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On the Controllability of Some Singularly Perturbed Nonlinear Systems

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The controllability of a large scale dynamic system which depends singularly upon a small parameter λ is considered. When $\lambda = 0$, the large scale system degenerates into a reduced order subsystem representing its slow dynamics while neglecting the fast phenomena. Another subsystem, often called a boundary layer system, represents the fast dynamics. In this paper sufficient conditions are established under which the controllability of the overall large scale system is inferred from the same property of the two subsystems.

1. INTRODUCTION

Concepts of controllability and observability have been playing a central role in modern control theory ever since they were introduced. On the other hand, presently there is a definite focus on developing decentralized control methods for large scale systems [1]. Singular perturbation methods [2] offer tools to separate a system into subsystems. Thus it is important to characterize the controllability of a high order system of singularly perturbed type in terms of the controllability of its subsystems. In this paper we examine this problem. Previously, Kokotović and Haddad [3] and Chow [4] considered this problem for linear time invariant case while Sannuti [5] considered the same for linear time varying case. Here, we consider a nonlinear system of the type

$$\dot{\mathbf{x}} = \mathbf{g}(\mathbf{x}, \mathbf{z}, \mathbf{u}, t) \tag{1a}$$

$$\lambda\dot{\mathbf{z}} = \mathbf{G}_1(\mathbf{x}, t) + \mathbf{D}(\mathbf{x}, t)\,\mathbf{z} + \mathbf{E}(\mathbf{x}, t)\,\mathbf{u} \triangleq \mathbf{G}(\mathbf{x}, \mathbf{z}, \mathbf{u}, t), \tag{1b}$$

where $\mathbf{x}$ and $\mathbf{z}$ are n- and m-dimensional state vectors respectively, $\mathbf{u}$ is an r-dimensional control vector, overdot denotes d/dt, and λ is a small positive parameter. The system defined by (1) is called the full system. The reduced system of the full system is defined by setting $\lambda = 0$ in (1),

$$\dot{x} = \mathbf{g}(x, z, u, t), \tag{2a}$$

$$0 = \mathbf{G}(x, z, u, t). \tag{2b}$$

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Thus with the matrix $\mathbf{D}(\mathbf{x}, t)$ invertible for each $\mathbf{x}$ and t, we can solve for z from (2b),

$$z = -\mathbf{D}^{-1}(x, t)\,(\mathbf{G}_1(x, t) + \mathbf{E}(x, t)\,u) \triangleq \mathbf{F}(x, u, t), \tag{3}$$

and rewrite (2) as

$$\dot{x} = \mathbf{g}(x, \mathbf{F}(x, u, t), u, t) \triangleq \mathbf{f}(x, u, t). \tag{4}$$

In addition to the reduced system (4), we define an auxiliary system often called a boundary layer system,

$$\frac{d\mathbf{z}}{d\rho} = \mathbf{D}(\mathbf{x}, t)\,\mathbf{z} + \mathbf{E}(\mathbf{x}, t)\,\mathbf{u}, \tag{5}$$

in which $\mathbf{x}$ and t are fixed parameters and ρ is an independent time variable.

We will say that the system (1) is completely controllable if for each t_0 and t_1 there exists some control $\mathbf{u}(t)$ defined over the interval $[t_0, t_1]$ such that the solution of (1) satisfies any prescribed boundary conditions,

$$\mathbf{x}(t_0) = \mathbf{x}_0, \qquad \mathbf{x}(t_1) = \mathbf{x}_1, \tag{6a}$$

$$\mathbf{z}(t_0) = \mathbf{z}_0, \qquad \mathbf{z}(t_1) = \mathbf{z}_1. \tag{6b}$$

The aim of the paper is to find conditions in terms of the subsystems (4) and (5) such that the full system (1) is completely controllable. We accomplish this by first constructing a formal solution $X(t)$, $Z(t)$, and $U(t)$ which satisfy the system (1) and the boundary conditions (6) within $O(\lambda)$, under the hypotheses that the reduced and boundary layer systems (4) and (5) are completely controllable. Then the existence of a solution of (1) and (6) is rigorously established by combining the successive approximation methods employed earlier by Sannuti [6] and Lukes [7].

