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Traveling wave solutions of the (2 + 1)-dimensional Zoomeron equation and the Burgers equations via the MSE method and the Exp-function method

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Abstract The modified simple equation (MSE) method is promising for finding exact traveling wave solutions of nonlinear evolution equations (NLEEs) in mathematical physics. In this letter, we investigate solutions of the (2 + 1)-dimensional Zoomeron equation and the (2 + 1)-dimensional Burgers equation by using the MSE method and the Exp-function method. The competence of the methods for constructing exact solutions has been established.

1. Introduction

The study of the traveling wave solutions for nonlinear evolution equations (NLEEs) plays an important role to look into the internal mechanism of intricate physical phenomena. Most of the physical phenomena such as, fluid mechanics, quantum mechanics, electricity, plasma physics, chemical kinematics, propagation of shallow water waves, and optical fibers are modeled by nonlinear evolution equation and the appearance of solitary wave solutions in nature is somewhat frequent. But, the nonlinear processes are one of the major challenges and not easy to control because the nonlinear characteristic of the system abruptly changes due to some small changes in valid parameters including time. Thus, the issue becomes more complicated and hence ultimate solution is needed. Therefore, the study of exact solutions of NLEEs plays a vital role to understand the physical mechanism of nonlinear phenomena. Advance nonlinear techniques are significant to solve inherent nonlinear problems, particularly those are involving dynamical systems and related areas. In recent years, there become significant improvements in finding the exact solutions of NLEEs. Many effective and powerful methods have been established and improved, such as, the Hirota's bilinear transformation method [1,2], the tanh-function method [3,4], the (G'/G)-expansion method [5–13], the Exp-function method [14–18],

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the homogeneous balance method [19,20], the F-expansion method [21], the Adomian decomposition method [22], the homotopy perturbation method [23], the extended tanh-function method [24,25], the auxiliary equation method [26], the Jacobi elliptic function method [27], the Weierstrass elliptic function method [28], the modified Exp-function method [29], the modified simple equation method [30–33], and so on.

The objective of this article is to look for new use relating to the MSE method and Exp-function method for solving the (2 + 1)-dimensional Zoomeron equation and the (2 + 1)-dimensional Burgers equation and demonstrate the advantage and straightforwardness of these methods. Burgers equation is a fundamental partial differential equation from fluid mechanics. It occurs in various areas of applied mathematics, such as modeling of gas dynamics and traffic flow [34,35].

The article is organized as follows: In Section 2, the MSE method and Exp-function method are discussed. In Section 3, we exert these methods to the nonlinear evolution equations pointed out above, in Section 4, interpretation and graphical representation of results, and in Section 5 conclusions are given.

2. The methodology

In this section, we will discuss the MSE method and the Exp-function method.

2.1. The MSE method

Suppose the nonlinear evolution equation is in the form

$$\wp(u, u_t, u_x, u_{xx}, u_{tt}, \ldots) = 0 \tag{2.1}$$

where $\wp$ is a polynomial of $u(x,\ t)$ and its partial derivatives wherein the highest order derivatives and nonlinear terms are involved. The focal steps of the MSE method are as follows [30–33]:

Step 1: Suppose the traveling wave transformation [13]

$$u(x,t) = u(\xi), \quad \xi = k(x \pm \omega t) \tag{2.2}$$

where ω is the speed of traveling wave, k is the wave number, permits us to reduce Eq. (2.1) into the following ordinary differential equation (ODE):

$$P(u, u', u'', \ldots) = 0, \tag{2.3}$$

where P is a polynomial in $u(\xi)$ and its derivatives, forasmuch $u'(\xi) = \frac{du}{d\xi}$.

Step 2: Suppose the solution of Eq. (2.3) can be written in the form

$$u(\xi) = A_0 + \sum_{i=1}^{N} A_i \left[\frac{\Phi'(\xi)}{\Phi(\xi)}\right]^i. \tag{2.4}$$

where N is a positive integer and $A_i(i = 1, 2, 3, \ldots, N)$ are arbitrary constants to be determined, such that $A_N \neq 0$ and $\Phi(\xi)$ is an unknown function to be determined afterward.

Step 3: We determine the positive integer N appearing in Eq. (2.4) by balancing the highest order derivatives and the highest order nonlinear terms occurring in Eq. (2.3).

Step 4: We substitute Eq. (2.4) into Eq. (2.3) and then we account the function $\Phi(\xi)$. As a result of this substitution, we get a polynomial of $\Phi^{-j}(\xi)$ with the derivatives of $\Phi(\xi)$. We equate all the coefficients of Φ^{-j} to zero, where $j \geqslant 0$. This procedure yields a system of equations whichever can be solved to find A_i and $\Phi(\xi)$. Substituting the values of A_i and $\Phi(\xi)$ into Eq. (2.4) completes the determination of the solution of Eq. (2.1).

2.2. The Exp-function method

We now present the Exp-function method for solving the nonlinear partial differential equation of the form of Eq. (2.1).

Step 1: Suppose the solution of Eq. (2.3) can be expressed in the following form [14–18]:

$$u(\xi) = \frac{\sum_{n=-c}^{d} A_n \exp(n\xi)}{\sum_{m=-p}^{q} B_m \exp(m\xi)} = \frac{A_{-c}\exp(-c\xi) + \cdots + A_d \exp(d\xi)}{B_{-p}\exp(-p\xi) + \cdots + B_q \exp(q\xi)}, \tag{2.5}$$

where c, d, p and q are positive integers which are unknown to be determined, A_n and B_m are unknown constants. Eq. (2.5) can be rewritten in the following equivalent form:

$$u(\xi) = \frac{A_c \exp(c\xi) + \cdots + A_{-d}\exp(-d\xi)}{B_p \exp(p\xi) + \cdots + B_{-q}\exp(-q\xi)}. \tag{2.6}$$

This equivalent presentation plays an important and fundamental role for finding the solitary wave solutions of NLEEs [14–18].

