Design of bounded feedback controls for linear dynamical systems by using common Lyapunov functions

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Abstract For a linear dynamical system, we address the problem of devising a bounded feedback control, which brings the system to the origin in finite time. The construction is based on the notion of a common Lyapunov function. It is shown that the constructed control remains effective in the presence of small perturbations. © 2011 The Chinese Society of Theoretical and Applied Mechanics. [doi:10.1063/2.1101301]

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Consider a linear autonomous dynamical system

\[ \dot{x} = Ax + Bu, \quad x \in V = \mathbb{R}^n, \quad u \in U = \mathbb{R}^m, \] (1)

such that the Kalman controllability condition is met. We want to build a bounded feedback control \( u = u(x) \), which brings an arbitrary state \( x_0 \) to the origin in finite time, provided that \( |x_0| \) is small enough. In other words, the corresponding phase curves of equation \( \dot{x} = Ax + Bu(x) \) with the initial conditions \( x(0) = x_0 \) gets to 0 in finite time. Note that, given a bound \( |u| \leq C \) on control, it is generally impossible to steer any given initial state into the origin.

The problem of feedback control design has been studied, in particular, by V. Korobov, [1] and his paper is a starting point for ours, though our arguments can be hardly put into a direct comparison with that of Ref. [1]. In principle, to get to the zero one can fix \( u \equiv 0 \), and then, by using common Lyapunov functions, the gauge \( \|u\| \leq C \) can be put into a direct comparison with that of Ref. [1]. In principle, to get to the zero one can fix \( u \equiv 0 \), and then, by using common Lyapunov functions, the gauge \( \|u\| \leq C \) can be put into a direct comparison with that of Ref. [1].

The obvious drawback of this approach consists in the great difficulties of implementation: the amount of computations required is prohibitive for a numerical simulation. Therefore we need the feedback control to be devised in such a way as to be easily implementable (constructive). One can see a posteriori that our control algorithm does not require much memory or computational power. To implement it one needs just basic operations of linear algebra plus finding the only root of a scalar monotone function of one variable. Our control is more smooth than the minimum–time one: its only singular point is zero, while the singular locus of optimal control is a singular hypersurface. Moreover, the feedback control \( u \) is locally equivalent to the minimum-time control \( u_{\min} \).

First, we simplify our control system (1). Note that the feedback control problem does not change essentially under transformation \( A \mapsto D^{-1}AD, \; B \mapsto D^{-1}B, \; u \mapsto u \) does not affect the problem. By using these transformations one can bring system (1) to the canonical Brunovsky form [4–6] — a set of independent subsystems of the form \( z^{(k)} = u; z, u \in \mathbb{R}^1 \). Now it suffices to bring each subsystem \( z^{(k)} = u \) to zero by a bounded feedback control.

Thus, the general problem (1) reduces to the case

\[ A = \begin{pmatrix} 0 & 1 \\ 0 & \ddots \\ \vdots & \ddots & 1 \\ 0 & \cdots & 0 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} 0 \\ \vdots \\ \vdots \\ 1 \end{pmatrix}. \] (2)

We introduce a scalar function \( T = T(x) \) which is specified below. System (1), (2) is related to a distinguished function matrix

\[ \delta(T) = \text{diag}(T^{-n}, T^{-n+1}, \ldots, T^{-1}) \]

such that

\[ \delta A \delta^{-1} = T^{-1} A, \quad \delta B = T^{-1} B, \quad \frac{d}{dT} \delta = T^{-1} M \delta, \] (3)

where \( M = -\text{diag}(n, n-1, \ldots, 1) \). This implies immediately that for \( y = \delta x \) we have

\[ \dot{y} = T^{-1} \left( Ay + Bu + \dot{T} M y \right). \] (4)

Here we present the main novelty of the paper: a construction of a common Lyapunov function for two specific stable matrices. Our feedback controls are based on the existence of this function.

In notations (2) consider stable matrices \( \breve{A} \) of the form

\[ \breve{A} = A + BC, \] (5)

where the row-vector \( C = (c_1, \ldots, c_n) \) is regarded as a \( 1 \times n \) matrix. In other words,

\[ \breve{A} = \begin{pmatrix} 0 & 1 & 0 & \ldots & 0 \\ 0 & 0 & 1 & \ldots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \ldots & 1 \end{pmatrix}, \]

\[ c_1 \quad c_2 \quad c_3 \ldots \quad c_n \]

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and the polynomial $f(x) = x^n - \sum_{i=1}^{n} c_i x^{i-1}$ is stable, i.e., all its roots have a negative real part.

**Theorem 1** One can choose the vector $C$ in such a way that the stable matrices $\tilde{A}$ and $M$ possess a common quadratic Lyapunov function: There exist symmetric positive definite matrices $Q, P,$ and $R$ such that

$$Q\tilde{A} + \tilde{A}^* Q = -P, \quad QM + MQ = -R$$

(6)

**Remark.** The vector $C$ can be defined as follows

$$f_{\lambda}(x) = \prod_{k=1}^{n} (x - e^{\lambda k}) = x^n - \sum_{i=1}^{n} c_i x^{i-1}.$$  

Then $C$ fits the theorem, provided that $\lambda > 0$ is large enough.

