T_\sigma x \geq 0, \ x \in \mathbb{R}^n\}, \]

where \(X_\sigma(A^V, b^V)\) is a convex set for any \(\sigma \in \{ \pm 1 \}^n\), regardless of the rank of \(A^V\).

Let us recall Proposition 3.1. in [5] that is formulated for a right-hand side which consists of only one vector. Applying it to the \((n+1) \times (n+1)\) system with the extended matrix

\[
\begin{pmatrix}
A^V & -b^V & 0 \\
0 & 1 & 1
\end{pmatrix},
\]

we obtain the result mentioned.

The basic problem of describing the solution set of a system \(A^V x = b^V\) is optimizing a linear function on \(X(A^V, b^V)\).

**Problem I.**

\[ c^T x \rightarrow \max \]

\[ \text{s.t.} \ x \in X(A^V, b^V). \]

The feasible set of this problem is nonconvex in general as is well known. Provided \(A^V\) is a regular vague matrix, \(X(A^V, b^V)\) is compact and connected as the continuous map of \(b^V\).

Let \(c \in \mathbb{R}^n\) and \(\sigma \in \{ \pm 1 \}^n\) be given.

**Problem II.** Find \(z^* \in \mathbb{R}^n, \ A_* \in A^V\) such that

\[ (A_*)^T z^* = c, \quad (1) \]

\[ T_\sigma A^T z^* \geq T_\sigma c \quad \forall A \in A^V. \quad (2) \]

If \(A^V\) is a regular vague matrix, then Problem II has a solution and \(z^*\) is determined uniquely. (The proof is similar to the proofs of Theorem 2 in [4] and Theorem 1 in [8].)
Lemma 1. Let \((z(c, \sigma), A(c, \sigma))\) be a solution of Problem II for \(c \in \mathbb{R}^n\) and \(\sigma \in \{-1, 1\}^n\). Then
\[
e^T x \leq b^T z(c, \sigma) \quad \forall x \in X_\sigma(A^V, b).
\] (3)

Proof. Relation (2) yields
\[
e^T x = (c^T T)(T^T x) \leq (z(c, \sigma))^T A T \sigma x = (z(c, \sigma))^T A x
\]
\[= b^T z(c, \sigma) \quad \forall A \in A^V \text{ such that } x = A^{-1} b \in \mathbb{R}^n. \]

The following theorem shows that Problem I can be managed by solving a set of Problems II.

Theorem 1. Assume that \(A^V\) is regular and let \(S\) be a list of sign vectors such that
\[
X(A^V, b^V) \subset \bigcup \{\mathbb{R}^n_\sigma \mid \sigma \in S\}. \quad (4)
\]
Then
\[
\max \{e^T x \mid x \in X(A^V, b^V)\} = \max_{\sigma \in S} \max \{b^T z(c, \sigma) \mid b \in b^V\}. \quad (5)
\]

Proof. Due to the regularity of \(A^V\), \(z(c, \sigma)\) exists for any \(c \in \mathbb{R}^n\), \(\sigma \in \{-1, 1\}^n\). According to Lemma 1, assumption (4) implies that
\[
e^T x \leq \max_{\sigma \in S} b^T z(c, \sigma) \quad \forall x \in X(A^V, b), \forall b \in b^V. \quad (6)
\]
Let \(x(b, \sigma)\) be the solution of \(A(c, \sigma)x = b\) for a given triad \((b, c, \sigma)\). Due to (1) we have \(b^T z(c, \sigma) = (z(c, \sigma))^T A(c, \sigma)x(b, \sigma) = e^T x(b, \sigma).\) Thus, for any \(b \in b^V, \sigma \in S\), the value \(b^T z(c, \sigma)\) is actually reached by \(e^T x\) in \(X(A^V, b^V)\). \(\square\)

Provided \(A^V\) is regular, we can utilize the fact that \(X(A^V, b)\) is connected. Applying Lemma 1 to \(c = e^j\), the \(j^{th}\) unit vector, we obtain a guide for constructing the list \(S\). The idea consists in sequential adding suspicious neighbours to a partial list.

