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Optimal consumption of the finite time horizon Ramsey problem

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ABSTRACT

In this paper, we study the stochastic Ramsey problem related to an economic growth model with the CES production function in a finite time horizon. By changing variables, the Hamilton–Jacobi–Bellman equation associated with this optimization problem is transformed. By the viscosity solution technique, we show the existence of a classical solution of the transformed Hamilton–Jacobi–Bellman equation, and then give an optimal consumption policy of the original problem.

1. Introduction

We are concerned with the stochastic Ramsey problem with a finite time horizon $T > 0$ in order to choose a consumption rate processes $c = \{c(t)\} \in \mathcal{A}_T$ maximizing the discounted expected utilities with a constant discount rate $\alpha \geqslant 0$:

$$J_T(c : x) = \mathbb{E}\left[\int_0^{\tau^{x,c}\wedge T} e^{-\alpha t} U\big(c(t)\big)\,dt + e^{-\alpha(\tau^{x,c}\wedge T)} g\big(X^{x,c}_{\tau^{x,c}\wedge T}\big)\right], \tag{1.1}$$

where a controlled process $X^{x,c}$ is governed by the stochastic differential equation (SDE, for short)

$$dX_t = \big[f(X_t) - c_t\big]\,dt + \sigma X_t\,dW_t, \quad t \geqslant 0, \qquad X_0 = x \geqslant 0; \tag{1.2}$$

$\{W_t\}_{t\geqslant 0}$ is a standard one-dimensional Brownian motion on a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$, endowed with a filtration $\mathbb{F} = \{\mathcal{F}_t\}_{t\geqslant 0}$ which is the $\mathbb{P}$-augmentation of the filtration generated by the Brownian motion W; $\sigma > 0$ is a diffusion constant; $f \in C^1(\mathbf{R}_+)$ is a concave function with $f(0) = 0$ and $f'(\infty) > -\infty$; $\mathcal{A}_T$ denotes the collection of all consumption policies $c(\cdot)$ which are $\mathbf{R}_+$-valued, $\mathbb{F}$-progressively measurable processes satisfying $\int_0^T c(t)\,dt < \infty$ a.s.;

$$\tau^{x,c} = \inf\big\{t \geqslant 0\colon\ X^{x,c}(t) = 0\big\}.$$

We assume that the utility function U has the following properties:

$$U \in C(\mathbf{R}_+) \cap C^2(0,\infty), \qquad U''(c) < 0 < U'(c), \qquad U'(\infty) = U(0) = 0, \tag{1.3}$$

and $g \in C(\mathbf{R}_+)$ is a non-decreasing concave function with $g(0) = 0$.

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The economic growth model in connection with the Ramsey problem has been studied by [4] and [9]. Recently, [10] treats this problem for the constant elasticity of substitution (CES) production function $F(z, y)$ with $\partial F(0+, y)/\partial z < \infty$ producing the commodity for the capital stock $z \geqslant 0$ and the labor force $y > 0$ in the infinite time horizon case. The capital stock $z(t)$ and the labor supply $y(t)$ at time t are governed by

$$dy(t) = \mu y(t)\,dt - \sigma y(t)\,dW_t, \qquad y(0) = y > 0, \quad \mu \neq 0,$$
$$dz(t) = \big[F\big(z(t), y(t)\big) - \nu z(t) - c(t)y(t)\big]\,dt, \qquad z(0) = z \geqslant 0.$$

Changing the variables by $X(t) = z(t)/y(t)$ and $f(x) = F(x, 1) - (\mu + \nu - \sigma^2)x$ for $x = z/y$, we observe that the optimal consumption in the growth model can be obtained by a reduction to the stochastic Ramsey problem (1.1)–(1.2) with $g \equiv 0$. However, [10] contains some incomplete proofs, which will be made correct later, for the existence results on viscosity solutions of the associated Hamilton–Jacobi–Bellman (HJB, for short) equations of *elliptic* type.

The purpose of this paper is to present a synthesis of optimal consumption policy $c^* \in \mathcal{A}_T$ for the stochastic Ramsey problem (1.1)–(1.2). The associated HJB equation is given by the *parabolic* PDE

$$\alpha u = u_t + \frac{\sigma^2}{2}x^2 u_{xx} + f(x)u_x + \widetilde{U}(u_x), \quad t \in (0, T),\ x > 0,$$
$$u(t, 0) = 0, \qquad u(T, x) = g(x), \quad t \in [0, T],\ x \geqslant 0, \tag{1.4}$$

where the subscripts denote the partial derivatives and $\widetilde{U}(x)$ is the Legendre transform of $-U(-x)$, i.e. $\widetilde{U}(x) = \max_{c \geqslant 0}[U(c) - cx]$, $x > 0$. The difficulty in solving the problem lies in the fact that (1.4) is degenerate and $\widetilde{U}$ is non-Lipschitz. Changing variables by

$$V(t, x) = u(T - t, x), \quad t \in [0, T],\ x \geqslant 0,$$

we have

$$\alpha V = -V_t + \frac{\sigma^2}{2}x^2 V_{xx} + f(x)V_x + \widetilde{U}(V_x), \quad t \in (0, T),\ x > 0,$$
$$V(t, 0) = 0, \qquad V(0, x) = g(x), \quad t \in [0, T],\ x \geqslant 0. \tag{1.5}$$

Our method consists in finding a smooth solution V using the comparison results for solutions of one-dimensional SDEs. By the viscosity solution technique, we show that the transformed HJB equation (1.5) admits a unique solution V, and the restrictive conditions for the existence of u are relaxed. The optimal consumption rate $c^* \in \mathcal{A}_T$ can be represented in a feedback form.

The remainder of this paper is organized in the following way: In the next section, we show the existence of viscosity solutions V of (1.5) with $T = \infty$. In Section 3, we derive the $C^{1,2}$-regularity of u, and we give a synthesis of the optimal consumption $c^* \in \mathcal{A}_T$ in terms of u. Section 4 is devoted to an application of our results to the infinite time horizon problem.