Step 2: For determining the values of c and p, we balance the linear term of the highest order to the highest order nonlinear term, and for determining the values of d and q, we balance the lowest order linear term to the lowest order nonlinear term in Eq. (2.3). This completes the determination of the values of c, d, p and q.

Step 3: Putting the values of c, d, p and q into Eq. (2.6) and then substituting Eq. (2.6) into Eq. (2.3) and simplifying, we obtain

$$\sum_j C_j \exp(j\eta) = 0. \tag{2.7}$$

Setting each coefficient $C_j = 0$, yields a set of algebraic equations for A_c's and B_p's.

Step 4: Suppose the unknown A_c's and B_p's can be obtained by solving the set algebraic equations obtained in step 3. Substituting these values into Eq. (2.6) we obtain the exact traveling wave solutions of Eq. (2.1).

3. Applications

3.1. The (2 + 1)-dimensional Zoomeron equation

In this subsection, we will exert the MSE method and Exp-function method to find the exact solutions of Zoomeron equation. Let us consider the Zoomeron equation

$$\left(\frac{u_{xy}}{u}\right)_{tt} - \left(\frac{u_{xy}}{u}\right)_{xx} + 2(u^2)_{xt} = 0 \tag{3.1}$$

where $u(x,\ y,\ t)$ is the amplitude of the relative wave mode.

The traveling wave transformation

$$u(x,y,t) = u(\xi), \quad \xi = x + y - \omega t \tag{3.2}$$

reduces Eq. (3.1) into the following ODE:

$$(\omega^2-1)u''-2\omega u^3+Ru=0 \tag{3.3}$$

where R is a constant of integration.

3.1.1. Solutions for Zoomeron equation via MSE method

Balancing the highest order derivative u'' and nonlinear term of the highest order u^3, yields $N=1$.

Through the MSE method, for $N=1$ Eq. (2.4) becomes

$$u(\xi)=A_0+A_1\left(\frac{\Phi'}{\Phi}\right) \tag{3.4}$$

where A_0 and A_1 are constants such that $A_1 \neq 0$, and $\Phi(\xi)$ is an unidentified function to be determined.

Substituting Eq. (3.4) into Eq. (3.3) and equating the coefficients of Φ^0, Φ^{-1}, Φ^{-2}, Φ^{-3} to zero, yields

$$-2\omega A_0^3+RA_0=0. \tag{3.5}$$

$$(\omega^2-1)\Phi'''-(6\omega A_0^2-R)\Phi'=0. \tag{3.6}$$

$$3(\omega^2-1)\Phi''+6\omega A_0A_1\Phi'=0 \tag{3.7}$$

$$(2A_1-2\omega A_1^3-2\omega^2A_1)\Phi'^3=0. \tag{3.8}$$

Solving Eq. (3.5), we obtain

$$A_0=0,\pm\sqrt{\left(\frac{R}{2\omega}\right)}$$

Again solving Eq. (3.8), we obtain

$$A_1=\pm\sqrt{\left(\frac{(\omega^2-1)}{\omega}\right)} \quad \text{since } A_1\neq 0.$$

From Eqs. (3.6) and (3.7), we obtain

$$\Phi'(\xi)=MA\exp(-LM\xi) \tag{3.9}$$

Integrating Eq. (3.9) with respect to ξ, yields

$$\Phi(\xi)=\frac{A}{-L}\exp(-LM\xi)+B \tag{3.10}$$

where $L=-\left(\frac{6\omega A_0^2-R}{\omega^2-1}\right)$, $M=-\left(\frac{\omega^2-1}{2\omega A_0A_1}\right)$ and A, B are constants of integration. Now using Eqs. (3.9)and (3.10); Eq. (3.4) yields the following exact solution

$$u(\xi)=A_0-A_1\left(\frac{LMA\exp(-LM\xi)}{A\exp(-LM\xi)-LB}\right). \tag{3.11}$$

Case-I: When $A_0=0$, Eq. (3.11) yields an absurd solution. Hence, the case is discarded.

Case-II: When $A_0=\pm\sqrt{\frac{R}{2\omega}}$, substituting the values of A_0, A_1, L, M Eq. (3.11) yield the following exact solution

$$u(\xi)=1-\frac{2A\left\{\cosh\left(\sqrt{\frac{R}{2(\omega^2-1)}}\xi\right)-\sinh\left(\sqrt{\frac{R}{2(\omega^2-1)}}\xi\right)\right\}}{\left(\frac{2RB}{\omega^2-1}+A\right)\cosh\left(\sqrt{\frac{R}{2(\omega^2-1)}}\xi\right)+\left(\frac{2RB}{\omega^2-1}-A\right)\sinh\left(\sqrt{\frac{R}{2(\omega^2-1)}}\xi\right)}. \tag{3.12}$$