We assume that the functions $\mathbf{g}$, $\mathbf{G}$ and $\mathbf{f}$ are twice continuously differentiable with respect to all their arguments in an appropriately defined domain. Using subscripts to denote matrices of partial derivatives in the usual way, we define $g_x(t) \equiv \mathbf{g}_x(x(t), z(t), u(t), t)$ with similar definitions for $g_z(t)$, $g_u(t)$, $f_x(t)$, $f_u(t)$ and $G_x(t)$. Whenever there is no ambiguity, the arguments of the functions will not be shown explicitly. The absolute value or norm of a vector or a matrix denotes the sum of the absolute values of its elements.

2. Construction of a Formal Solution

We will construct a formal solution under the following two hypotheses:

(H1) The reduced system (4) is completely controllable, i.e., for each t_0

and t_1 there exists a control $u(t)$ defined over the interval $[t_0, t_1]$ such that the solution of (4) satisfies the boundary conditions

$$x(t_0) = \mathbf{x}_0 \qquad \text{and} \qquad x(t_1) = \mathbf{x}_1 .$$

(H2) The rank of the matrix

$$[D(t), D(t)\, E(t), \ldots, D^{m-1}(t)\, E(t)]$$

is m for each fixed $t \in [t_0, t_1]$, where

$$D(t) = \mathbf{D}(x(t), t) \qquad \text{and} \qquad E(t) = \mathbf{E}(x(t), t).$$

We note that the reduced system has no boundary conditions related to (6b), i.e., $z(t) = \mathbf{F}(x, u, t)$ will not be equal to the prescribed boundary conditions $\mathbf{z}_0$ at t_0 and $\mathbf{z}_1$ at t_1. Thus the system (1) is expected in general to exhibit boundary layer phenomena at the end points t_0 and t_1 as λ tends to zero. We impose the hypothesis (H2) to construct formally these boundary layers.

Consider the following two systems called the left and right boundary layer systems respectively:

$$dz_l/d\tau = D(t_0)\, z_l + E(t_0)\, u_l , \qquad z_l(0) = \mathbf{z}_0 - z(t_0), \tag{7}$$

$$dz_r/d\sigma = -D(t_1)\, z_r - E(t_1)\, u_r , \qquad z_r(0) = \mathbf{z}_1 - z(t_1). \tag{8}$$

Here τ and σ are stretched time coordinates $\tau = (t - t_0)/\lambda$ and $\sigma = (t_1 - t)/\lambda$. An immediate consequence of (H2) is to guarantee the existence of matrices L_0 and L_1 such that $D_l = D(t_0) + E(t_0)\, L_0$ and $D_r = -D(t_1) - E(t_1)\, L_1$ are stable, i.e., each eigen value of D_l and D_r has a real part $\leqslant -\gamma < 0$. With

$$u_l(\tau) = L_0 z_l(\tau) \qquad \text{and} \qquad u_r(\sigma) = L_1 z_r(\sigma),$$

an exponentially decaying solution can be constructed for both (7) and (8),

$$z_l(\tau) = \exp[D_l \tau]\, (\mathbf{z}_0 - z(t_0)), \qquad 0 \leqslant \tau < \infty,$$

$$z_r(\sigma) = \exp[D_r \sigma]\, (\mathbf{z}_1 - z(t_1)), \qquad 0 \leqslant \sigma < \infty.$$

Now let us define a formal solution,

$$\begin{aligned} X(t) &= x(t), \\ Z(t) &= z(t) + z_l((t - t_0)/\lambda) + z_r((t_1 - t)/\lambda), \\ U(t) &= u(t) + u_l((t - t_0)/\lambda) + u_r((t_1 - t)/\lambda). \end{aligned} \tag{9}$$

We intend to show that (1) along with the boundary conditions (6) has a solution and that X, Z, and U approximate $\mathbf{x}$, $\mathbf{z}$, and $\mathbf{u}$ within $O(\lambda)$. However we need an additional hypothesis on a linearized system of the reduced system.

(H3) The linear system (10),

$$\dot{\alpha} = f_x(t)\,\alpha + f_u(t)\,\nu \tag{10}$$

(where α and ν are treated as state and control variables respectively) is completely controllable. More specifically, we assume that $w(t_1, t_0)$,

$$w(t_1, t_0) = \int_{t_0}^{t_1} \psi_1(t_0, t) f_u(t) f'_u(t)\, \psi'_1(t_0, t)\, dt,$$

is nonsingular. The prime denotes the matrix transpose and $\psi_1(t, t_0)$ is the fundamental matrix of $f_x(t)$ given by

$$\dot{\psi}_1(t, t_0) = f_x(t)\, \psi_1(t, t_0), \qquad \psi_1(t_0, t_0) = I.$$

Here and elsewhere I denotes an identity matrix of appropriate dimension. Note that if the reduced system (4) is linear then (H1) implies (H3).

