DISCRETE APPLIED MATHEMATICS

ELSEVIER Discrete Applied Mathematics 60 (1995) 93–98

Interval orders based on weak orders[☆]

Kenneth P. Bogart[a,*,1], Joseph Bonin[b,2], Jutta Mitas[c]

[a] Department of Mathematics and Computer Science, Dartmouth College, Hanover, NH 03755, USA
[b] George Washington University, Washington, DC 20052, USA
[c] Technische Hochschule Darmstadt, Darmstadt, Germany

Received 29 September 1992; revised 7 October 1994

Abstract

One definition of an interval order is as an order isomorphic to that of a family of nontrivial intervals of a linearly ordered set with $[a,b] < [c,d]$ if $b \leqslant c$. Fishburn's theorem states that an order is an interval order if and only if it has no four-element restriction isomorphic to the ordered set (shown in Fig. 1) "$\underline{2} + \underline{2}$". We show that an order is isomorphic to a family of nontrivial intervals of a weak order, ordered as above, if and only if it has no restriction to one of the four ordered sets (shown in Fig. 2) "$\underline{3} + \underline{2}$", "$\underline{2} + N$", a six-element crown or a six-element fence.

1. Introduction

An *order* is an irreflexive, transitive (and thus asymmetric) relation. We use "$<$" to stand for an order and "$\leqslant$" to stand for the associated reflexive, antisymmetric, transitive relation which is the union of $<$ and $=$. As is common, when the set to which $<$ applies can be determined from context, we use this same symbol to represent orders on two different sets. An *interval order* is an order isomorphic to that of a family of nontrivial intervals (trivial intervals are empty or have one point) of a linearly ordered set with $[a,b] < [c,d]$ if $b \leqslant c$.

Fishburn's theorem [2], proved in this form in [1], states that an order is an interval order if and only if it has no four-element restriction isomorphic to the ordering of "$\underline{2} + \underline{2}$" shown in Fig. 1; that is there are not four elements a, b, c and d with $a < b$, $c < d$ and no other comparisons.

A *weak order* is one that has no three-element restriction isomorphic to the order "$\underline{1} + \underline{2}$" shown in Fig. 1. Thus the diamond ordering shown in Fig. 1 is a weak

* Corresponding author.
☆ This paper is based on research begun at ARIDAM VI.
[1] Research Supported by ONR contract N0014-91-J-1019.
[2] Research partially supported by George Washington University UFF grant.

0166-218X/95/$09.50 © 1995—Elsevier Science B.V. All rights reserved
SSDI 0166-218X(95)00110-3

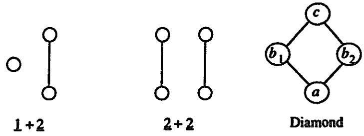

Fig. 1.

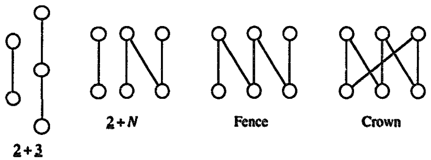

Fig. 2.

ordering. We show that an ordering is isomorphic to a set of nontrivial intervals of a weak ordering if and only if it has no restriction isomorphic to one of the orderings shown in Fig. 2.

We use the phrase *weak interval ordering* to mean an ordering of nontrivial intervals chosen from a weak ordering with $[a, b] < [c, d]$ if $b \leqslant c$ (there should be little chance of confusing this with an interval order which is also a weak order). The order $\underline{2} + \underline{2}$ is a weak interval order: take the intervals $[a, b_1]$, $[a, b_2]$, $[b_1, c]$, and $[b_2, c]$ of the diamond shown in Fig. 1.

This paper is a first step in an attempt to describe which classes of ordered sets have natural descriptions as sets of intervals of other classes of – presumably simpler – ordered sets. Weak orders are a natural place to begin this effort for two reasons: they are the most straightforward generalization of linear orders, and there is a reasonable likelihood for applications. Consider, for example the scheduling of meetings in rooms some distance apart with a discrete set of possible starting and stopping times, say every fifteen minutes. We can postulate that someone can travel from one room to another in one time period, but cannot participate in two meetings, one of which ends when the other starts, unless they are in the same room. Then the elements of our weak ordering are ordered pairs of a room and a time, and (room 1, time 1) is less than (room 2, time 2) if time 1 is less than time 2. Thus a meeting in room i ending at time j can have someone in common with a meeting starting in room i' at a time j' with $j' \geqslant j$ if and only if the pair (i, j) precedes or equals the pair (i', j'). (Note that it is traditional to define $[a, b] < [c, d]$ if $b < c$; our definition is motivated both by the

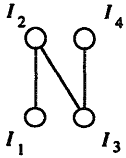

Fig. 3.

possible application and by the fact that while our definition and the traditional one are equivalent for intervals chosen from linear orders, they are not equivalent for intervals chosen from nonlinear orders, and our definition gives rise to interesting new classes of orders.)