Since A and B are arbitrarily constants, therefore, if we set $A=\frac{2RB}{(\omega^2-1)}$, from Eq. (3.12), we obtain

$$u_{1,2}(x,y,t)=\pm\sqrt{\frac{R}{2\omega}}\times\tanh\left(\sqrt{\frac{R}{2(\omega^2-1)}}(x+y-\omega t)\right). \tag{3.13}$$

Again setting $A=-\frac{2RB}{(\omega^2-1)}$, Eq. (3.12) reduces to:

$$u_{3,4}(x,y,t)=\pm\sqrt{\frac{R}{2\omega}}\times\coth\left(\sqrt{\frac{R}{2(\omega^2-1)}}(x+y-\omega t)\right). \tag{3.14}$$

If $R<0$, using hyperbolic identities Eqs. (3.13) and (3.14) yields

$$u_{5,6}(x,y,t)=\pm\sqrt{\frac{R}{2\omega}}\tan\left(\sqrt{\frac{R}{2(\omega^2-1)}}(x+y-\omega t)\right) \tag{3.15}$$

and $$u_{5,6}(x,y,t)=\pm\sqrt{\frac{R}{2\omega}}\times\cot\left(\sqrt{\frac{R}{2(\omega^2-1)}}(x+y-\omega t)\right) \tag{3.16}$$

respectively.

3.1.2. Solutions for Zoomeron equation via Exp-function method

Now, we apply the Exp-function method to construct the generalized traveling wave solutions of Zoomeron Eq. (3.1).

According to Step 1 of Section 2.2, the solution of Eq. (3.3) can be written in the form (2.6). To determine the values of c and p, according to Step 2, we balance the linear term u'' of the highest order in Eq. (3.3) to the highest order nonlinear term u^3. With the aid of Maple, yields to the result $p=c$.

To determine the values of q and d, we balance the linear term u'' of lowest order in Eq. (3.3) with lowest order nonlinear term u^3, which leads to the result $q=d$.

We can arbitrarily choose the values of c and d, but the final solution does not depend upon the choices of them. We are interested to obtain nontrivial solutions of Eq. (3.3). By a nontrivial solution, we mean the solution except the solution $u=a$, where a is an arbitrary constant.

Now suppose $p=c=1$ and $q=d=1$. Since there are some free variables, for simplicity, we suppose $B_1=1$. Hence, we obtain

$$u(\xi)=\frac{A_1\exp(\xi)+A_0+A_{-1}\exp(-\xi)}{B_1\exp(\xi)+B_0+B_{-1}\exp(-\xi)}. \tag{3.17}$$

Now, substituting Eq. (3.17) into Eq. (3.3) and by employing the computer algebra, such as Maple, we obtain the following seven algebraic equations.

$$-2\omega A_{-1}^3+RA_{-1}B_{-1}^2=0.$$

$$\omega^2A_0B_{-1}^2-\omega^2A_{-1}B_{-1}B_0+A_{-1}B_0B_{-1}-A_0B_{-1}^2-6\omega A_0A_{-1}^2+2RA_{-1}B_0B_{-1}+RA_0B_{-1}^2=0.$$

$$2RA_0B_0B_{-1}+2RA_{-1}B_{-1}-\omega^2A_0B_{-1}B_0-A_{-1}B_0^2+4\omega^2A_1B_{-1}^2-4\omega^2A_{-1}B_{-1}+\omega^2A_{-1}B_0^2$$

$$-6\omega A_1A_{-1}^2-6\omega A_0^2A_{-1}+A_0B_{-1}B_0-4A_1B_{-1}^2+4A_{-1}B_{-1}+RA_1B_{-1}^2+RA_{-1}B_0^2=0.$$

$$2RA_1B_0B_{-1}+2RA_{-1}B_0-12\omega A_1A_0A_{-1}+6A_0B_{-1}-2\omega A_0^3+3\omega^2A_1B_0B_{-1}-6\omega^2A_0B_{-1}$$

$$-3A_{-1}B_0-3A_1B_{-1}B_0+3\omega^2A_{-1}B_0+2RA_0B_{-1}+RA_0B_0^2=0.$$

$$-4A_{-1}-6\omega A_1^2A_{-1}+4\omega^2A_{-1}-4\omega^2A_1B_{-1}+4A_1B_{-1}+RA_1B_0^2+\omega^2A_1B_0^2+A_0B_0$$

$$+2RA_0B_0-6\omega A_1A_0^2+RA_{-1}+2RA_1B_{-1}-A_1B_0^2-\omega^2A_0B_0=0.$$

$$-\omega^2A_1B_0+RA_0+2RA_1B_0+A_1B_0-A_0-6\omega A_1^2A_0+\omega^2A_0=0.$$

$$RA_1-2\omega A_1^3=0.$$

Solving the above algebraic equations for A_{-1}, A_0, A_1, B_{-1}, B_0, ω, R, we get the following two valid sets.

Set 1 $R=1-\omega^2,\quad \omega=\omega,\quad A_{-1}=0,\quad A_0=A_0,$

$$A_1=0,\quad B_{-1}=-\frac{\omega A_0^2}{4(\omega^2-1)},\quad B_0=0$$

Set 2 $R=\frac{1}{2}(\omega^2-1),\quad \omega=\omega,\quad A_{-1}=0,$

$$A_0=\mp\frac{B_0}{2}\sqrt{\frac{\omega^2-1}{\omega}},\quad A_1=\pm\frac{1}{2}\sqrt{\frac{\omega^2-1}{\omega}},\quad B_{-1}=0, B_0=B_0$$

For the values of Set 1 and Set 2, Eq. (3.17) gives the following traveling wave solutions.

$$u_1(\xi)=\frac{4A_0(\omega^2-1)}{(4\omega^2-4-\omega A_0^2)\cosh(\xi)+(4\omega^2-4+\omega A_0^2)\sinh(\xi)} \tag{3.18}$$

and
$$u_{2,3}(\xi)=\pm\frac{1}{2}\sqrt{\frac{\omega^2-1}{\omega}}\frac{1+\tanh(\xi)-B_0\sec h(\xi)}{1+\tanh(\xi)+B_0\sec h(\xi)} \tag{3.19}$$

respectively, where $\xi=x+y-\omega t$.