3. Preliminaries

In this section we consider some preliminary results useful for the perturbation analysis to follow in the next section. We first observe as in the previous section that under the hypothesis (H2), there exists a matrix $L(t)$ such that $\hat{D} = D(t) + E(t)L(t)$ has all eigenvalues with negative real parts for each fixed $t \in [t_0, t_1]$.

Lemma 1. *Define*

$$\begin{aligned} \hat{g}_x &= g_x(t) + g_u(t)\, L(t)\, D^{-1}(t)\, G_x(t), \\ \hat{g}_z &= g_z(t) + g_u(t)\, L(t), \\ \hat{G}_x &= G_x(t) + E(t)\, L(t)\, D^{-1}(t)\, G_x(t). \end{aligned} \tag{11}$$

Then the system of equations

$$\begin{aligned} \lambda \dot{M} &= \lambda(\hat{g}_x - \hat{g}_z N)\, M - M(\hat{D} + \lambda N \hat{g}_z) - \hat{g}_z\,, \\ \lambda \dot{N} &= \hat{D} N - \lambda N(\hat{g}_x - \hat{g}_z N) - \hat{G}_x\,, \end{aligned}$$

has a bounded solution for all $t \in [t_0, t_1]$ and for all λ sufficiently small. Further M and N satisfy the limits

$$\lim_{\lambda \to 0} M = -\hat{g}_z \hat{D}^{-1}, \qquad \lim_{\lambda \to 0} N = \hat{D}^{-1} \hat{G}_x .$$

Proof. This is a special case of the equations dealt in [8]. Since $\hat{D}(t)$ is a stable matrix, application of the standard theorem of Tikhonov [2] will yield the result.

LEMMA 2. *Equations* (11) *imply the following matrix identities:*

$$\hat{g}_x - \hat{g}_z \hat{D}^{-1} \hat{G}_x = g_x - g_z D^{-1} G_x = f_x , \tag{12}$$

$$g_u - \hat{g}_z \hat{D}^{-1} E = (g_u - g_z D^{-1} E)(I - L\hat{D}^{-1} E) = f_u (I + LD^{-1} E)^{-1}. \tag{13}$$

Proof. From the identity

$$\hat{D}^{-1} \hat{D} = \hat{D}^{-1}(D + EL) = (D + EL)\, \hat{D}^{-1} = I,$$

we first note that

$$\hat{D}^{-1} = D^{-1} - \hat{D}^{-1} ELD^{-1} = D^{-1} - D^{-1} EL\hat{D}^{-1}. \tag{14}$$

Then consider

$$\begin{aligned} \hat{D}^{-1} \hat{G}_x &= \hat{D}^{-1} G_x + \hat{D}^{-1} ELD^{-1} G_x \\ &= (D^{-1} - \hat{D}^{-1} ELD^{-1})\, G_x + \hat{D}^{-1} ELD^{-1} G_x = D^{-1} G_x . \end{aligned} \tag{15}$$

It is easy now to show the first half of identity (12) from (15). The second half of (12) follows from the definition of the function **f**. Similarly,

$$\begin{aligned} \hat{g}_z \hat{D}^{-1} &= g_z (D^{-1} - D^{-1} EL\hat{D}^{-1}) + g_u L\hat{D}^{-1} \\ &= g_z D^{-1} - g_z D^{-1} EL\hat{D}^{-1} + g_u L\hat{D}^{-1}, \\ g_u - \hat{g}_z \hat{D}^{-1} E &= g_u - g_z D^{-1} E + g_z D^{-1} EL\hat{D}^{-1} E - g_u L\hat{D}^{-1} E \\ &= (g_u - g_z D^{-1} E)(I - L\hat{D}^{-1} E). \end{aligned}$$

This establishes the first part of identity (13). Using (14), it is easy to check

$$I - L\hat{D}^{-1} E = (I + LD^{-1} E)^{-1}. \tag{16}$$

Equation (16) and the definition of the function **f** establish the second part of (13). We remark that identities similar to (12) and (13) were used earlier in [4].