Lemma 2. Assume that \(A^V\) is regular and let the following conditions be satisfied for a list \(S\) of sign-vectors:
\[
\exists \sigma \in S: X_\sigma(A^V, b^V) \neq \emptyset, \quad (7)
\]
\[
\max \{b^T z(-\sigma_j e^j, \sigma) \mid b \in b^V\} < 0
\]
\[\quad \forall \sigma \in S, \quad j \in J \text{ such that } (\sigma_1, \ldots, -\sigma_j, \ldots, \sigma_n) \notin S. \quad (8)
\]
Then, inclusion (4) holds.

Proof. Let us choose \(\sigma \in S, \sigma^* \notin S\) such that \(\sigma^* = (\sigma_1, \ldots, -\sigma_j, \ldots, \sigma_n)\). According to (3), assumption (8) implies that \(-\sigma_j x_j < 0,\) and hence, \(|x_j| > 0 \forall x \in\)
Thus, we have a hint for completion of $S$: if $\max \{b^T z(−σ_j e^j, σ) | b \in b^V \} ≥ 0$ for a pair $(σ \in S, j \in J)$, the vector $σ^* = (σ_1, \ldots, −σ_j, \ldots, σ_n)$ is added to $S$. Let us note that the list $S$ can contain redundant items.

The methodology of constructing $S$ can be interpreted as an application of the well-known graph search method for finding connected components [3]. A similar approach is used in [1, 2], where a method for finding the precise list of the relevant orthants is described. It consists in solving a sequence of linear programming problems of the size $2n \times n$. This method, however, is proposed for systems of interval linear equations only and can be hardly applied to more general vague systems.

3. Systems with pattern-shaped columns

Problem II can be solved by using the COM mentioned above. It is an iterative method which requires solving a linear equation system in each step. Here, we will discuss a specific case of this problem. Let $p$ be a constant positive vector and $d^V$ be a vague vector such that $0 \in d^V$ and $d^V = −d^V$. Let us consider the following vague matrix, defined as a pseudoproduct of $d^V$, $p^T$:

$$d^V \circ p^T = \{(d^1, \ldots, d^n) | d^j \in p_j d^V, \ j \in J\}.$$ 

Provided $d^V \neq \emptyset$, the maximal rank of the matrices belonging to $d^V \circ p^T$ is equal to $n$ because their columns are chosen from $p_j d^V$ independently of each other. On the other hand, $d^V p^T = \{(p_1 d, \ldots, p_n d) | d \in d^V\}$ consists of matrices the rank of which is less than or equal to 1. Evidently, we have $d^V p^T \subset d^V \circ p^T$.

Definition 1. For any “central matrix” $A_0$, 

$$A^P = A_0 + d^V \circ p^T \quad (9)$$

is called a vague matrix with pattern-shaped columns (abbreviated: pattern-shaped matrix).

The simplest example of $A^P$ is the following interval matrix:

$$A^I = (a_{i,j}), \quad a^0_{i,j} − r_i p_j ≤ a_{i,j} ≤ a^0_{i,j} + r_i p_j; \quad r_i ≥ 0, \ p_j > 0. \quad (10)$$

The columns $A^j$ of a more sophisticated pattern-shaped elliptic matrix $A^E$ are defined as follows:

$$A^j = \{(a^j - a^{0j})^T H (a^j - a^{0j}) ≤ (p_j)^2\},$$

where $H$ is a given positive definite matrix and $a^{0j}$ is the $j$th column of $A_0$. 

$X_σ(A^V, b^V)$. It means that $R^n_σ \cap X_σ(A^V, b^V) = \emptyset \ \forall σ \in S$. Since $X(A^V, b^V)$ is a connected set, there are no solution parts $X_σ(A^V, b^V)$ but those being registered in $S$. □
In order to formulate Problem II for a pattern-shaped vague matrix, let us consider a representative \( A \in A^P \), i.e., \( A = A_0 + D, \ D \in d^Y \circ p^T \). For a solution \( (z^*, A_* = A_0 + D_*) \) of Problem II, we have \( (A_0 + D_*)^T z^* = c \) and \( T_\sigma (A_0 + D)^T z^* \geq T_\sigma (A_0 + D_*)^T z^* = T_\sigma c \ \forall D \in d^Y \circ p^T \). Reducing the latter relation, we obtain the following.