3.2. The (2 + 1)-dimensional Burgers equation

In this subsection, we will avail the MSE method and Exp-function method to look for the exact solutions and then the solitary wave solutions to the (2 + 1)-dimensional Burgers equation

$$u_t-uu_x-u_{xx}-u_{yy}=0 \tag{3.20}$$

where
$$u=u(x,y,t),\quad \xi=k(x+y-\omega t),\quad u(x,y,t)=u(\xi) \tag{3.21}$$

By means of the traveling wave transformation (3.21), Eq. (3.20) reduces to the following ODE:

$$-\omega u-uu'-2ku''=0. \tag{3.22}$$

Integrating Eq. (3.22) with respect to ξ and regarding integrating constant to zero, we obtain

$$2\omega u+u^2+4ku'=0. \tag{3.23}$$

3.2.1. Solutions for Burgers equation via MSE method

Through the MSE method, balancing the highest order derivative u' and nonlinear term u^2, we obtain $N=1$.

Therefore, Eq. (2.4) takes the shape

$$u(\xi)=A_0+A_1\left(\frac{\Phi'(\xi)}{\Phi(\xi)}\right), \tag{3.24}$$

where A_0 and A_1 are constants such that $A_1\neq 0$, and $\Phi(\xi)$ is an unidentified function to be determined. It is easy to make out that

$$u'=A_1\left[\frac{\Phi''}{\Phi}-\left(\frac{\Phi'}{\Phi}\right)^2\right] \tag{3.25}$$

$$u^2=A_0^2+2A_0A_1\left(\frac{\Phi'}{\Phi}\right)+A_1^2\left(\frac{\Phi'}{\Phi}\right)^2 \tag{3.26}$$

Substituting the values of u, u' and u^2 from Eqs. (3.24)–(3.26), into Eq. (3.23) and then equating the coefficients of Φ^0, Φ^{-1}, Φ^{-2} to zero, we respectively obtain

$$2\omega A_0+A_0^2=0. \tag{3.27}$$

$$2k\Phi''+(\omega+A_0)\Phi'=0. \tag{3.28}$$

$$(A_1^2-4kA_1)(\Phi')^2=0. \tag{3.29}$$

From Eq. (3.27), we obtain

$A_0=0,-2\omega$

And from Eq. (3.29), we obtain

$A_1=4k,$ since $A_1\neq 0$

Solving Eq. (3.28), we obtain

$$\frac{\Phi''}{\Phi'}=-l \tag{3.30}$$

where $l=\left(\frac{A_0+\omega}{2k}\right)$.Integrating Eq. (3.30), we obtain

$$\Phi'(\xi)=c_1\exp(-l\xi) \tag{3.31}$$

where c_1 is a constant of integration.

Integrating Eq. (3.31) with respect to ξ, we obtain

$$\Phi(\xi)=c_2-\frac{c_1}{l}\exp(-l\xi) \tag{3.32}$$

where c_2 is a constant of integration.

Substituting the values of Φ and Φ' into Eq. (3.24), yields

$$u(\xi)=A_0+A_1\left(\frac{c_1l\exp(-l\xi)}{c_2l-c_1\exp(-l\xi)}\right). \tag{3.33}$$

Case-I: When $A_0=0$, Eq. (3.33) becomes

$$u(\xi)=A_1\left(\frac{c_1l\exp(-l\xi)}{c_2l-c_1\exp(-l\xi)}\right). \tag{3.34}$$

We can arbitrarily choose the constants c_1 and c_2. Therefore, setting $c_1=\frac{\omega c_2}{2k}$ and substituting the values of A_1 and l into Eq. (3.34), we obtain

$$u_1(x,y,t) = -\omega\left\{1-\coth\left(\frac{\omega}{4}(x+y-\omega t)\right)\right\}, \quad \omega > 0. \quad (3.35)$$

Again if we set $c_1 = \frac{-\omega c_2}{2k}$, Eq. (3.34) reduces to:

$$u_2(x,y,t) = -\omega\left\{1-\tanh\left(\frac{\omega}{4}(x+y-\omega t)\right)\right\}, \quad \omega > 0. \quad (3.36)$$

If $\omega < 0$, Eqs. (3.35) and (3.36) yields

$$u_3(x,y,t) = -\omega\left\{1-\cot\left(\frac{\omega}{4}(x+y-\omega t)\right)\right\} \quad (3.37)$$

and $\quad u_4(x,y,t) = -\omega\left\{1-\tan\left(\frac{\omega}{4}(x+y-\omega t)\right)\right\}. \quad (3.38)$

Case-II: When $A_0 = -2\omega$, Eq. (3.33) becomes

$$u_5(x,y,t) = -2\omega + \frac{4\omega c_1(1+\tanh(\omega(x+y-\omega t)/2))}{2c_1(1+\tanh(\omega(x+y-\omega t)/2)) + c_2\omega\,\mathrm{sec}\,h(\omega(x+y-\omega t)/2)}. \quad (3.39)$$

The free parameters may imply some physical meaningful results in fluid mechanics, gas dynamics, and traffic flow.