2. The transformed HJB equations

2.1. Stochastic control problems

Let us consider the stochastic control problem

$$V(t, x) := \sup_{c \in \mathcal{C}} J_t(c : x) = \sup_{c \in \mathcal{C}} \mathbb{E}\left[\int_0^{\tau^{x,c} \wedge t} e^{-\alpha s} U(c_s)\,ds + e^{-\alpha t} g\big(X_t^{x,c}\big)\mathbb{1}_{\{\tau^{x,c} > t\}}\right], \quad t, x \geqslant 0,$$

where $\mathcal{C} = \bigcap_{T>0} \mathcal{A}_T$ and $\mathbb{1}$ is the indicator function. Clearly, $V(t, 0) = 0$ and $V(0, x) = g(x)$. By the comparison results for solutions of one-dimensional SDEs, see e.g. Proposition 5.2.18 of [5], we see that the mappings $x \mapsto X_t^{x,c}(\omega)$ and $x \mapsto \tau^{x,c}(\omega)$ are non-decreasing for a.e. (t, ω). Since $U \geqslant 0$ and g is non-decreasing, we note that

$$\text{the function } x \mapsto V(t, x) \text{ is non-decreasing for each } t \geqslant 0. \tag{2.1}$$

For each $\gamma \in \mathbf{R}$, we denote by $\mathcal{E}_\gamma(\cdot)$ the exponential martingale

$$\mathcal{E}_\gamma(t) = \exp\left[\gamma W_t - \frac{\gamma^2}{2}t\right], \quad t \geqslant 0.$$

For $t, x \geqslant 0$, $c \in \mathcal{C}$ and $\beta \in \mathbf{R}$, by Itô's formula, we then have

$$x + (\gamma+\sigma)\int_0^{t\wedge\tau^{x,c}} e^{-(\alpha+\beta)s}\mathcal{E}_\gamma(s)X_s\,dW_s = \int_0^{t\wedge\tau^{x,c}} e^{-(\alpha+\beta)s}\mathcal{E}_\gamma(s)c_s\,ds + e^{-(\alpha+\beta)(t\wedge\tau^{x,c})}\mathcal{E}_\gamma\big(t\wedge\tau^{x,c}\big)X_{t\wedge\tau^{x,c}} + \int_0^{t\wedge\tau^{x,c}} e^{-(\alpha+\beta)s}\mathcal{E}_\gamma(s)\big[(\alpha+\beta-\sigma\gamma)X_s - f(X_s)\big]\,ds. \tag{2.2}$$

Since the function $x \mapsto f(x)/x$ is non-increasing and $f(0)=0$, we note that

$$l := f'(\infty) \leqslant \frac{f(x)}{x} \leqslant f'(0+), \quad x>0. \tag{2.3}$$

Thus, if $\beta \geqslant f'(0+) - \alpha + \sigma\gamma$, then the local martingale in the left-hand side of (2.2) is super-martingale, and thereby

$$\mathbb{E}\left[\int_0^{t\wedge\tau^{x,c}} e^{-(\alpha+\beta)s}\mathcal{E}_\gamma(s)c_s\,ds\right] \leqslant x, \quad t, x \geqslant 0,\ c \in \mathcal{C}. \tag{2.4}$$

In this section and the next section, we set

$$\gamma := 0 \quad \text{and} \quad \beta := \big(f'(0+) - \alpha\big)^+,$$

and thus (2.4) holds with $\mathcal{E}_\gamma(\cdot) \equiv 1$. Further, by (2.2) and Jensen's inequality, we have

$$\mathbb{E}\big[e^{-\alpha t}g\big(X_t^{x,c}\big)\mathbb{1}_{\{\tau^{x,c}>t\}}\big] \leqslant e^{-\alpha t}g\big(\mathbb{E}\big[X^{x,c}_{\tau^{x,c}\wedge t}\big]\big) \leqslant e^{-\alpha t}g\big(xe^{(\alpha+\beta)t}\big), \quad t, x \geqslant 0,\ c \in \mathcal{C}. \tag{2.5}$$

Proposition 2.1. *For all $t, x, z \geqslant 0$ and $y > 0$,*

$$V(t,x) \geqslant \mathbb{E}\left[\int_0^t e^{-\alpha s}U\big(xze^{(l-z)s}\mathcal{E}_\sigma(s)\big)\,ds + e^{-\alpha t}g\big(xe^{(l-z)t}\mathcal{E}_\sigma(t)\big)\right], \tag{2.6}$$

$$V(t,x) \leqslant \widetilde{U}(y)\int_0^t e^{-\alpha s}\,ds + xye^{\beta t} + e^{-\alpha t}g\big(xe^{(\alpha+\beta)t}\big). \tag{2.7}$$

Proof. Clearly, the above inequalities with $x=0$ hold true because $V(t,0) = U(0) = g(0) = 0$. Let $\widetilde{c}(s) := z\widetilde{X}(s) = zxe^{(l-z)s}\mathcal{E}_\sigma(s)$ for $x>0$. Then, by (2.3),

$$d\widetilde{X}(s) = (l-z)\widetilde{X}(s)\,ds + \sigma\widetilde{X}(s)\,dW_s \leqslant \big[f\big(\widetilde{X}(s)\big) - \widetilde{c}_s\big]\,ds + \sigma\widetilde{X}(s)\,dW_s.$$

Thus the comparison results show $0 < \widetilde{X} \leqslant X^{x,\widetilde{c}}$ a.e. and $\tau^{x,\widetilde{c}} \wedge t = t$ a.s. which gives (2.6).

Since $\mathcal{E}_\gamma(\cdot) \equiv 1$, by (2.4) and (2.5), we also have

$$\begin{aligned} V(t,x) &\leqslant \sup_{c\in\mathcal{C}}\mathbb{E}\left[\int_0^{\tau^{x,c}\wedge t} e^{-\alpha s}\big[U\big(c(s)\big) - yc(s)\big]\,ds\right] + y\sup_{c\in\mathcal{C}}\mathbb{E}\left[\int_0^{\tau^{x,c}\wedge t} e^{-\alpha s}c(s)\,ds\right] + e^{-\alpha t}g\big(xe^{(\alpha+\beta)t}\big) \\ &\leqslant \widetilde{U}(y)\int_0^t e^{-\alpha s}\,ds + xye^{\beta t} + e^{-\alpha t}g\big(xe^{(\alpha+\beta)t}\big), \quad y>0. \qquad \square \end{aligned}$$

Remark 2.2. We recall that the decreasing convex function $\widetilde{U}$ on $(0,\infty)$ has the properties

$$\widetilde{U}(0+) = U(\infty), \qquad \widetilde{U}(\infty) = U(0) = 0, \qquad U(x) = \inf_{y>0}\big[\widetilde{U}(y) + xy\big], \quad x \geqslant 0.$$

Thus we know from (2.6)–(2.7) that

$$V(t,0+) = 0, \qquad V(0+,x) = g(x), \quad t, x \geqslant 0,$$

$$V(t,\infty) = U(\infty)\int_0^t e^{-\alpha s}\,ds + e^{-\alpha t}g(\infty), \quad t>0.$$

2.2. Auxiliary control problems and a dynamic programming principle

Let us next introduce the auxiliary control problems:

$$V_n(t,x) := \sup_{c\in\mathcal{C}_n} J_t(c:x), \quad t,x \geqslant 0,\ n \geqslant 1, \tag{2.8}$$

where $\mathcal{C}_n = \{c \in \mathcal{C}:\ 0 \leqslant c \leqslant n \text{ a.e.}\}$.