**Problem III.** Find a pair \( (z^*, D_*) \), \( D_* \in d^Y \circ p^T \) such that
\[
(A_0 + D_*)^T z^* = c, \tag{11}
\]
\[
T_\sigma D^T z^* \geq T_\sigma D_*^T z^* \quad \forall D \in d^Y \circ p^T. \tag{12}
\]

**Lemma 3.** Let Problem III have a solution \( (z^*, D_*) \). Then there exists a solution \( z, Y \) such that \( Y = D_* p^T \) for \( p^T \) such that
\[
(A_0 + C^T D_*) z^* = c; \tag{14}
\]
\[
\sup_{x \in X(A^P, b)} c^T x \leq b^T z^* \leq \sup_{x \in X(A^P, b)} c^T x \quad \forall b \in \mathbb{R}^n.
\]

This plain lemma has an important implication: Problem III can be reduced to a simpler problem with a rank-one error matrix:
\[
A^R_\omega = A_0 + d^Y \circ p^T T_\sigma. \tag{13}
\]
As \( d^Y \equiv -d^Y \), we have \( A^R_\omega = A^R_{\omega}\). Thus, in a given pattern-shaped matrix, there exist \( 2^{n-1} \) inner matrices of type (13).

Let us denote
\[
c^\sigma = T_\sigma c, \quad p^\sigma = T_\sigma p \quad \text{and} \quad A_{\gamma \sigma} = A_0 + y (p^\sigma)^T \quad \text{for} \ y \in d^Y. \tag{14}
\]

According to Lemma 3, Problem III is equivalent to the following.

**Problem IV.** Find \( z^* \in \mathbb{R}^n, \ y^* \in d^Y \) such that
\[
A_{\gamma \sigma}^T z^* = c, \tag{15}
\]
\[
y^T z^* \geq (y^*)^T z^* \quad \forall y \in d^Y. \tag{16}
\]

Using Lemmas 3 and 1, we can formulate the following.

**Corollary 1.** If \((z^*, y^*)\) is a solution of Problem IV, then
\[
\sup \{ c^T x \mid x \in X_\sigma (A^P, b) \} \leq b^T z^* \leq \sup \{ c^T x \mid x \in X(A^P, b) \} \quad \forall b \in \mathbb{R}^n.
\]
Linear equation systems with restricted-rank error matrices are studied in [6]. Let us reformulate some of the results in terms of our problem.

**Lemma 4.** If $A^{R1}_0$ is regular, then for an arbitrary $b \in \mathbb{R}^n$, the expression $(p^\sigma)^T x$ is sign-stable for all $x$'s from $X(A^{R1}_0, b)$.

**Proof.** Let us choose a $y \in d^Y$. Using the well-known “rank-one change” formula, we have

$$A_{y\sigma}^{-1} = A_0^{-1} - \left(1 + (p^\sigma)^T A_0^{-1} y\right)^{-1} A_0^{-1} y (p^\sigma)^T A_0^{-1}.$$  \hfill (17)

Thus, the solution $x$ of $A y x = b$ can be expressed as follows:

$$x = A_{y\sigma}^{-1} b = x^0 - \frac{(p^\sigma)^T x^0}{1 + (q^\sigma)^T y} A_0^{-1} y,$$

where

$$x^0 = A_0^{-1} b, \quad q^\sigma = (A_0^T)^{-1} p^\sigma.$$

Hence,

$$\left(p^\sigma\right)^T x = \left(p^\sigma\right)^T x^0 - \left(p^\sigma\right)^T x^0 \left(q^\sigma\right)^T y = \frac{(p^\sigma)^T x^0}{1 + (q^\sigma)^T y} \quad \forall x \in X(A^{R1}_0, b).$$

Regularity of $A^{R1}_0$ and $0 \in d^Y$ guarantee that the denominator of this fraction is positive for $y \in d^Y$. \hfill \square

**Theorem 2.** Assume that
(i) $A^{R1}_0$ is regular;
(ii) $X_\sigma(A^{R1}_0, b) \neq \emptyset$ for a given $b$;
(iii) $(z^*, y^*)$ is a solution of Problem IV.

Then the solution $x^*$ of the equation $(A_0 + y^* (p^\sigma)^T) x = b$ maximizes $c^T x$ on $X(A^{R1}_0, b)$.