2. Four examples which are not weak interval orderings

We return to the examples in Fig. 2. If the ordering $\underline{3} + \underline{2}$ were represented by intervals

$$I_j = [l_j, r_j]$$

with $I_1 < I_2 < I_3$ and $I_4 < I_5$, then for I_4 to be incomparable to I_3, at least one point of I_4 would have to be incomparable to l_3 or between l_3 and r_3 or incomparable to r_3. Then all points of I_5 would have to be greater than or equal to l_3 and thus greater than r_1 (because $l_3 > l_2 \geqslant r_1$), so that $I_1 < I_5$ as well.

Now suppose a set $I_1, \ldots, I_6$ of intervals from a weak order contains an "N" with $I_1 < I_2$, $I_3 < I_2$, I_4, and no other comparisons among $\{I_1, I_2, I_3, I_4\}$ as in Fig. 3.

Suppose l_2 is the left-hand endpoint of I_2 and r_3 is the right-hand endpoint of I_3. If $l_2 = r_3$, then $r_1 \leqslant l_2 \leqslant r_3 \leqslant l_4$ so that $I_1 < I_4$ as well. Thus $l_2 > r_3$. To show that the remaining three ordered sets in Fig. 2 do not arise from intervals of a weak order, assume we have $I_5 < I_6$ as in Fig. 4.

Since none of the three orders has an element covered by three elements, I_3 and I_6 must be incomparable. Then at least one point of I_6 is less than r_3 or incomparable to r_3. Thus all points of I_5 are less than or incomparable to r_3 so all points of I_5 are below l_2, a contradiction.

We summarize in Theorem 1.

Theorem 1. *None of the orders $\underline{3} + \underline{2}$, $N + \underline{2}$, a six-element fence, or a six-element crown has a representation as intervals from a weakly ordered set.*

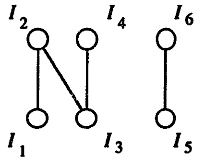

Fig. 4.

3. The predecessor–successor order of a weak interval ordering

One of the approaches to representing (linear) interval orders uses the fact that the predecessor sets, and predecessor–successor sets, defined by

$$P(x) = \mathrm{Pred}(x) = \{y \mid y < x\}$$

and

$$PS(x) = \mathrm{PredSuc}(x) = \bigcap_{z:z>x} \mathrm{Pred}(z),$$

respectively, are linearly ordered by set inclusion. For any ordered set $(V, <)$ we define the family $\mathscr{P}$ of predecessor sets and the family $\mathscr{P}\mathscr{S}$ of predecessor–successor sets in the same way, we refer to the ordered set $\mathscr{P} \cup \mathscr{P}\mathscr{S}$ with the set inclusion order as the predecessor–successor structure of $(V, <)$ and we denote it by $\mathscr{P}\mathscr{S}(V, <)$. It is straightforward [1] to show that the map that associates the interval

$$[\mathrm{Pred}(x), \mathrm{PredSuc}(x)]$$

to each element x of X is an isomorphism from $(V, <)$ to a set of nontrivial intervals of the predecessor–successor structure with $[a, b] < [c, d]$ if $b \leqslant c$. For this reason and by analogy with (linear) interval orders, it is natural to expect that the predecessor–successor structure of a weak interval order is a weak order. This is the case, and it means that given a weak interval order as a relation on a set of points rather than the usual relation on a set of intervals, we may discover from the ordered set itself a weak order from which we may choose intervals to represent the ordered set.

In order to prove this, we begin with a lemma which restricts the number of cases we must consider in order to demonstrate that $\mathscr{P}\mathscr{S}(V, <)$ is (or is not) a weak order.

Lemma 2. *If $\mathscr{P}\mathscr{S}(V, <)$ contains sets X, Y and Z with $Z \subset Y$, $Y \neq Z$, and X incomparable to both, then we may assume $X = P(x)$, $Y = P(y)$, $Z = PS(z)$.*

Proof. If $X = PS(x)$, then for each $w > x$, $P(w)$ is not a subset of Y or Z (for then $PS(x)$ would be). If $X = PS(x)$, then for some $w > x$, $P(w)$ does not contain Y or Z as a subset, for otherwise $PS(x)$ would. Thus there is a set $X' = P(x')$ such that X' is incomparable with both Y and Z. Similarly there is a set $Y' = P(y')$ such that $Z \subset Y'$, $Z \neq Y'$ and X' is incomparable with both Y' and Z. If $Z = P(z)$, then Z cannot be empty (for then it would be a subset of X') so there must be a $z' \in Z$ such that $z' \notin X'$ and therefore $PS(z')$, which contains z', cannot be a subset of X' but is a subset of Z and therefore Y'. Thus there is a $Z' = PS(z')$ which is a subset of Y', not equal to Y' and incomparable with X'. Replacing X, Y, and Z by the sets X', Y' and Z' proves the lemma. □

Theorem 3. *If $(V, <)$ is isomorphic to a set of nontrivial intervals of a weak order ordered by $[a,b] < [c,d]$ if $b \leqslant c$, then $\mathscr{PS}(V, <)$ is a weakly ordered set.*