LEMMA 3. *Let*

$$A_1 = \hat{g}_x - \hat{g}_z N, \qquad A_2 = \hat{D} + \lambda N \hat{g}_z,$$

$$B_1 = g_u + ME + \lambda M N g_u, \qquad B_2 = E + \lambda N g_u,$$

$$\dot{\psi}_1(t, t_0) = A_1 \psi_1(t, t_0), \qquad \psi_1(t_0, t_0) = I,$$

$$\lambda \dot{\psi}_2(t, t_0) = A_2 \psi_2(t, t_0), \qquad \psi_2(t_0, t_0) = I,$$

$$\psi(t, t_0) = \begin{bmatrix} \psi_1(t, t_0) & 0 \\ 0 & \psi_2(t, t_0) \end{bmatrix}, \qquad B = \begin{bmatrix} B_1 \\ B_2/\lambda \end{bmatrix},$$

$$\phi(t) = \int_{t_1+t_0-t}^{t_1} \psi(t_1, s)\, B(s)\, ds,$$

$$C(t) = [C_1(t)\; C_2(t)] = \int_{t_1+t_0-t}^{t_1} \phi'(s)\, ds - \frac{t - t_0}{t_1 - t_0} \int_{t_0}^{t_1} \phi'(s)\, ds,$$

$$S(t) = \begin{bmatrix} S_{11}(t) & S_{12}(t) \\ S_{21}(t) & S_{22}(t) \end{bmatrix} = \int_{t_0}^{t} \psi(t, s)\, B(s)\, C(s)\, ds,$$

$$S^* = \int_{t_0}^{t_1} \phi(s)\, \phi'(s)\, ds - \frac{1}{t_1 - t_0} \left(\int_{t_0}^{t_1} \phi(s)\, ds \right) \left(\int_{t_0}^{t_1} \phi'(s)\, ds \right).$$

Then the matrix S^* is symmetric, positive definite and is equal to $S(t_1)$. The matrices $\phi(t)$, $C(t)$ and $S(t)$ are bounded as λ tends to zero. In particular,

$$C_2(t) = \lambda C^*_2(t), \qquad S_{12}(t) = \lambda P_1(t), \qquad S_{22}(t) = \lambda P_2(t) \tag{17}$$

for λ sufficiently small where $C^*_2(t)$, $P_1(t)$ and $P_2(t)$ are bounded as λ tends to zero. Furthermore, $S^{-1}(t_1)$ is of the form

$$S^{-1}(t_1) = \begin{bmatrix} \theta_1 & \theta_2 \\ \theta'_2 & \theta_3/\lambda \end{bmatrix},$$

where θ_i, $i = 1$ to 3, are bounded as λ tends to zero.

Proof. Consider the linear system

$$\begin{aligned} \dot{\alpha}_1 &= A_1 \alpha_1 + B_1 \nu, \\ \lambda \dot{\alpha}_2 &= A_2 \alpha_2 + B_2 \nu. \end{aligned} \tag{18}$$

Lemmas 1 and 2 imply that

$$\lim_{\lambda \to 0} A_1 = f_x, \qquad \lim_{\lambda \to 0} A_2 = \hat{D},$$

$$\lim_{\lambda \to 0} B_1 = f_u (I + LD^{-1}E)^{-1}, \qquad \lim_{\lambda \to 0} B_2 = E.$$

Hypothesis (H2) implies that the rows of $\psi_2(t, t_0)\, B_2(t)$ are linearly independent and (H3) implies that the rows of $\psi_1(t, t_0)\, B_1(t)$ are linearly independent for λ sufficiently small and for $t \in [t_0, t_1]$. This then guarantees that the rows of $\psi(t, t_0)\, B(t)$ are linearly independent [5] on $[t_0, t_1]$ and hence (18) is completely controllable for λ sufficiently small. Then Lemmas (2.1) and (2.2) of Lukes [7] show that S^* is symmetric positive definite and is equal to $S(t_1)$. We remark that Lukes proves the required results for time invariant case. However, with the modified definitions for ϕ, C and S as given here, extension of Lemmas (2.1) and (2.2) of Lukes for time varying case is straight forward (see Appendix). Since A_2 is a stable matrix, the boundedness and the particular form (17) of the matrices $\phi(t)$, $C(t)$ and $S(t)$ follow from the observation [9] that

$$\psi_2(t_1, s) = \exp[A_2(t_1)\,(t_1 - s)/\lambda] + O(\lambda), \qquad s \leqslant t_1,$$

and that

$$\int_{t_0}^{t} \psi_2(t, s)\, \frac{\Gamma(s)}{\lambda}\, ds = -A_2^{-1}(t)\, \Gamma(t) + A_2^{-1}(t_0)\, \Gamma(t_0) \exp[A_2(t_0)\,(t - t_0)/\lambda] + O(\lambda)$$

for any $\Gamma(t)$ continuous at t_0, Finally, noting that $S(t_1)$ is symmetric and positive definite, the form of $S^{-1}(t_1)$ is a consequence of the well-known formula [10] for inverting a partitioned matrix.