**Proof.** Let us assume that $b \neq 0$ (in the opposite case, the theorem holds trivially). Due to (ii) there exists an $x^\sigma \in X_\sigma(A^{R1}_0, b)$. Since $x^\sigma \neq 0$, we have $(p^\sigma)^T x^\sigma = p^T T x^\sigma > 0$. Let us consider $y \in d^Y, x \in \mathbb{R}^n$ such that $(A_0 + y (p^\sigma)^T) x = b$. Due to Lemma 4, $(p^\sigma)^T x > 0$. According to (15) and (16), we have

$$c^T x = (z^*)^T \left(A_0 + y^* (p^\sigma)^T\right) x \leq (z^*)^T \left(A_0 + y (p^\sigma)^T\right) x = (z^*)^T b = (z^*)^T \left(A_0 + y^* (p^\sigma)^T\right) x^* = c^T x^*.$$ \hfill (18)

Thus, $c^T x \leq c^T x^*$ holds for an arbitrary $x \in X(A^{R1}_0, b)$. Since $x^*$ belongs to the same set, the theorem is proved. \hfill \square
Using (17), we obtain

\[
(A_T^T)^{-1}y = (A_0^T)^{-1}y - \frac{(A_0^T)^{-1}p^\sigma y^T(A_T^T)^{-1}c}{1 + (q^\sigma)^T y}
\]

where \(z = (A_0^T)^{-1}c\).

For a given \(z^0 \in \mathbb{R}^n\), let us construct sequences \(z^k \in \mathbb{R}^n, y^k \in d^V\) as follows:

\[
(z^k)^T y^k = \min \left\{ (z^k)^T y \mid y \in d^V \right\},
\]

\[
z^{k+1} = z + \alpha_k q^\sigma,
\]

where \(\alpha_k = -\frac{(z^k)^T y^k}{1 + (q^\sigma)^T y^k}\). \hspace{1cm} (20)

Due to (19), we have

\[
z^{k+1} = (A_T^T)^{-1} c \quad \text{for} \quad A_k = A_0 + y^k (p^\sigma)^T.
\]

Formulae (20) describe a simultaneous version of COM, because the same error vector \(y^k\) is used to modify all the columns of the current matrix.

**Theorem 3.** Let \(A^R_{R1}\) be regular. Then the following assertions hold:

1. there exists a solution \((z^*, D^*)\) of Problem III;
2. the vector \(z^*\) is determined uniquely;
3. the sequence \(z^k\) converges to \(z^*\) regardless of the choice of the initial vector \(z^0\).

**Proof.** (i) Let \(B\) be the set of all \(b\)'s such that \((p^\sigma)^T x > 0\) for all \(x \in X(A^R_{R1}, b)\). Due to Lemma 4, \(B\) is a nonempty open set. Let us choose \(b \in B\) arbitrarily and consider \(x^k = A_k^{-1} b\). Since

\[
(z^k)^T y^k (p^\sigma)^T x^k \geq (z^k)^T y^k (p^\sigma)^T x^k
\]

holds due to (20), the identity \(c = A_k^T z^k + 1 = A_{k-1}^T z^k\) implies that

\[
(z^{k+1})^T A_k x^k = (z^k)^T A_{k-1} x^k
\]

which implies

\[
(z^{k+1})^T A_k x^k = (z^k)^T A_{k-1} x^k
\]

and hence

\[
(z^{k+1})^T b \geq (z^k)^T b \quad \forall b \in B \quad \text{and} \quad k \geq 1.
\]

The sequence \(y^k \in d^V\) is bounded and therefore \(z^k\) is also bounded according to (21). Let us consider two accumulation points \(\tilde{z}, \tilde{z}\) of \(z^k\). Due to (22) we have \((\tilde{z} - \tilde{z})^T b = \)
Let \( y^* \) be an accumulation point of \( \{y^k\} \). Then \((z^*)^T y^* = \min \{ (z^k)^T y \mid y \in d^V \} \) and \((A_0 + y^* (p^*)^T)^T z^* = c\), which means that \((z^*, y^*)\) solves Problem IV. The assertions 1 and 3 are proved.