3.2.2. Solutions for Burgers equation via Exp-function method

Now, we apply the Exp-function method to construct the generalized traveling wave solutions of Burgers Eq. (3.20).

According to the parallel course of action discussed in Section 3.1.2, the solution of Eq. (3.23) can be written in the form of

$$u(\xi) = \frac{A_1\exp(\xi) + A_0 + A_{-1}\exp(-\xi)}{\exp(\xi) + B_0 + B_{-1}\exp(-\xi)}. \quad (3.40)$$

Now, substituting Eq. (3.40) into Eq. (3.23) and by employing the computer algebra, such as Maple, we obtain the following seven algebraic equations.

$$-A_{-1}^2 - 2\omega A_{-1}B_{-1} = 0,$$

$$-4kA_0B_{-1} - 2A_0A_{-1} - 2\omega A_0B_{-1} - 2\omega A_{-1}B_0 + 4kA_{-1}B_0 = 0,$$

$$-2A_1A_{-1} + 8kA_{-1} - 2\omega A_1B_{-1} - 2\omega A_0B_0$$

$$-A_0^2 - 2\omega A_{-1} - 8kA_1B_{-1} = 0,$$

$$-2A_1A_0 + 4kA_0 - 2\omega A_1B_0 - 4kA_1B_0 - 2\omega A_0 = 0,$$

$$-2\omega A_1 - A_1^2 = 0$$

Solving the above algebraic equations for $A_{-1}, A_0, A_1, B_{-1}, B_0, \omega, R$, we get the following five valid sets.

Set 1 $\quad k = -\frac{\omega}{2}, \quad \omega = \omega, \quad A_{-1} = 0, \quad A_0 = 0, \quad A_1 = -2\omega, \quad B_{-1} = 0, \quad B_0 = B_0$

Set 2 $\quad k = -\frac{\omega}{4}, \quad \omega = \omega, \quad A_{-1} = 0, \quad A_0 = 0, \quad A_1 = -2\omega, \quad B_{-1} = B_{-1}, \quad B_0 = 0$

Set 3 $\quad k = \frac{A_1}{4}, \quad \omega = -\frac{A_1}{2}, \quad A_{-1} = 0, \quad A_0 = A_0, \quad A_1 = A_1, \quad B_{-1} = B_{-1}, \quad B_0 = \frac{B_{-1}A_1^2 + A_0^2}{A_1A_0}$

Set 4 $\quad k = \frac{\omega}{4}, \quad \omega = \omega, \quad A_{-1} = -2\omega B_{-1}, \quad A_0 = 0, \quad A_1 = 0, \quad B_{-1} = B_{-1}, \quad B_0 = 0$

Set 5 $\quad k = \frac{\omega}{2}, \quad \omega = \omega, \quad A_{-1} = \frac{A_0(A_0+2\omega B_0)}{\omega}, \quad A_0 = A_0, \quad A_1 = 0, \quad B_{-1} = -\frac{A_0(A_0+2\omega B_0)}{4\omega^2}, \quad B_0 = B_0$

Now substituting Set 1–Set 5 into Eq. (3.40), we deduce abundant traveling wave solutions of Eq. (3.20) as follows.

$$u_1(\xi) = -2\omega\left(\frac{\cosh(\xi)+\sinh(\xi)}{\cosh(\xi)+\sinh(\xi)+B_0}\right), \quad (3.41)$$

where $\xi = -\frac{\omega}{2}(x+y-\omega t)$.

$$u_2(\xi) = -2\omega\left(\frac{\cosh(\xi)+\sinh(\xi)}{(1+B_{-1})\cosh(\xi)+(1-B_{-1})\sinh(\xi)}\right), \quad (3.42)$$

where $\xi = -\frac{\omega}{4}(x+y-\omega t)$.

$$u_3(\xi) = \left(\frac{(A_1(\cosh(\xi)+\sinh(\xi))+A_0)A_0A_1}{(\cosh(\xi)+\sinh(\xi))A_0A_1 + B_{-1}A_1^2 + A_0^2 + (\cosh(\xi)-\sinh(\xi))A_0A_1B_{-1}}\right), \quad (3.43)$$

where $\xi = \frac{A_1}{4}(x+y-\omega t)$.

$$u_4(\xi) = -2\omega B_{-1}\left(\frac{\cosh(\xi)-\sinh(\xi)}{(1+B_{-1})\cosh(\xi)+(1-B_{-1})\sinh(\xi)}\right), \quad (3.44)$$

where $\xi = \frac{\omega}{4}(x+y-\omega t)$.

$$u_5(\xi) = 2\omega A_0\left(\frac{(A_0+2\omega B_0)(\cosh(\xi)-\sinh(\xi))+2\omega}{(4\omega^2 - A_0^2 - 2\omega A_0B_0)\cosh(\xi) + (4\omega^2 + A_0^2 + 2\omega A_0B_0)\sinh(\xi)}\right), \quad (3.45)$$

where $\xi = \frac{\omega}{2}(x+y-\omega t)$.

4. Explanations and graphical representations of the solutions

In this subsection, we will discuss the physical interpretation of the results of the (2 + 1)-dimensional Zoomeron Equation and the (2 + 1)-dimensional Burgers Equation.