Proposition 2.3. *For each $t, x \geqslant 0$, $V_n(t,x) \nearrow V(t,x)$ as $n \to \infty$.*

Proof. Fix an arbitrary $c \in \mathcal{C}$, and set $c_n := n \wedge c \in \mathcal{C}_n$. Since $X^{x,c} \leqslant X^{x,c_n}$ a.e. and $\tau^{x,c} \leqslant \tau^{x,c_n}$ a.s., Fatou's lemma shows

$$\lim_{n\to\infty} V_n(t,x) \geqslant \lim_{n\to\infty} J_t(c_n:x) \geqslant \lim_{n\to\infty} \mathbb{E}\left[\int_0^{\tau^{x,c}\wedge t} e^{-\alpha s} U\big(c_n(s)\big)\,ds + e^{-\alpha t} g\big(X_t^{x,c}\big)1_{\{\tau^{x,c}>t\}}\right] \geqslant J_t(c:x)$$

which means $\lim_{n\to\infty} V_n(t,x) \geqslant V(t,x)$. Since $V_n \leqslant V$, we have the assertion. □

Let us define

$$\boldsymbol{w}(t) := \mathbb{E}\big[\big(e^{f'(0+)t}\mathcal{E}_\sigma(t) - 1\big)^+\big] + \mathbb{E}\big[\big(e^{(\sigma^2 - f'(\infty))t}\mathcal{E}_\sigma(t) - 1\big)^+\big], \quad t \geqslant 0.$$

Then we see $\boldsymbol{w}(t) \to 0$ as $t \downarrow 0$.

Lemma 2.4. *For $0 \leqslant s \leqslant t$, $0 < x \leqslant y$ and $\xi > 0$, V_n satisfies the following properties:*

$$0 \leqslant V_n(t,y) - V_n(t,x) \leqslant t\left[U(\xi) + \frac{y-x}{y}U'\left(\xi\frac{x}{y}\right)n\right] + e^{-\alpha t}\frac{y-x}{x} g\big(xe^{(\alpha+\beta)t}\big), \tag{2.9}$$

$$V_n(s,x) - V_n(t,x) \leqslant g\big(xe^{(\alpha+\beta)s}\big)\big[\alpha e^{-\alpha s}(t-s) + e^{-\alpha t}\boldsymbol{w}(t-s)\big], \tag{2.10}$$

$$V_n(t,x) - V_n(s,x) \leqslant U(n)(t-s) + e^{-\alpha s} g\big(xe^{(\alpha+\beta)s}\big)\boldsymbol{w}(t-s). \tag{2.11}$$

Proof. 1. Let $c \in \mathcal{C}_n$ be arbitrary, and set $z := x/y \in (0,1)$, $\widehat{c} := zc$ and $\widehat{X} := zX^{y,c}$. Since $x \mapsto f(x)/x$ is non-increasing,

$$d\widehat{X}_s = \big[zf(\widehat{X}_s/z) - \widehat{c}(s)\big]ds + \sigma\widehat{X}_s\,dW_s \leqslant \big[f(\widehat{X}_s) - \widehat{c}(s)\big]ds + \sigma\widehat{X}_s\,dW_s.$$

By the comparison results we have $zX^{y,c} \leqslant X^{x,\widehat{c}}$ a.e. and $\tau^{y,c} \leqslant \tau^{x,\widehat{c}}$ a.s. Let g'_+ be the right-hand derivative of the concave function g. Since $xg'_+(x) \leqslant g(x)$, $x > 0$, we obtain

$$\begin{aligned}
J_t(c:y) - V_n(t,x) &\leqslant \mathbb{E}\left[\int_0^{\tau^{y,c}\wedge t} e^{-\alpha s}\big\{U\big(c(s)\big) - U\big(\widehat{c}(s)\big)\big\}\,ds + e^{-\alpha t}\big\{g\big(X_t^{y,c}\big)1_{\{\tau^{y,c}>t\}} - g\big(X_t^{x,\widehat{c}}\big)1_{\{\tau^{x,\widehat{c}}>t\}}\big\}\right] \\
&\leqslant U(\xi)\int_0^t e^{-\alpha s}\,ds + \mathbb{E}\left[\int_0^{\tau^{y,c}\wedge t} e^{-\alpha s}\big\{U\big(c(s)\big) - U\big(zc(s)\big)\big\}1_{\{c(s)\geqslant\xi\}}\,ds\right] \\
&\quad + e^{-\alpha t}\mathbb{E}\big[\big\{g\big(X_t^{y,c}\big) - g\big(zX_t^{y,c}\big)\big\}1_{\{\tau^{y,c}>t\}}\big] \\
&\leqslant tU(\xi) + (1-z)\cdot\mathbb{E}\left[\int_0^{\tau^{y,c}\wedge t} e^{-\alpha s}\cdot c(s)U'\big(zc(s)\big)1_{\{c(s)\geqslant\xi\}}\,ds\right] \\
&\quad + e^{-\alpha t}(1-z)\mathbb{E}\big[X_t^{y,c}g'_+\big(zX_t^{y,c}\big)1_{\{\tau^{y,c}>t\}}\big] \\
&\leqslant tU(\xi) + t(1-z)nU'(z\xi) + e^{-\alpha t}\frac{1-z}{z}\mathbb{E}\big[g\big(zX^{y,c}_{\tau^{y,c}\wedge t}\big)\big].
\end{aligned}$$

Thus (2.9) follows from the above estimate and (2.5).

2. Let $c \in \mathcal{C}_n$ be arbitrary, and set $\widetilde{c}(v) := c(v)1_{\{v\leqslant\tau^{x,c}\wedge s\}} \in \mathcal{C}_n$. Since $f(x) \geqslant xf'(\infty)$ for $x \geqslant 0$, the comparison results show

$$X_v^{x,\widetilde{c}} \geqslant X_s^{x,c}e^{f'(\infty)(v-s)+\sigma(W_v-W_s)-(\sigma^2/2)(v-s)} =: X_s^{x,c}H_{s,v}^0 \quad \text{on } \big\{\tau^{x,c} > s\big\}$$

for all $v \geqslant s$. This means $\{\tau^{x,\widetilde{c}} > t\} = \{\tau^{x,c} > s\}$ a.s. Since $U \geqslant 0$, we get