Proof. Suppose, to the contrary, that there are three predecessor or predecessor–successor sets X, Y and Z such that $Z \subset Y$, $Y \neq Z$, and X is incomparable (relative to the subset relation) to both. By Lemma 2 we may assume $X = P(x)$, $Y = P(y)$ and $Z = PS(z)$ for some x, y and z. Recall that x, y and z may be represented by intervals in a weak ordering. Since x and y have incomparable predecessor sets, they must have incomparable left-hand endpoints. Thus for z to be a predecessor of y but not x, the right-hand endpoint of z must equal the left-hand endpoint l_y of y. Thus for any successor t of z, either $l_t = l_y$ or $l_t > l_y$. Thus $PS(z) = P(y)$, a contradiction. Therefore the subset ordering on $PS(V, <)$ must be a weak ordering. □

4. The forbidden restrictions

Our next theorem completes the characterization of weak interval orders.

Theorem 4. *If $(V, <)$ is an ordered set that is not isomorphic to a set of intervals of a weak ordering with $[a,b] < [c,d]$ if and only if $b \leqslant c$, then there is a subset U of V such that the restriction of $(V, <)$ to U is isomorphic to one of the four ordered sets in Fig. 2.*

Proof. By the remarks before Lemma 2 (that any order can be represented by intervals in its predecessor–successor structure), $\mathscr{PS}(V, <)$ is not weakly ordered and so by Lemma 2, there are elements x, y and z such that $PS(z) \subset P(y)$, $PS(z) \neq P(y)$ and $P(x)$ is incomparable (relative to the subset order) to both $PS(z)$ and $P(y)$. Since $P(x)$ is not a subset of $P(y)$, there must be an element $x' < x$ such that $x' \notin P(y)$. It is straightforward to check that x' is then incomparable with y and z. Further, since $PS(z) \subset P(y)$, z must be less than y, and since $PS(z) \neq P(y)$, there must be a $y' < y$ such that $y' \notin PS(z)$. In particular, y' cannot be less than z; however, it might be either incomparable with or greater than z. This means that the restriction of $(V, <)$ to $\{x, x', z, y, y'\}$ is one of the orders pictured in Fig. 5. In case 1, the set $\{x, x', y, y', z\}$ is the desired set U, since case 1 is the diagram of $\underline{3} + \underline{2}$. In cases 2, 3 and 4, since

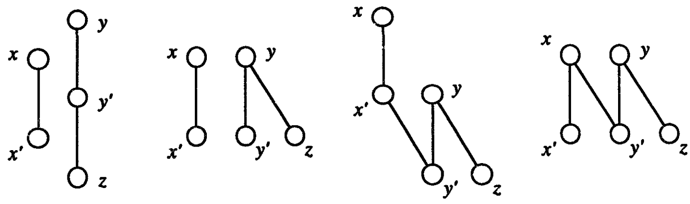

Fig. 5.

$y' \notin PS(z)$ we have a $z' > z$ with $y' \nless z'$. If z' were a predecessor of either y or y', then $\{x, x', y, z, z'\}$ would form a $\underline{3} + \underline{2}$. Thus we may assume that z' is incomparable with y and with y'. In case 3, the set $\{x, x', y', z, z'\}$ is the desired set U since the restriction of $(V, <)$ to this set must be isomorphic to $\underline{3} + \underline{2}$.

In case 2, z' could be above x' but not x, giving the six-element fence, z' could be above x in which case the restriction of $(V, <)$ to $\{x', x, z', y, y'\}$ would be isomorphic to $\underline{3} + \underline{2}$, or z' could be incomparable to both x and x', giving the ordering $\underline{2} + N$.

In case 4, z' could be incomparable to x and x', giving the six-element fence; z' could be over x', giving the six-element crown, but if z' were above x, it would be above y', and we have assumed z' and y' are incomparable.

Thus in all cases, one of the four orderings shown in Fig. 2 must be a restriction of $(V, <)$. □

This completes the characterization of weak interval orders. Note that with the more traditional relation of $<$ for intervals, namely $[a, b] < [c, d]$ if and only if $b < c$ (an equivalent definition for intervals chosen from linearly ordered sets), the family of interval orders based on weak orders will be identical to the family of interval orders based on linear orders because the first family can have no restriction isomorphic to $\underline{2} + \underline{2}$. Even the interval orders based on interval orders will themselves be interval orders. This leaves us with two natural questions. First, with the less than relation used in this paper, what other families of orders give rise to families of interval orders with similar forbidden restriction characterizations? Second, with the more traditional definition of the less than relation, are there other natural families of interval orders with similar characterizations?

References

[1] K.P. Bogart, An obvious proof of Fishburn's representation theorem for interval orders, Discrete Math. 118 (1993) 239–242.
[2] P.C. Fishburn, Interval Graphs and Interval Orders (Wiley, New York, 1985).