The (2 + 1)-dimensional Zoomeron Equation:

(i) Applying the MSE method, the (2 + 1)-dimensional Zoomeron Equation provides the traveling wave solutions from Eqs. (3.13)–(3.16). In these equations the arbitrary constant $R \neq 0$. The shape of Eqs. (3.13) and (3.15) are represented in Figs. 1 and 2 respectively with wave speed $\omega = 2$, $y = 0$ and $R = 9$ within the interval $-3 \leqslant x, t \leqslant 3$. Fig. 1 represents kink wave and Fig. 2 represents periodic wave.

(ii) And applying Exp-function method, the (2 + 1)-dimensional Zoomeron Equation provides the traveling wave solutions Eqs. (3.18) and (3.19). In Eq. (3.18), the constant $A_0 \neq 0$. Eqs. (3.18) and (3.19) are hyperbolic functions solutions. Fig. 3 represents the bell-shaped profile of Eq. (3.18) with $A_0 = 1$, $y = 0$, $\omega = 0.75$ within $-3 \leqslant x, t \leqslant 3$.

The (2 + 1)-dimensional Burgers equation:

(i) Applying the MSE method, the (2 + 1)-dimensional Burgers Equation provides the traveling wave solutions

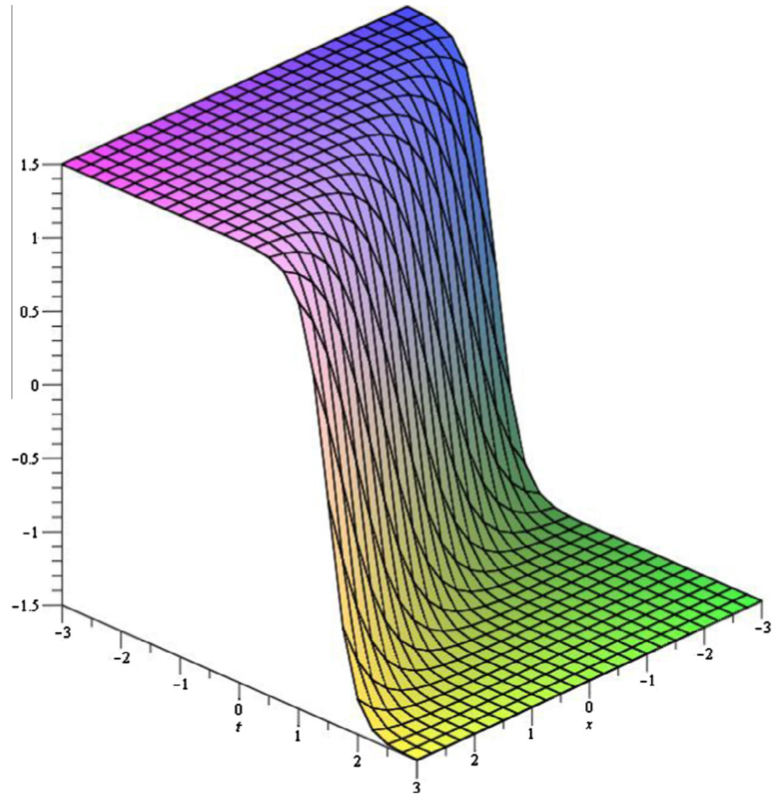

Figure 1 Profile of Eq. (3.13) with $R = 9$, $y = 0$, $\omega = 2$ within $-3 \leqslant x, t \leqslant 3$.

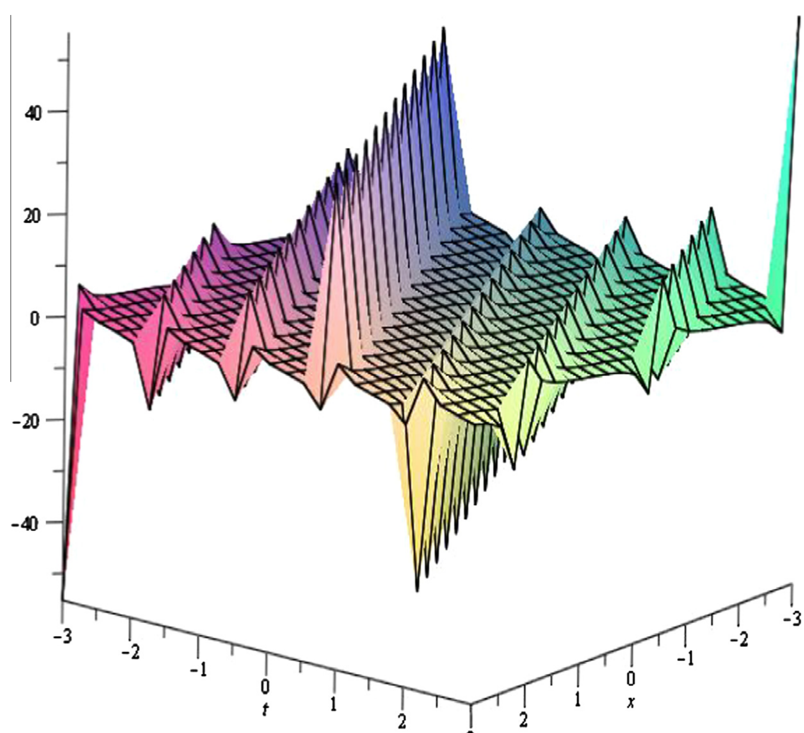

Figure 2 Profile of Eq. (3.15) with $R = 9$, $y = 0$, $\omega = 2$ within $-3 \leqslant x, t \leqslant 3$.