Now we can define a bounded feedback control $u$ which brings the system (1), (2) to zero in finite time. Put $y = \delta(T)x$, $u = (C, y)$, where (the row-vector) $C$ is chosen in Theorem 1. We define the function $T$ by $T(0) = 0$ and

$$(Qy, y) = 1, \quad \text{where } y = \delta(T)x \text{ if } x \neq 0.$$  

(7)

The definition is correct, since for a fixed $x \neq 0$ the analytic function $\phi(T) = (Q \delta(T)x, \delta(T)x)$ decreases as $T$ increases, and tends to infinity as $T \to 0$, and to zero as $T \to \infty$. Indeed, by virtue of Theorem 1

$$\frac{d}{dT}\phi(T) = T^{-1} ((QM + MQ)y, y) < 0$$

(8)

Moreover, $T$ depends on $x$ analytically if $x \neq 0$, and the condition (7) guarantee the boundedness of $u(x) = (C, y(x))$.

Now it follows from Eqs. (7) and (4) that

$$(y, (Q\tilde{A} + \tilde{A}^* Q)y + \tilde{T}(QM + MQ)y) = 0,$$

or

$$\tilde{T} = -\frac{(y, (Q\tilde{A} + \tilde{A}^* Q)y)}{(y, (QM + MQ)y)}.$$  

In view of the Lyapunov equations (6)

$$\tilde{T} \leq -c,$$

where $c = c(Q)$ is a positive constant. This implies that up to the zero $T$ decreases with a speed separated from 0. Therefore, the motion ends in the zero in finite time $\tau(x)$ which can be estimated as $\tau(x) = O(T(x))$. In its turn, $T(x)$ can be estimated as $O(\tau_{\text{min}}(x))$ so that the time required for getting into zero is of the same order of magnitude as the minimal one. The result of this section can be stated as follows:

**Theorem 2** Suppose $Q(x) = (Qx, x)$ is a common quadratic Lyapunov function for two stable matrices $\tilde{A} = A + BC$ and $M$. Then, condition (7) defines a bounded feedback control $u(x) = (C, y(x)) = (C, \delta(T(x))x)$ bringing any state vector of system (1), (2) to zero in finite time. This time has the same order of magnitude as the minimal one.

**Remark.** Note that the proposed control is global: it is bounded in the whole phase space and brings any initial state of system (1), (2) to zero in finite time. It also remains effective for the system

$$x^{(n)} = u + v$$

under small perturbation $v$.

One can generalize the above first method of control as follows: We again put $y = \delta(T)x$, $u = (C, y)$, but define the function $T$ by condition

$$T^{-2\beta} \langle Qy, y \rangle = 1,$$

(9)

where $\beta \geq 0$ is a new parameter. Introduction of the new parameter does not spoil our previous arguments essentially. The function $\phi_{\beta}(T) = T^{-2\beta} \langle \delta(T)x, \delta(T)x \rangle$ tends to infinity as $T \to 0$, and to zero as $T \to \infty$. Moreover,

$$\frac{d}{dT} \phi_{\beta}(T) = T^{-1-2\beta} \langle (QM + MQ)y, y \rangle.$$  

(10)

where $M_{\beta} = M - \beta I$. If the matrix $Q$ defines a quadratic Lyapunov function for the stable matrix $M_{\beta}$, then we see from (10) that $\phi_{\beta}(T)$ decreases as $T$ increases. This allows us to define the function $T = T(x)$. Similarly to our arguments in the previous section it follows from Eqs. (9) and (4) that

$$\tilde{T} = -\frac{(y, (Q\tilde{A} + \tilde{A}^* Q)y)}{(y, (QM_{\beta} + MQ_{\beta})y)}.$$  

(11)

If the matrix $Q$ defines a common quadratic Lyapunov function for two stable matrices $\tilde{A} = A + BC$ and $M_{\beta} = M - \beta I$ then the above arguments prove that the controlled motion ends in the zero in finite time $\tau(x) = O(T(x))$.

The result of this section can be stated as follows:

**Theorem 3** Suppose $Q(x) = (Qx, x)$ is a common quadratic Lyapunov function for two stable matrices $\tilde{A} = A + BC$ and $M_{\beta} = M - \beta I$. Then, condition (9) defines a bounded feedback control $u(x) = (C, y(x)) = C\delta(T(x))x$ bringing any state vector of the system (1)-(2) to zero in finite time.

**Remark.** Note that Theorem 2 is based on a rather deep Theorem 1. On the contrary, conditions of Theorem 3 can be easily verified in many cases, e.g. if $\beta$ is large, without appealing to any deep result. On the other hand, the time for getting to zero needed by Theorem 3 can be much greater than that in Theorem 2.
The second method of control has an advantage in that it still works under smooth perturbations
\[ \dot{x} = Ax + f(x) + Bu, \quad f(x) = O(|x|^2) \]  \tag{12}
of the control system.

**Theorem 4** Suppose \( Q(x) = (Qx, x) \) is a common quadratic Lyapunov function for two stable matrices \( \tilde{A} = A + BC \) and \( M_\beta = M - \beta I \), and \( \beta > n - 3 \). Then, condition (9) defines a bounded feedback control
\[ u(x) = (C, y(x)) = (C, \delta(T(x))x) \]
bringing any state vector close to zero of the system (12), (2) to zero in finite time.

**Remark.** Thus, the second approach is locally applicable to a nonlinear control system
\[ \dot{x} = F(x) + Bu, \quad F(x) \in C^2 \]
which can be represented in the form (12) in the vicinity of an equilibrium state.

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