$$\begin{aligned}
J_s(c:x) - V_n(t,x) &\leqslant J_s(c:x) - J_t(\widetilde{c}:x) \leqslant \mathbb{E}\big[e^{-\alpha s} g\big(X_s^{x,c}\big)\mathbb{1}_{\{\tau^{x,c}>s\}} - e^{-\alpha t} g\big(X_t^{x,\widetilde{c}}\big)\mathbb{1}_{\{\tau^{x,\widetilde{c}}>t\}}\big] \\
&\leqslant \big(e^{-\alpha s} - e^{-\alpha t}\big)\mathbb{E}\big[g\big(X_{\tau^{x,c}\wedge s}^{x,c}\big)\big] + e^{-\alpha t}\mathbb{E}\big[\big\{g\big(X_s^{x,c}\big) - g\big(X_s^{x,c} H_{s,v}^0\big)\big\}\mathbb{1}_{\{\tau^{x,c}>s\}}\big] \\
&\leqslant \alpha e^{-\alpha s}(t-s) g\big(x e^{(\alpha+\beta)s}\big) + e^{-\alpha t}\mathbb{E}\big[g'_+\big(X_s^{x,c} H_{s,v}^0\big) X_s^{x,c}\big(1 - H_{s,v}^0\big)^+ \mathbb{1}_{\{\tau^{x,c}>s\}}\big] \\
&\leqslant \textbf{1st term} + e^{-\alpha t}\mathbb{E}\bigg[g\big(X_s^{x,c} H_{s,t}^0\big)\bigg(\frac{1}{H_{s,t}^0} - 1\bigg)^+ \mathbb{1}_{\{\tau^{x,c}>s\}}\bigg] \\
&\leqslant \textbf{1st term} + e^{-\alpha t}\mathbb{E}\big[g\big(X_{\tau^{x,c}\wedge s}^{x,c}\big)\big]\mathbb{E}\bigg[\bigg(\frac{1}{H_{s,t}^0} - 1\bigg)^+\bigg] \\
&\leqslant g\big(x e^{(\alpha+\beta)s}\big)\big[\alpha e^{-\alpha s}(t-s) + e^{-\alpha t}\boldsymbol{w}(t-s)\big] \qquad (2.12)
\end{aligned}$$

where we used the fact that $H_{s,t}^0$ is independent of $\mathcal{F}_s$. Hence we have (2.10).

3. Let $c \in \mathcal{C}_n$ be arbitrary, and set $\widetilde{c}(v) := c(v)\mathbb{1}_{\{v \leqslant \tau^{x,c}\}}$ and $Y_t^c := X_t^{x,\widetilde{c}}/x$ for $x > 0$. Since $f(x) \leqslant x f'(0+)$ for $x \geqslant 0$, $Y_t^c \leqslant e^{f'(0+)t}\mathcal{E}_\sigma(t)$ a.s. by the comparison results. This implies

$$\frac{X_t^{x,\widetilde{c}}}{X_s^{x,c}} \leqslant e^{f'(0+)(t-s)+\sigma(W_t - W_s)-(\sigma^2/2)(t-s)} =: H_{s,t}^1 \quad \text{on } \big\{\tau^{x,c} > s\big\}.$$

Thus

$$\begin{aligned}
J_t(c:x) - V_n(s,x) &\leqslant J_t(c:x) - J_s(c:x) \\
&\leqslant U(n)(t-s) + e^{-\alpha s}\mathbb{E}\big[\big\{g\big(X_t^{x,c}\big) - g\big(X_s^{x,c}\big)\big\}\mathbb{1}_{\{\tau^{x,c}>t\}}\big] \\
&\leqslant U(n)(t-s) + e^{-\alpha s}\mathbb{E}\big[g'_+\big(X_s^{x,c}\big)\big(X_t^{x,c} - X_s^{x,c}\big)^+ \mathbb{1}_{\{\tau^{x,c}>t\}}\big] \\
&\leqslant U(n)(t-s) + e^{-\alpha s}\mathbb{E}\big[g\big(X_s^{x,c}\big)\big(H_{s,t}^1 - 1\big)^+ \mathbb{1}_{\{\tau^{x,c}>s\}}\big] \\
&= U(n)(t-s) + e^{-\alpha s}\mathbb{E}\big[g\big(X_{\tau^{x,c}\wedge s}^{x,c}\big)\big]\mathbb{E}\big[\big(H_{s,t}^1 - 1\big)^+\big],
\end{aligned}$$

which yields (2.11). □

Next we will establish a dynamic programming principle for V_n.

Theorem 2.5. *The following property holds*:

For all $t, x \geqslant 0$ and an $\mathbb{F}$-stopping time θ with $\theta \leqslant t$ a.s.,

$$V_n(t,x) = \sup_{c \in \mathcal{C}_n} \mathbb{E}\Bigg[\int_0^{\theta\wedge\tau^{x,c}} e^{-\alpha s} U\big(c(s)\big)\,ds + e^{-\alpha(\theta\wedge\tau^{x,c})} V_n\big(t - \theta\wedge\tau^{x,c}, X^{x,c}\big(\theta\wedge\tau^{x,c}\big)\big)\Bigg]. \qquad (2.13)$$

This property holds true even if θ depends on a control $c \in \mathcal{C}_n$.

Proof. Since $V_n(t,0) = 0$, $V_n(0,x) = g(x)$ and $\tau^{0,c} = 0$ a.s., the assertion is trivial if $t = 0$ or $x = 0$. Fix arbitrary $t, x > 0$ and an $\mathbb{F}$-stopping time θ with $\theta \leqslant t$ a.s. Denote $\tau_s^{x,c} := s \wedge \tau^{x,c}$. Then we observe

$$\begin{aligned}
V_n(t,x) &= \sup_{c\in\mathcal{C}_n}\mathbb{E}\Bigg[\int_0^{\tau_\theta^{x,c}} e^{-\alpha v} U\big(c(v)\big)\,dv + \mathbb{E}\Bigg[\int_\theta^{\tau_t^{x,c}} e^{-\alpha v} U\big(c(v)\big)\,dv + e^{-\alpha t} g\big(X_t^{x,c}\big)\mathbb{1}_{\{\tau^{x,c}>t\}} \big| \mathcal{F}_\theta\Bigg] \cdot \mathbb{1}_{\{\theta<\tau^{x,c}\}}\Bigg] \\
&= \sup_{c\in\mathcal{C}_n}\mathbb{E}\Bigg[\int_0^{\tau_\theta^{x,c}} e^{-\alpha v} U\big(c(v)\big)\,dv + \mathbb{1}_{\{\theta<\tau^{x,c}\}} \cdot e^{-\alpha\theta} J_{t-\theta}\big(\widehat{c} : X_\theta^{x,c}\big)\Bigg] \\
&\leqslant \sup_{c\in\mathcal{C}_n}\mathbb{E}\Bigg[\int_0^{\tau_\theta^{x,c}} e^{-\alpha v} U\big(c(v)\big)\,dv + \mathbb{1}_{\{\theta<\tau^{x,c}\}} \cdot e^{-\alpha\theta} V_n\big(t-\theta, X^{x,c}(\theta)\big)\Bigg] \\
&= \sup_{c\in\mathcal{C}_n}\mathbb{E}\Bigg[\int_0^{\tau_\theta^{x,c}} e^{-\alpha v} U\big(c(v)\big)\,dv + e^{-\alpha\tau_\theta^{x,c}} V_n\big(t-\tau_\theta^{x,c}, X^{x,c}\big(\tau_\theta^{x,c}\big)\big)\Bigg], \qquad (2.14)
\end{aligned}$$

where $\widehat{c}$ is the shifted process of c by θ, i.e. $\widehat{c}(s) = c(s+\theta)$.