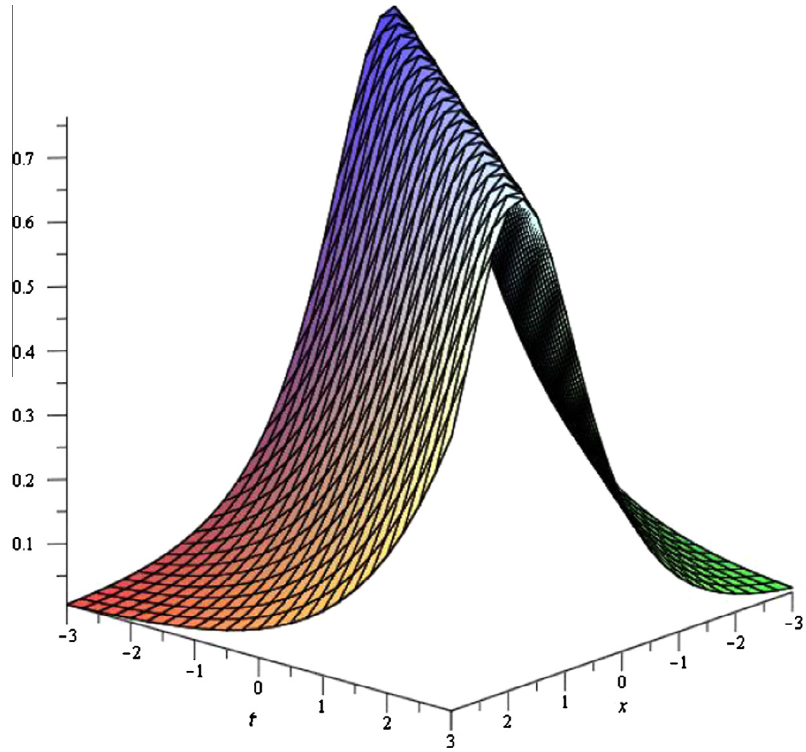

Figure 3 Profile of Eq. (3.18) with $A_0 = 1$, $y = 0$, $\omega = 0.75$ within $-3 \leqslant x, t \leqslant 3$.

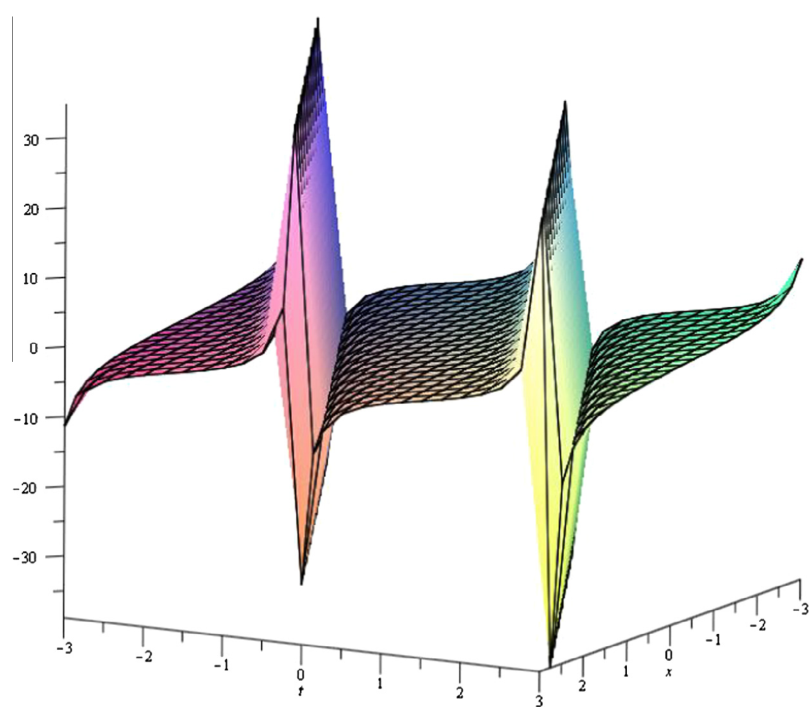

Figure 4 Profile of Eq. (3.36) with $y = 0$, $\omega = 2$ within $-3 \leqslant x, t \leqslant 3$.

from Eqs. (3.35), (3.36), (3.37), (3.38), (3.39). The shape of Eq. (3.38) is represented in Fig. 4 is a periodic wave solution with wave speed $\omega = 2$, $y = 0$ within the interval $-3 \leqslant x, t \leqslant 3$.

(ii) And applying Exp-function method, the (2 + 1)-dimensional Burgers equation provides the traveling wave solutions from Eqs. (3.41), (3.42), (3.43), (3.44), (3.45) which are expressed through hyperbolic functions. In