4. Main Result

THEOREM. *Under the hypotheses* (H1)–(H3), *the system* (1) *along with the boundary conditions* (6) *has a solution which is of the form*

$$\begin{aligned} \mathbf{x}(t) &= X(t) + x^*, \\ \mathbf{z}(t) &= Z(t) + z^*, \\ \mathbf{u}(t) &= U(t) + u^*, \end{aligned} \tag{19}$$

where x^*, z^* and u^* are all of $O(\lambda)$ uniformly throughout $t_0 \leqslant t \leqslant t_1$ for λ sufficiently small.

Proof. Existence of the variables x^*, z^* and u^* having the required property is shown through a method of successive approximations. We first need to develop a set of differential equations for the variables x^* and z^* with u^* as a control variable. Equations (1), (4) (7)–(9), and (19) imply that

$$\begin{aligned} \dot{x}^* &= g_x(t)\, x^* + g_z(t)\, z^* + g_u(t)\, u^* + g^*(x^*, z^*, u^*, \lambda, t), \\ \lambda \dot{z}^* &= G_x(t)\, x^* + D(t)\, z^* + E(t)\, u^* + G^*(x^*, z^*, u^*, \lambda, t), \end{aligned} \tag{20}$$

where the functions g^* and G^* are given by

$$g^* = \mathbf{g}(X + x^*, Z + z^*, U + u^*, t) - \mathbf{g}(x, z, u, t)$$
$$- g_x(t)\, x^* - g_z(t)\, z^* - g_u(t)\, u^*,$$

$$G^* = \mathbf{G}_1(X + x^*, t) + \mathbf{D}(X + x^*, t)\,(Z + z^*) + \mathbf{E}(X + x^*, t)\,(U + u^*)$$
$$- D(t_0)\, z_l(\tau) - E(t_0)\, u_l(\tau) - D(t_1)\, z_r(\sigma) - E(t_1)\, u_r(\sigma)$$
$$- G_x(t)\, x^* - D(t)\, z^* - E(t)\, u^* - \lambda \dot{z}.$$

Here τ and σ are stretched time variables $\tau = (t - t_0)/\lambda$ and $\sigma = (t_1 - t)/\lambda$. Now by conveniently linearizing g^* and G^* along the solution of the reduced system and then using the mean value theorem we see that g^* and G^* satisfy the following two properties:

Property 1.

$$|\, g^*(0, 0, 0, \lambda, t)| \leqslant K_0(\exp[-\gamma(t - t_0)/\lambda] + \exp[-\gamma(t_1 - t)/\lambda] + \lambda),$$
$$|\, G^*(0, 0, 0, \lambda, t)| \leqslant K_0\lambda.$$

Here K_0 and γ are positive constants.

Property 2. For each $\delta > 0$, there exists an $\epsilon(\delta) > 0$ such that for $|\, \hat{x}^* \,|$, $|\, \tilde{x}^* \,|$, etc., and $\lambda < \epsilon$,

$$|\, g^*(\hat{x}^*, \hat{z}^*, \hat{u}^*, \lambda, t) - g^*(\tilde{x}^*, \tilde{z}^*, \tilde{u}^*, \lambda, t)|$$
$$\leqslant \delta(|\, \hat{x}^* - \tilde{x}^* \,| + |\, \hat{z}^* - \tilde{z}^* \,| + |\, \hat{u}^* - \tilde{u}^* \,|),$$

$$|\, G^*(\hat{x}^*, \hat{z}^*, \hat{u}^*, \lambda, t) - G^*(\tilde{x}^*, \tilde{z}^*, \tilde{u}^*, \lambda, t)|$$
$$\leqslant \delta(|\, \hat{x}^* - \tilde{x}^* \,| + |\, \hat{z}^* - \tilde{z}^* \,| + |\, \hat{u}^* - \tilde{u}^* \,|).$$

Boundary conditions (6), and (19) imply that

$$\begin{aligned} x^*(t_0) &= 0, & x^*(t_1) &= 0, \\ z^*(t_0) &= -z_r((t_1 - t_0)/\lambda), & z^*(t_1) &= -z_l((t_1 - t_0)/\lambda). \end{aligned} \tag{21}$$

Since $z_l(\tau)$ and $z_r(\sigma)$ decay exponentially to zero away from $t = t_0$ and $t = t_1$ respectively, we have

$$z^*(t_0) = O(\lambda^p) \qquad \text{and} \qquad z^*(t_1) = O(\lambda^p),$$

for any positive integer p arbitrarily large.