Let $x_1, \varepsilon > 0$ and $c \in \mathcal{C}_n$ be arbitrary. To prove the reverse inequality, we take sequences

$$t_j := \frac{t}{M} j, \qquad x_{j+1} := x_1 + \frac{N - x_1}{M} j, \quad j = 0, \dots, M,$$

for large numbers $M, N \geqslant t$. Let us define the $\mathcal{F}_\theta$-measurable sets $\{A_{ij}\}$ as

$$A_{ij} = \big\{\omega \in \Omega\colon\ \theta \in [t_{i-1}, t_i),\ X(\theta) \in [x_j, x_{j+1})\big\}, \quad i, j = 1, \dots, M.$$

By means of (2.8), there is a $\{c_{ij}\} \subset \mathcal{C}_n$ such that

$$V_n(t - t_i, x_j) - \varepsilon \leqslant J_{t-t_i}(c_{ij} : x_j), \quad i, j = 1, \dots, M.$$

We may assume that $c_{ij}(v) = 0$ on $\{v \geqslant \tau_{t-t_i}^{x_j, c_{ij}}\}$. Define $\widehat{c} \in \mathcal{C}_n$ as

$$\widehat{c}(s) = c(s)\mathbb{1}_{\{s \leqslant \theta\}} + \sum_{i,j=1}^{M} c_{ij}(s - \theta)\mathbb{1}_{A_{ij}}\mathbb{1}_{\{s > \theta\}}.$$

Then we have

$$\begin{aligned}
K^{M,N} &:= J_t(\widehat{c} : x) - \mathbb{E}\Bigg[\int_0^{\tau_\theta^{x,c}} e^{-\alpha v} U\big(c(v)\big)\,dv\Bigg] \\
&= \mathbb{E}\big[e^{-\alpha t} g\big(X_t^{x,c}\big)\mathbb{1}_{\{\theta = t < \tau^{x,c}\}}\big] + \mathbb{E}\big[\mathbb{1}_{\{\theta < \tau_t^{x,c}\}} e^{-\alpha\theta} J_{t-\theta}\big(\widehat{c}(\theta + \cdot) : X_\theta^{x,c}\big)\big] \\
&\geqslant \mathbb{E}\big[e^{-\alpha\theta} V_n\big(t - \theta, X_\theta^{x,c}\big)\mathbb{1}_{\{\theta = t < \tau^{x,c}\}}\big] + \sum_{i,j=1}^{M} \mathbb{E}\big[\mathbb{1}_{\{\theta < \tau_t^{x,c}\}}\mathbb{1}_{A_{ij}} e^{-\alpha\theta} J_{t-\theta}(c_{ij} : x_j)\big] \\
&\geqslant \textbf{1st term} + \sum_{i,j=1}^{M} \mathbb{E}\big[\mathbb{1}_{\{\theta < \tau_t^{x,c}\}}\mathbb{1}_{A_{ij}} e^{-\alpha\theta}\big\{V_n(t - t_i, x_j) - \varepsilon\big\}\big] \\
&\quad - \sum_{i,j=1}^{M} \mathbb{E}\big[\mathbb{1}_{\{\theta < \tau_t^{x,c}\}}\mathbb{1}_{A_{ij}} e^{-\alpha\theta}\big\{J_{t-t_i}(c_{ij} : x_j) - J_{t-\theta}(c_{ij} : x_j)\big\}\big] \\
&\geqslant -\varepsilon + \mathbb{E}\big[e^{-\alpha\theta} V_n\big(t - \theta, X_\theta^{x,c}\big)\mathbb{1}_{\{\theta < \tau^{x,c}\}}\big] \\
&\quad - \sum_{i,j=1}^{M} \mathbb{E}\big[e^{-\alpha\theta}\big\{V_n\big(t - \theta, X_\theta^{x,c}\big) - V_n(t - t_i, x_j)\big\}\mathbb{1}_{A_{ij}}\mathbb{1}_{\{\theta < \tau_t^{x,c}\}}\big] \\
&\quad - \mathbb{E}\big[e^{-\alpha\theta} V_n\big(t - \theta, X_\theta^{x,c}\big) \cdot \mathbb{1}_{\{0 \leqslant X(\theta) < x_1\}}\big] - \mathbb{E}\big[e^{-\alpha\theta} V_n\big(t - \theta, X_\theta^{x,c}\big) \cdot \mathbb{1}_{\{X(\theta) \geqslant N\}}\big] \\
&\quad - \sum_{i,j=1}^{M} \mathbb{E}\big[e^{-\alpha\theta}\big\{J_{t-t_i}(c_{ij} : x_j) - J_{t-\theta}(c_{ij} : x_j)\big\}\mathbb{1}_{\{\theta < \tau_t^{x,c}\}}\mathbb{1}_{A_{ij}}\big] \\
&=: -\varepsilon + \textbf{2nd term} - \sum_{i,j=1}^{M} \mathbb{E}\big[e^{-\alpha\theta} K_{ij}\big] - K_0 - K_N - \sum_{i,j=1}^{M} \mathbb{E}\big[e^{-\alpha\theta} L_{ij}\big].
\end{aligned}$$

Moreover, Lemma 2.4 and (2.12) yield

$$\begin{aligned}
K_{ij} &\leqslant \big[\big\{V_n(t - \theta, x_{j+1}) - V_n(t - \theta, x_j)\big\} + \big\{V_n(t - \theta, x_j) - V_n(t - t_i, x_j)\big\}\big]\mathbb{1}_{A_{ij}}\mathbb{1}_{\{\theta < \tau^{x,c}\}} \\
&\leqslant tU(x_1)\mathbb{1}_{A_{ij}} + \left[\frac{tn}{x_1}U'\left(\frac{x_1^2}{N}\right) + \frac{g(Ne^{(\alpha+\beta)t})}{x_1}\right]\frac{N}{M}\mathbb{1}_{A_{ij}} + \left[U(n)\frac{N}{M} + g\big(Ne^{(\alpha+\beta)t}\big)\boldsymbol{w}\left(\frac{N}{M}\right)\right]\mathbb{1}_{A_{ij}}, \\
L_{ij} &\leqslant g\big(Ne^{(\alpha+\beta)t}\big)\left[\alpha\frac{N}{M} + \boldsymbol{w}\left(\frac{N}{M}\right)\right]\mathbb{1}_{A_{ij}}.
\end{aligned}$$