(ii) Let two solutions \((z^1, D_1)\) of Problem III be given. According to Lemma 3, we can find \( y_1, y_2 \) in \( d^V \) such that \( D_i = y_i (p^*)^T, i = 1, 2 \). Let us choose a \( b \in B \) and denote \( A_i = A_0 + D_i, x^i = A_i^{-1} b, z^i = (A_i^T)^{-1} c \). According to Theorem 2, both of \( x^1, x^2 \) maximize \( c^T x \) on \( X(A_0^1, b) \) and therefore \( c^T x^1 = c^T x^2 \). Since \((z^i)^T A_i x^i = (z^i)^T b = c^T x^i \), we have \( b^T z^1 = b^T z^2 \ \forall b \in B \). Consequently, \( z^1 = z^2 \).  

**Corollary 2.** If \( A^P \) is a regular interval matrix, then process (20) terminates in a finite number of steps.

Notice that the regularity of \( A^P \) is not required in Theorem 3. We assume that the regularity of a “thinner” inaccurate matrix \( A_{\sigma}^R \subset A^P \) only. If this assumption is not satisfied, iterative process (20) can terminate in oscillations. This situation is characterized, as a rule, by alternating the sign of the determinant of the current matrix \( A_{\sigma} \). As follows from (17), \( \text{sgn} (\text{det} A_{\sigma}) = \text{sgn} (\text{det} A_0) \text{sgn} (1 + y^T q^\sigma). \) Thus, the sign-stability of \( \text{det} A_{\sigma} \) can be easily checked in the course of the computations.

If \( A^P \) is regular, then all the inner matrices \( A_{\sigma}^R \) are also regular and algorithm (20) works for any \( \sigma \in \{\pm 1\}^n \). Repeating this process for all the orthants \( R_\sigma, \sigma \in S \), where \( S \) satisfies (4), we obtain the maximal value of \( c^T x \) on \( X(A^P, b^V) \) by applying Theorem 1. Before starting iterations, it is necessary to compute the inverse matrix \( A_{\sigma}^{-1} \) which is subsequently used for computing the vectors \( q^\sigma \) and \( \tilde{z} \). In order to find the interval hull of \( X(A^V, b^V) \), we need to choose \( c = \pm e^i, i = 1, \ldots, n \), sequentially. For this purpose, the iterative process must be realized at most \((2n \times s)\)-times, where \( s \) is the cardinality of \( S \).

Let us briefly discuss two forms of the pattern \( d^V \).

(a) **Cubic columns:**

\[
d^V = \left\{ y \mid |y_i| \leq r, \ i \in J \right\}.
\]

(23)

A solution of the optimization problem in (20) is evident: \( y^k = -r \text{ sgn } z^k_i \). Thus, the respective modification of iteration process (20) reads

\[
z^k_{i+1} = \tilde{z}_i + \alpha_k q^\sigma_i,
\]

where \( \alpha_k = r \left( \sum_j \tilde{z}_j \text{ sgn } z^k_j \right) / \left( 1 - r \sum_j q^\sigma_j \text{ sgn } z^k_j \right), \)

\[
i = 1, \ldots, n, \quad k = 0, 1, \ldots
\]

(24)

This method, of course, can be easily modified for the case of a more general interval matrix, which can be brought to the cubic matrix by scaling.
(b) **Ball-shaped columns:**

\[
d^V = \{y \mid y^Ty \leq r^2\}.
\]

A solution of the problem \(\min((z^k)^T y \mid \sum_j (y_j^k)^2 - r^2 \leq 0)\) is given by the explicit formula

\[
y_i^k = -\beta_k z_i^k \quad \text{for} \quad \beta_k = \begin{cases} \frac{r}{\sqrt{\sum_j (z_j^k)^2}} & \text{if } z_i^k \neq 0, \\ 0 & \text{otherwise}, \end{cases}
\]

which can be readily verified. Substituting (26) into (20), we obtain

\[
z_i^{k+1} = \tilde{z}_i + \alpha_k q_i^\sigma
\]

where \(\alpha_k = \beta_k \left(\sum_j \tilde{z}_j z_j^k\right) \left(1 - \beta_k \sum_j q_j^\sigma z_j^k\right)^{-1}, \quad i = 1, \ldots, n, \quad k = 0, 1, \ldots\)

The described algorithm produces a solution of Problem IV, a specific case of Problem II. Then we can apply the results of Section 2.