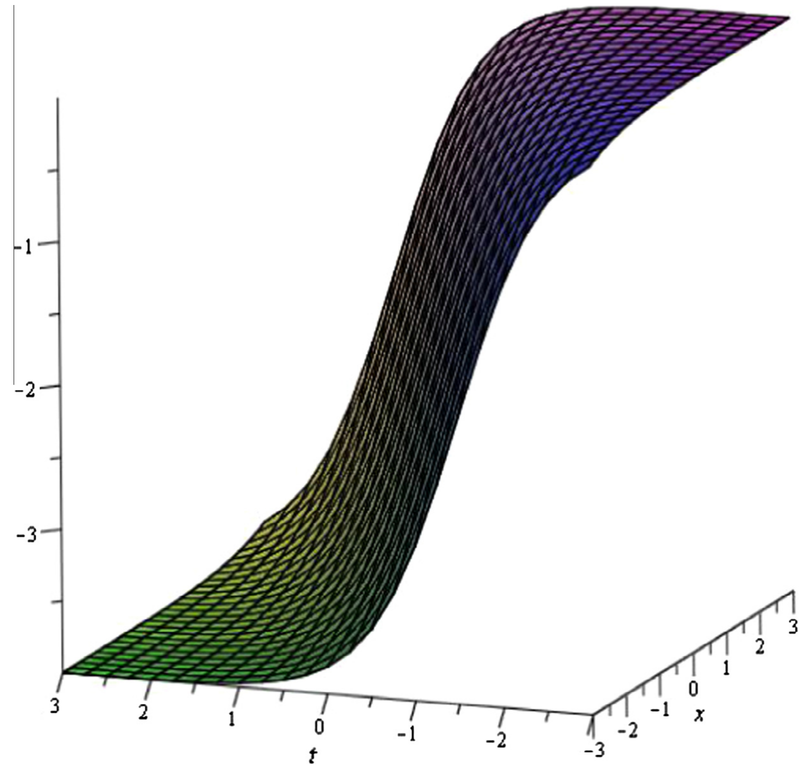

Figure 5 Profile of Eq. (3.42) with $B_{-1} = 1$, $y = 0$, $\omega = 2$ within $-3 \leqslant x, t \leqslant 3$.

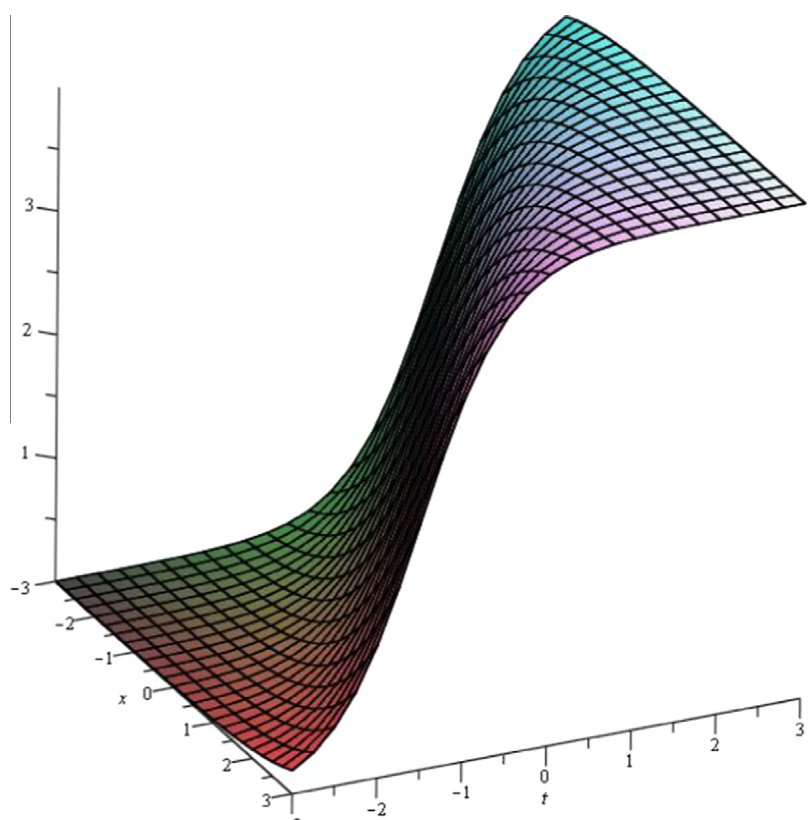

Figure 6 Profile of Eq. (3.44) with $B_{-1} = 1$, $y = 0$, $\omega = -2$ within $-3 \leqslant x, t \leqslant 3$.

Eq. (3.41) the constant $B_0 \neq 0$, in Eq. (3.43) the constants A_0, $A_1 \neq 0$, in Eq. (3.44) the constant $B_{-1} \neq 0$ and in Eq. (3.45) the constant $A_0 \neq 0$. Figs. 5 and 6 represent kink profile of Eqs. (3.44) and (3.45) with $B_{-1} = 1$, $y = 0$, $\omega = 2$ and $B_{-1} = 1$, $y = 0$, $\omega = -2$ respectively within the interval $-3 \leqslant x, t \leqslant 3$.

Furthermore, the graphical demonstrations of some obtained solutions are shown in Figs. 1–6 in the following subsection.

Some of our obtained traveling wave solutions are represented in the following figures with the aid of commercial software Maple:

5. Conclusions

In this article, we considered the (2 + 1)-dimensional Zoomeron equation and the (2 + 1)-dimensional Burgers equation. We put forth the modified simple equation (MSE) method and Exp-function method for finding exact solutions of these equations. It is significant to observe that comparing the MSE method with the Exp-function method, we assert that the MSE method is direct, easy, concise, and straightforward. To calculate the coefficients A_0, A_1, A_2, etc. it need not use any computer algebra when the MSE method is used. On the other hand, the Exp-function method must needs computer algebra to compute the coefficients A_0, A_1, A_2, etc. The MSE method can be applied to many other nonlinear evolution equations in mathematical physics. This study shows that the MSE method is quite efficient and practically well suited to be used in finding exact solutions of NLEEs.

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