By (2.7) and the standard results on solutions of SDEs with random coefficients, we also have

$$K_0 \leqslant \mathbb{E}\big[V_n(t-\theta, x_1)\big] \leqslant t\widetilde{U}\left(\frac{1}{\sqrt{x_1}}\right) + e^{\beta t}\sqrt{x_1} + g\big(x_1 e^{(\alpha+\beta)t}\big),$$

$$\begin{aligned} K_N &\leqslant \mathbb{E}\Big[\mathbb{1}_{\{\sup_{0\leqslant s\leqslant t}|X^{x,c}(s)|\geqslant N\}}\Big(t\widetilde{U}(1) + g(1) + e^{\beta t}\big(1+g'_+(1)\big)\sup_{0\leqslant s\leqslant t}\big|X^{x,c}(s)\big|\Big)\Big] \\ &\leqslant \boldsymbol{c}_0(t)\Big\{1+\mathbb{E}\Big[\sup_{0\leqslant s\leqslant t} X^{x,c}(s)^2\Big]^{1/2}\Big\}\mathbb{P}\Big\{\sup_{0\leqslant s\leqslant t}\big|X^{x,c}(s)\big|\geqslant N\Big\}^{1/2} \\ &\leqslant \frac{\boldsymbol{c}_1(t,x,n)}{N} \end{aligned}$$

for positive constants $\boldsymbol{c}_0$ and $\boldsymbol{c}_1$, where we used the inequality $g(x) \leqslant g(1) + g'_+(1)x$. Combining the above estimates, we obtain

$$\begin{aligned} \lim_{M\to\infty} K^{M,N} &\geqslant -\varepsilon + \mathbb{E}\big[e^{-\alpha\theta} V_n\big(t-\theta, X^{x,c}(\theta)\big)\cdot \mathbb{1}_{\{\theta<\tau^{x,c}\}}\big] \\ &\quad - \left[tU(x_1) + t\widetilde{U}\left(\frac{1}{\sqrt{x_1}}\right) + e^{\beta t}\sqrt{x_1} + g\big(x_1e^{(\alpha+\beta)t}\big) + \frac{\boldsymbol{c}_1}{N}\right]. \end{aligned}$$

Letting $\varepsilon, x_1 \downarrow 0$ and $N \to \infty$, we have the part "$\geqslant$" of the equality (2.13). □

2.3. Viscosity solutions

Given a real-valued function w on $\mathbf{R}_+^2$, we shall denote by w^* (resp. w_*) its upper (resp. lower) semi-continuous envelope, i.e.

$$w^*(t,x) = \lim_{\varepsilon\downarrow 0}\sup\big\{w(s,y):\ |t-s|+|x-y|\leqslant\varepsilon,\ s,y>0\big\}, \quad t,x\geqslant 0, \tag{2.15}$$

and $w_* = -(-w)^*$.

Definition 2.6. Let $\mathcal{G}$ be an $\mathbf{R}$-valued continuous function on $\mathbf{R}_+ \times \mathbf{R} \times (0,\infty) \times \mathbf{R}$, and consider the non-linear PDE

$$w_t(t,x) + \mathcal{G}\big(x, w(t,x), w_x(t,x), w_{xx}(t,x)\big) = 0, \quad t,x>0. \tag{2.16}$$

Assume further that $a \mapsto \mathcal{G}(x,r,q,a)$ is non-increasing. Let w be a locally bounded $\mathbf{R}$-valued function on $(0,\infty)^2$ and φ be a smooth $\mathbf{R}$-valued function on $(0,\infty)^2$.

(i) w is called a viscosity super-solution to (2.16) if

$$\varphi_t(t_0,x_0) + \mathcal{G}\big(x_0, w_*(t_0,x_0), \varphi_x(t_0,x_0), \varphi_{xx}(t_0,x_0)\big) \geqslant 0$$

for any local minimizer $(t_0,x_0) \in (0,\infty)^2$ of $(w_* - \varphi)$ on $(0,\infty)^2$.

(ii) w is called a viscosity sub-solution to (2.16) if

$$\varphi_t(t_0,x_0) + \mathcal{G}\big(x_0, w^*(t_0,x_0), \varphi_x(t_0,x_0), \varphi_{xx}(t_0,x_0)\big) \leqslant 0$$

for any local maximizer $(t_0,x_0) \in (0,\infty)^2$ of $(w^* - \varphi)$ on $(0,\infty)^2$.

(iii) w is called a viscosity solution to (2.16) if it satisfies the above requirements (i) and (ii).

Let us define the differential operator $\mathcal{L}$ as

$$\mathcal{L}\varphi = \frac{\sigma^2}{2}x^2\varphi_{xx} + f(x)\varphi_x - \alpha\varphi.$$

Lemma 2.7. *V_n is a continuous viscosity solution to the corresponding HJB equation*

$$(V_n)_t(t,x) - \mathcal{L}V_n(t,x) - \widetilde{U}_n\big((V_n)_x(t,x)\big) = 0, \quad t,x>0,$$

where $\widetilde{U}_n(x) = \max_{0\leqslant c\leqslant n}[U(c) - cx]$.

Proof. By (2.7) and Lemma 2.4 we first know that V_n is continuous on $\mathbf{R}_+^2$. Let $(t_0, x_0) \in (0,\infty)^2$ and φ be an $\mathbf{R}$-valued smooth test function on $\mathbf{R}_+^2$.

1. Assume that $0 = (V_n - \varphi)(t_0, x_0) = \min_{\mathbf{R}_+^2}(V_n - \varphi)$. Fix an arbitrary $c \in [0, n]$. For a large number m, let us define

$$\theta_m = \frac{1}{m} \wedge \inf\left\{s \geqslant 0:\ \left|X^{x_0,c}(s) - x_0\right| \geqslant \frac{x_0}{2}\right\} < \frac{t_0}{2} \wedge \tau^{x_0,c}.$$

Since $\varphi \leqslant V_n$, it follows from Theorem 2.5 that

$$\begin{aligned}
0 &= V_n(t_0, x_0) - \varphi(t_0, x_0) \\
&\geqslant \mathbb{E}\left[\int_0^{\theta_m} e^{-\alpha s} U(c)\,ds + e^{-\alpha\theta_m} V_n\big(t_0 - \theta_m, X^{x_0,c}(\theta_m)\big)\right] - \varphi(t_0, x_0) \\
&\geqslant \mathbb{E}\left[\int_0^{\theta_m} e^{-\alpha s} U(c)\,ds + e^{-\alpha\theta_m} \varphi\big(t_0 - \theta_m, X^{x_0,c}(\theta_m)\big)\right] - \varphi(t_0, x_0) \\
&= \mathbb{E}\left[\int_0^{\theta_m} e^{-\alpha s}\big[U(c) - c\cdot\varphi_x\big(t_0 - s, X^{x_0,c}(s)\big) + \mathcal{L}\varphi\big(t_0 - s, X^{x_0,c}(s)\big) - \varphi_t\big(t_0 - s, X^{x_0,c}(s)\big)\big]\,ds\right] \\
&= \frac{1}{m}\mathbb{E}\left[\int_0^1 \mathbb{1}_{\{s/m \leqslant \theta_m\}} e^{-\alpha(s/m)}\left[U(c) - c\cdot\varphi_x\left(t_0 - \frac{s}{m}, X^{x_0,c}\left(\frac{s}{m}\right)\right)\right.\right. \\
&\qquad \left.\left. + \mathcal{L}\varphi\left(t_0 - \frac{s}{m}, X^{x_0,c}\left(\frac{s}{m}\right)\right) - \varphi_t\left(t_0 - \frac{s}{m}, X^{x_0,c}\left(\frac{s}{m}\right)\right)\right]ds\right] \\
&=: \frac{1}{m} L_m.
\end{aligned}$$

The standard results about solutions of the SDEs give

$$0 \leqslant \mathbb{E}[1 - m\theta_m] \leqslant \mathbb{P}\big\{\theta_m < m^{-1}\big\} \leqslant \mathbb{P}\Big\{\sup_{0 \leqslant s \leqslant m^{-1}} \left|X^{x_0,c}(s) - x_0\right| \geqslant \frac{x_0}{2}\Big\} \leqslant \frac{\mathbf{c}_0}{m},$$

where $\mathbf{c}_0 = \mathbf{c}_0(x_0, n)$ is a positive constant. Therefore, after passing to a subsequence, $m\theta_m \xrightarrow{m\to\infty} 1$ a.s. Hence the dominated convergence theorem shows

$$0 \geqslant \lim_{m\to\infty} L_m = \big[U(c) - c\varphi_x(t_0, x_0)\big] + \mathcal{L}\varphi(t_0, x_0) - \varphi_t(t_0, x_0).$$

This implies the super-viscosity property of V_n.