Now we consider a transformation of variables so that the linear part of (20) can be written in a block diagonal form,

$$\begin{aligned}\alpha &= x^* + \lambda MNx^* + \lambda Mz^*,\\ \beta &= Nx^* + z^*,\\ \nu &= u^* - LD^{-1}G_x x^* - Lz^*.\end{aligned}$$

Here the matrices L, M and N are as defined in Lemma 1. The inverse transformation is given by

$$\begin{aligned}x^* &= \alpha - \lambda M\beta,\\ z^* &= -N\alpha + \beta + \lambda NM\beta,\\ u^* &= \nu + LD^{-1}G_x x^* + Lz^*.\end{aligned}$$

This change of variables transforms the system of equations (20) into the following form,

$$\begin{aligned}\dot{\alpha} &= A_1\alpha + B_1\nu + \mathbf{h}_1(\alpha, \beta, \nu, \lambda, t),\\ \lambda\dot{\beta} &= A_2\beta + B_2\nu + \mathbf{h}_2(\alpha, \beta, \nu, \lambda, t),\end{aligned} \tag{22}$$

where A_i and B_i are as defined in Lemma 3 and

$$\begin{aligned}\mathbf{h}_1(\alpha, \beta, \nu, \lambda, t) &= (I + \lambda MN)g^* + MG^*,\\ \mathbf{h}_2(\alpha, \beta, \nu, \lambda, t) &= \lambda Ng^* + G^*.\end{aligned} \tag{23}$$

In view of (23) and Properties 1 and 2, we see that $\mathbf{h}_i$ satisfy analogous properties a and b:

Property a.

$$\begin{aligned}|\mathbf{h}_1(0, 0, 0, \lambda, t)| &\leqslant K_1(\exp[-\gamma(t - t_0)/\lambda] + \exp[-\gamma(t_1 - t)/\lambda] + \lambda),\\ |\mathbf{h}_2(0, 0, 0, \lambda, t)| &\leqslant K_1\lambda,\end{aligned}$$

where K_1 is a positive constant.

Property b. For each $\delta > 0$, there exists an $\epsilon > 0$ such that for $|\hat{\alpha}|$, $|\tilde{\alpha}|$, etc., and $\lambda < \epsilon$,

$$\begin{aligned}&|\mathbf{h}_i(\hat{\alpha}, \hat{\beta}, \hat{\nu}, \lambda, t) - \mathbf{h}_i(\tilde{\alpha}, \tilde{\beta}, \tilde{\nu}, \lambda, t)|\\ &\quad\leqslant \delta(|\hat{\alpha} - \tilde{\alpha}| + |\hat{\beta} - \tilde{\beta}| + |\hat{\nu} - \tilde{\nu}|), \qquad i = 1, 2.\end{aligned}$$

Further, the boundary conditions (20) imply that

$$\alpha(t_0) = O(\lambda^p), \qquad \alpha(t_1) = O(\lambda^p), \qquad \beta(t_0) = O(\lambda^p), \qquad \beta(t_1) = O(\lambda^p). \tag{24}$$

Now we are in a position to show that the variables α, β, and ν satisfying the set of equations (22) and the boundary conditions (24) exist and are all of $O(\lambda)$. For this purpose we first convert (22) into a system of integral equations and then use a successive approximation scheme. The procedure is identical to the method used by Lukes [7] even though his theorems are not directly applicable. As can easily be verified, the system of equations (22) with auxiliary conditions, $\alpha(t_0) = \alpha_0$, $\alpha(t_1) = \alpha_1$, $\beta(t_0) = \beta_0$ and $\beta(t_1) = \beta_1$, is equivalent to the following system of integral equations:

$$
\begin{aligned}
Y &= S^{-1}(t_1)\begin{bmatrix}\alpha_1 - \psi_1(t_1, t_0)\,\alpha_0\\ \beta_1 - \psi_2(t_1, t_0)\,\beta_0\end{bmatrix} - S^{-1}(t_1)\int_{t_0}^{t_1}\begin{bmatrix}\psi_1(t_1, s)\,h_1(s)\\ \psi_2(t_1, s)\,\dfrac{h_2}{\lambda}(s)\end{bmatrix} ds,\\
\nu(t) &= C(t)\,Y,\\
\begin{bmatrix}\alpha(t)\\ \beta(t)\end{bmatrix} &= \begin{bmatrix}\psi_1(t, t_0)\,\alpha_0\\ \psi_2(t, t_0)\,\beta_0\end{bmatrix} + S(t)\,Y + \int_{t_0}^{t}\begin{bmatrix}\psi_1(t, s)\,h_1(s)\\ \psi_2(t, s)\,\dfrac{h_2}{\lambda}(s)\end{bmatrix} ds,
\end{aligned}
\tag{25}
$$

where ψ_i, C and S are as defined in Lemma 3 and

$$h_j(s) = \mathbf{h}_j(\alpha(s), \beta(s), \nu(s), \lambda, s), \qquad j = 1, 2.$$

Successive iterates are defined by $\alpha^0(t) \equiv 0$, $\beta^0(t) \equiv 0$, $\nu^0(t) \equiv 0$ for all $t_0 \leqslant t \leqslant t_1$, and for each integer $i \geqslant 0$,

$$
\begin{aligned}
Y^i &= S^{-1}(t_1)\begin{bmatrix}\alpha_1 - \psi_1(t_1, t_0)\,\alpha_0\\ \beta_1 - \psi_2(t_1, t_0)\,\beta_0\end{bmatrix} - S^{-1}(t_1)\int_{t_0}^{t_1}\begin{bmatrix}\psi_1(t_1, s)\,h_1^i(s)\\ \psi_2(t_1, s)\,\dfrac{h_2^i}{\lambda}(s)\end{bmatrix} ds,\\
\nu^{i-1}(t) &= C(t)\,Y^i,\\
\begin{bmatrix}\alpha^{i-1}(t)\\ \beta^{i+1}(t)\end{bmatrix} &= \begin{bmatrix}\psi_1(t, t_0)\,\alpha_0\\ \psi_2(t, t_0)\,\beta_0\end{bmatrix} + S(t)\,Y^i + \int_{t_0}^{t}\begin{bmatrix}\psi_1(t, s)\,h_1^i(s)\\ \psi_2(t, s)\,\dfrac{h_2^i}{\lambda}(s)\end{bmatrix} ds,
\end{aligned}
$$

where

$$h_j^i(s) = \mathbf{h}_j(\alpha^i(s), \beta^i(s), \nu^i(s), \lambda, s), \qquad j = 1, 2.$$

Taking α_0, β_0, α_1 and β_1 all of $O(\lambda)$, noting that

$$|\psi_2(t, s)| \leqslant K_2 \exp[-\gamma(t - s)/\lambda], \qquad t_0 \leqslant s \leqslant t \leqslant t_1, \tag{26}$$

for some positive constant γ and K_2, and using Property a and the properties of

the matrices $C(t)$, $S(t)$ and $S^{-1}(t_1)$, it is easy to verify that α^1, β^1 and ν^1 are all $O(\lambda)$. Then using Property b and (26), one can easily get that the estimate

$$\begin{aligned} &| \alpha^{i+1} - \alpha^i | + | \beta^{i+1} - \beta^i | + | \nu^{i+1} - \nu^i | \\ &\quad \leqslant K\delta(| \alpha^i - \alpha^{i-1} | + | \beta^i - \beta^{i-1} | + | \nu^i - \nu^{i-1} |), \end{aligned}$$

is uniformly valid throughout $t_0 \leqslant t \leqslant t_1$, where K is an appropriately selected positive constant. Thus we find that the successive iterates are well defined for λ sufficiently small and that there exists a $\delta > 0$ such that the sequence of successive approximations converges uniformly to a solution of (22) and (24). Also, the estimates that α, β, ν are all of $O(\lambda)$ follow from the iteration. This proves the theorem.

5. Conclusions

Controllability of a large scale system of singularly perturbed type is analyzed. Sufficient conditions are given under which the controllability of the overall system is inferred from the same property of its two subsystems.

Appendix

The following lemma extends some of the results of Lemmas (2.1) and (2.2) of Lukes [7] for linear time varying systems.