**Example.** We shall solve both the cubic and the ball-shaped forms of the system \((A_0 + d^V \circ p^T)x = b\) given by the matrix

\[
\begin{pmatrix}
A_0 & b \\
p^T & \sim
\end{pmatrix} = \begin{pmatrix}
4 & 5 & -12 & \vdots & 6 \\
10 & -5 & 15 & \vdots & 4 \\
-3 & 8 & 6 & \vdots & -2 \\
1 & 1 & 2 & \vdots & \sim
\end{pmatrix}
\]

with the parameter \(r = 2\) in defining relations (23) and (25). The current orthant will be denoted by the symbolic signs of the respective coordinates, i.e., \((+ + +)\) instead of \((+1, +1, +1)\), etc.

(a) **Cubic form:** Since the “central” solution \(x^0\) belongs to the orthant \((+ - +)\), we start with \(S = \{(+ + +)\}. The sequential applications of process (24) to \(\tilde{x} = (A_0^R)^{-1}c\) for \(c = \pm e^i, \ i = 1, 2, 3\), give the following bounds of the variables \(x_1, x_2, x_3\) on the set \(X_\sigma(A^R, b)\):

\[
(+ - +): \uparrow 2.963211 0.26087 2.682275 \\
\downarrow 0.44712 -0.434014 -0.056147
\]

As \(x_2, x_3\) can alternate their signs, the list \(S\) is to be extended by the orthants \((+ + +)\) and \((++ -)\). Applying the same process to these orthants, we obtain:

\[
(+ + +): \uparrow 5.95604 0.857142 6.087907 \\
\downarrow 0.49636 \ldots -0.002047
\]

\[
(+ + -): \uparrow 1.141899 \ldots \ldots \\
\downarrow 0.47753 -0.374888 -0.190164
\]
The missing items have not to be computed: we already know that the respective components are sign-alternating. We must test another orthant:

\[
\begin{align*}
+ &-: \uparrow \begin{pmatrix} 0.864475 & -0.13523 \\ 0.65972 & \cdots \end{pmatrix} \\
&-C &-: \downarrow \begin{pmatrix} -0.008032 \\ 0 \end{pmatrix}
\end{align*}
\]

The latter results seem to be strange: the upper bound of the nonnegative component \(x_2\) is negative. It means that the respective set \(X_\sigma(A^P, b)\) is empty. Thus, \((+ -)\) represents a superfluous item of the list \(S\).

Evidently, there are no other candidates for adding to \(S\). As a result, we have obtained the exact componentwise bounds of the solution of the interval system described above:

\[
(\underline{x}, \overline{x}) = \begin{pmatrix} 0.447117, & 5.956039 \\
-0.43401, & 0.857142 \\
-0.19016, & 6.087907 \end{pmatrix}.
\]

(b) *Ball-shaped form*: As follows from the following results, the list \(S\) consists of two items only:

\[
\begin{align*}
(+ +): & \uparrow \begin{pmatrix} 1.238697 & -0.01793 \\ 0.553900 & -0.355101 \\ -0.062936 & 0.735751 \end{pmatrix} \\
&-C &-: \downarrow \begin{pmatrix} 0.905359 & -0.12434 \\ 0.624480 & -0.283677 \end{pmatrix}
\end{align*}
\]

Hence,

\[
(\underline{x}, \overline{x}) = \begin{pmatrix} 0.553904, & 1.238697 \\
-0.355101, & -0.01793 \\
-0.062936, & 0.735751 \end{pmatrix}.
\]

In order to achieve six valid digits, process (27) does not exceed six steps in this example.

4. Conclusion

The results of this paper provide an operative instrument for several problems which arise in connection with solving square linear equation systems with a given absolute tolerance of inputs. For example, a fixed upper bound can be imposed upon:

- individual input errors (interval matrix);
- the sum of the column component errors (octaedric matrix (see [5]));
- the sum of the column component error squares (ball matrix).

Even more sophisticated pattern types cause no principal difficulties.

The number of algebraic operations which are to be performed in a current step of the iterative process increases linearly with the size \(n\).
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