2. Suppose that $0 = (V_n - \varphi)(t_0, x_0) = \max_{\mathbf{R}_+^2}(V_n - \varphi)$. To prove the sub-viscosity property of V_n, we assume to the contrary that $-2\varepsilon := -\varphi_t(t_0, x_0) + \mathcal{L}\varphi(t_0, x_0) + \widetilde{U}_n(\varphi_x(t_0, x_0)) < 0$. Then there exists a $\delta > 0$ such that

$$-\varphi_t(t, x) + \mathcal{L}\varphi(t, x) + \widetilde{U}_n\big(\varphi_x(t, x)\big) \leqslant -\varepsilon$$

for all $(t, x) \in [t_0 - \delta, t_0 + \delta] \times [x_0 - \delta, x_0 + \delta] \subset (0,\infty)^2$. Let $c \in \mathcal{C}_n$ be arbitrary, and define

$$\begin{aligned}
&\theta^c = \delta \wedge \inf\big\{s \geqslant 0:\ \left|X^{x_0,c}(s) - x_0\right| \geqslant \delta\big\}, \\
&\theta_0 = \min\big[\inf\big\{s \geqslant 0:\ X^{x_0,0}(s) \geqslant x_0 + \delta\big\},\ \inf\big\{s \geqslant 0:\ X^{x_0,n}(s) \leqslant x_0 - \delta\big\}\big].
\end{aligned}$$

By the comparison results, we note $0 < \delta \wedge \theta_0 \leqslant \theta^c$ a.s. Since $\varphi \geqslant V_n$, by Itô's formula, we obtain

$$\begin{aligned}
\varphi(t_0, x_0) = \mathbb{E}&\left[\int_0^{\theta^c} e^{-\alpha s} U\big(c(s)\big)\,ds + e^{-\alpha\theta^c}\varphi\big(t_0 - \theta^c, X^{x_0,c}\big(\theta^c\big)\big)\right] \\
&- \mathbb{E}\left[\int_0^{\theta^c} e^{-\alpha s}\big[U\big(c(s)\big) - c(s)\varphi_x\big(t_0 - s, X^{x_0,c}(s)\big)\big]\,ds\right]
\end{aligned}$$

$$
\begin{aligned}
&- \mathbb{E}\left[\int_0^{\theta^c} e^{-\alpha s}\left[\mathcal{L}\varphi\left(t_0-s, X^{x_0,c}(s)\right) - \varphi_t\left(t_0-s, X^{x_0,c}(s)\right)\right] ds\right] \\
&\geqslant \mathbb{E}\left[\int_0^{\theta^c} e^{-\alpha s} U\left(c(s)\right) ds + e^{-\alpha\theta^c} V_n\left(t_0-\theta^c, X^{x_0,c}\left(\theta^c\right)\right)\right] + \varepsilon\mathbb{E}\left[\int_0^{\theta^c} e^{-\alpha s} ds\right] \\
&\geqslant \mathbb{E}\left[\int_0^{\theta^c} e^{-\alpha s} U\left(c(s)\right) ds + e^{-\alpha\theta^c} V_n\left(t_0-\theta^c, X^{x_0,c}\left(\theta^c\right)\right)\right] + \varepsilon e^{-\delta}\mathbb{E}[\delta \wedge \theta_0].
\end{aligned}
$$

By the arbitrariness of $c \in \mathcal{C}_n$ and Theorem 2.5, we see that the above inequality provides the contradiction: $\varepsilon e^{-\delta}\mathbb{E}[\delta \wedge \theta_0] \leqslant 0$. □

Theorem 2.8. *V is a viscosity solution to the associated HJB equation*

$$V_t(t,x) - \mathcal{L}V(t,x) - \widetilde{U}\left(V_x(t,x)\right) = 0, \quad t, x > 0 \tag{2.17a}$$

with the boundary conditions

$$V^*(t,0) = V_*(t,0) = 0, \qquad V^*(0,x) = V_*(0,x) = g(x), \quad t, x \geqslant 0. \tag{2.17b}$$

Proof. Thanks to Proposition 2.3, it is easy to check

$$
\begin{aligned}
V_*(t,x) &= \lim_{m\to\infty} \inf\left\{V_n(s,y)\colon n \geqslant m,\ |t-s| + |x-y| \leqslant m^{-1},\ s, y > 0\right\}, \\
V^*(t,x) &= \lim_{m\to\infty} \sup\left\{V_n(s,y)\colon n \geqslant m,\ |t-s| + |x-y| \leqslant m^{-1},\ s, y > 0\right\}.
\end{aligned}
$$

In view of Lemma 2.7, the viscosity properties follow from the arguments in §6 of [1] because $\widetilde{U}_n \nearrow \widetilde{U}$ as $n \to \infty$.

Taking (2.7) into account, we see

$$
\begin{aligned}
V^*(t,0) &\leqslant \varlimsup_{(t_0,x)\to(t,0)} \left[t_0\widetilde{U}(y) + xye^{\beta t_0} + g\left(xe^{(\alpha+\beta)t_0}\right)\right] = t\widetilde{U}(y) \xrightarrow{y\to\infty} 0, \\
V^*(0,x) &\leqslant \varlimsup_{(t,x_0)\to(0,x)} \left[t\widetilde{U}(y) + x_0 ye^{\beta t} + g\left(x_0 e^{(\alpha+\beta)t}\right)\right] = xy + g(x) \xrightarrow{y\downarrow 0} g(x), \\
V_*(0,x) &\geqslant \varliminf_{(t,x_0)\to(0,x)} J_t(0 : x_0) \geqslant \varliminf_{(t,x_0)\to(0,x)} \mathbb{E}\left[e^{-\alpha t} g\left(x_0 e^{lt}\mathcal{E}_\sigma(t)\right)\right] \geqslant g(x).
\end{aligned}
$$

Hence we have (2.17b). □

Theorem 2.9. *Let u (resp. v) be an upper (resp. lower) semi-continuous sub-solution (resp. super-solution) of* (2.17a) *and*

$$u(t,0) = v(t,0) = 0, \qquad u(0,x) = v(0,x) = g(x), \quad t, x \geqslant 0, \tag{2.18}$$

$$\sup_{t,x\geqslant 0} \frac{|u(t,x)| + |v(t,x)|}{1 + t + xe^{\beta t}} < \infty. \tag{2.19}$$

Then $u \leqslant v$ on $\mathbf{R}_+^2$.