Lemma. *Consider a completely controllable linear system,*

$$\dot{x} = A(t)\, x + B(t)\, u, \tag{*}$$

and let

$$\dot{\psi}(t, t_0) = A(t)\, \psi(t, t_0), \qquad \psi(t_0, t_0) = I,$$

$$\phi(t) = \int_{t_1+t_0-t}^{t_1} \psi(t_1, s)\, B(s)\, ds,$$

$$S^* = \int_{t_0}^{t_1} \phi(s)\, \phi'(s)\, ds - \frac{1}{t_1 - t_0} \left(\int_{t_0}^{t_1} \phi(s)\, ds\right) \left(\int_{t_0}^{t_1} \phi'(s)\, ds\right),$$

$$C(t) = \int_{t_1+t_0-t}^{t_1} \phi'(s)\, ds - \frac{t - t_0}{t_1 - t_0} \int_{t_0}^{t_1} \phi'(s)\, ds,$$

$$S(t) = \int_{t_0}^{t} \psi(t, s)\, B(s)\, C(s)\, ds.$$

Then the matrix S^* is positive definite and is equal to $S(t_1)$.

Proof. It is obvious that S^* is symmetric. To show that S^* is positive definite, we compute the quadratic form,

$$\begin{aligned}\Gamma' S^* \Gamma &= \int_{t_0}^{t_1} |\phi'(s)\,\Gamma|^2\,ds - \frac{1}{t_1 - t_0}\left|\int_{t_0}^{t_1} \phi'(s)\,\Gamma\,ds\right|^2 \\ &\geqslant \int_{t_0}^{t_1} |\phi'(s)\,\Gamma|^2\,ds - \frac{1}{t_1 - t_0}\left(\int_{t_0}^{t_1} |\phi'(s)\,\Gamma|\,ds\right)^2 \\ &\geqslant \int_{t_0}^{t_1} |\phi'(s)\,\Gamma|^2\,ds - \frac{1}{t_1 - t_0}\int_{t_0}^{t_1} d\sigma \int_{t_0}^{t_1} |\phi'(s)\,\Gamma|^2\,ds \\ &= 0\end{aligned}$$

by Schwarz inequality in which equality holds only if $|\phi'(s)\,\Gamma|$ is a constant for all $s \in [t_0, t_1]$. But clearly that constant would be zero and hence we would have $\phi'(s)\,\Gamma = 0$ for $t_0 \leqslant s \leqslant t_1$, i.e.,

$$\int_{t_1+t_0-t}^{t_1} B'(s)\,\psi'(t_1, s)\,\Gamma\,ds = 0 \qquad \text{for all } t_0 \leqslant t \leqslant t_1 .$$

However, this would then contradict the complete controllability criterion of the linear system (*). Hence $\Gamma' S^* \Gamma > 0$ for all $\Gamma \neq 0$.

Now let us compute $S(t_1)$,

$$\begin{aligned}S(t_1) = &\int_{t_0}^{t_1} \psi(t_1, s)\,B(s) \int_{t_1+t_0-s}^{t_1} \phi'(\sigma)\,d\sigma\,ds \\ &- \frac{1}{t_1 - t_0}\int_{t_0}^{t_1} \psi(t_1, s)\,B(s)\,(s - t_0) \int_{t_0}^{t_1} \phi'(\sigma)\,d\sigma\,ds.\end{aligned}$$

By interchanging the order of integration, we note that

$$\begin{aligned}\int_{t_0}^{t_1} \psi(t_1, s)\,B(s) \int_{t_1+t_0-s}^{t_1} \phi'(\sigma)\,d\sigma\,ds &= \int_{t_0}^{t_1}\int_{t_1+t_0-\sigma}^{t_1} \psi(t_1, s)\,B(s)\,ds\,\phi'(\sigma)\,d\sigma \\ &= \int_{t_0}^{t_1} \phi(s)\,\phi'(s)\,ds\end{aligned}$$

and that

$$\begin{aligned}\int_{t_0}^{t_1} \psi(t_1, s)\,B(s)\,(s - t_0)\,ds &= \int_{t_0}^{t} \psi(t_1, s)\,B(s) \int_{t_1+t_0-s}^{t_1} d\sigma\,ds \\ &= \int_{t_0}^{t_1}\int_{t_1+t_0-\sigma}^{t_1} \psi(t_1, s)\,B(s)\,ds\,d\sigma = \int_{t_0}^{t_1} \phi(\sigma)\,d\sigma.\end{aligned}$$

This then shows that $S(t_1) = S^*$.

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