Proof. Assume to the contrary that $2\zeta := u(t_0,x_0) - v(t_0,x_0) > 0$ for some $t_0, x_0 > 0$, and let us work towards a contradiction. For any $m \geqslant 1$ and $\eta > \sigma^2 + 2|f'(0+)| + 1$, we define

$$
\begin{aligned}
&\psi(t,x) = \left(1 + x^2\right)e^{\eta t}, \\
&\varphi(t,x,s,y) = u(t,x) - v(s,y) - \frac{m}{2}\left[(t-s)^2 + (x-y)^2\right] - \varepsilon\left[\psi(t,x) + \psi(s,y)\right], \\
&M_m = \sup_{\mathbf{R}_+^2\times\mathbf{R}_+^2} \varphi(t,x,s,y),
\end{aligned}
$$

where the constant $\varepsilon > 0$ is chosen so that

$$M_m \geqslant \varphi(t_0,x_0,t_0,x_0) = u(t_0,x_0) - v(t_0,x_0) - 2\varepsilon\psi(t_0,x_0) = \zeta.$$

Since $\eta > \beta = (f'(0+) - \alpha)^+$, (2.19) guarantees that $M_m = \varphi(t_m, x_m, s_m, y_m)$ for some $(t_m, x_m), (s_m, y_m) \in \mathbf{R}_+^2$, and

$$\zeta + \frac{m}{2}\big[(t_m - s_m)^2 + (x_m - y_m)^2\big] + \varepsilon\big[\psi(t_m, x_m) + \psi(s_m, y_m)\big] \leqslant u(t_m, x_m) - v(s_m, y_m),$$

which provides that $\{(t_m, x_m)\}_m$ and $\{(s_m, y_m)\}_m$ are located in a compact subset of $\mathbf{R}_+^2$. Therefore, after passing to a subsequence, $(t_m, x_m) \xrightarrow{m\to\infty} (t_*, x_*) \in \mathbf{R}_+^2$. Further, Lemma 3.1 in [1] gives

$$m\big[(t_m - s_m)^2 + (x_m - y_m)^2\big] \xrightarrow{m\to\infty} 0,$$
$$M_m \xrightarrow{m\to\infty} u(t_*, x_*) - v(t_*, x_*) - 2\varepsilon\psi(t_*, x_*) \geqslant \zeta.$$

Hence we know from (2.18) that $(t_*, x_*) \in (0,\infty)^2$ and so $(t_m, x_m), (s_m, y_m) \in (0,\infty)^2$ for sufficient large m. By virtue of Theorem 8.3 in [1], there exist $a, b \in \mathbf{R}$ such that

$$\begin{aligned}
&ax^2 - by^2 \leqslant 3m(x - y)^2, \quad x, y \in \mathbf{R},\\
&\big(p_m + \varepsilon\psi_t(t_m, x_m), q_m + \varepsilon\psi_x(t_m, x_m), a + \varepsilon\psi_{xx}(t_m, x_m)\big) \in \overline{\mathcal{P}}^{2,+}u(t_m, x_m),\\
&\big(p_m - \varepsilon\psi_t(s_m, y_m), q_m - \varepsilon\psi_x(s_m, y_m), b - \varepsilon\psi_{xx}(s_m, y_m)\big) \in \overline{\mathcal{P}}^{2,-}v(s_m, y_m),
\end{aligned} \tag{2.20}$$

where $p_m = m(t_m - s_m)$, $q_m = m(x_m - y_m)$ and $\overline{\mathcal{P}}^{2,+}w(z)$ (resp. $\overline{\mathcal{P}}^{2,-}w(z)$) is the closed superjet (resp. subjet) of the function w at the point z, see §8 in [1] for the definitions. By the viscosity properties of the functions u and v, (2.20) implies that

$$\begin{aligned}
0 &\geqslant \alpha u(t_m, x_m) + p_m - f(x_m)q_m - \frac{\sigma^2}{2}x_m^2 a - \widetilde{U}\big(q_m + \varepsilon\psi_x(t_m, x_m)\big)\\
&\quad + \varepsilon\left[\psi_t(t_m, x_m) - f(x_m)\psi_x(t_m, x_m) - \frac{\sigma^2}{2}x_m^2\psi_{xx}(t_m, x_m)\right]\\
&\geqslant \alpha u(t_m, x_m) + p_m - f(x_m)q_m - \frac{\sigma^2}{2}x_m^2 a - \widetilde{U}(q_m) + \varepsilon\psi(t_m, x_m),\\
0 &\leqslant \alpha v(s_m, y_m) + p_m - f(y_m)q_m - \frac{\sigma^2}{2}y_m^2 b - \widetilde{U}\big(q_m - \varepsilon\psi_x(s_m, y_m)\big)\\
&\quad - \varepsilon\left[\psi_t(s_m, y_m) - f(y_m)\psi_x(s_m, y_m) - \frac{\sigma^2}{2}y_m^2\psi_{xx}(s_m, y_m)\right]\\
&\leqslant \alpha v(s_m, y_m) + p_m - f(y_m)q_m - \frac{\sigma^2}{2}y_m^2 b - \widetilde{U}(q_m) - \varepsilon\psi(s_m, y_m),
\end{aligned}$$

where we use the monotonicity of $\widetilde{U}$. Putting these inequalities together, we get

$$\begin{aligned}
\alpha\big[u(t_m, x_m) - v(s_m, y_m)\big] + \varepsilon\big[\psi(t_m, x_m) + \psi(s_m, y_m)\big] &\leqslant \big[f(x_m) - f(y_m)\big]q_m + \frac{\sigma^2}{2}\big(x_m^2 a - y_m^2 b\big)\\
&\leqslant \big[|f'(0+)| + |f'(\infty)| + 2\sigma^2\big]\cdot m(x_m - y_m)^2 \xrightarrow{m\to\infty} 0,
\end{aligned}$$

which yields the contradiction: $\alpha\zeta + 2(1+\alpha)\varepsilon\psi(t_*, x_*) \leqslant 0$. □

Remark 2.10. By Theorems 2.8 and 2.9 and (2.7), we have $V^* \leqslant V_*$ on $\mathbf{R}_+^2$. Since $V_* \leqslant V \leqslant V^*$ on $\mathbf{R}_+^2$ by (2.15) and (2.17b), we see that V is continuous on $\mathbf{R}_+^2$.

Remark 2.11. The results in this section hold true even if $U(0) > 0$, exclusive of the continuity of $V(t,\cdot)$ at $x = 0$. Indeed, we can easily prove that

$$V^*(t, 0+) = V_*(t, 0+) = U(0)^+\frac{1 - e^{-\alpha t}}{\alpha}, \quad t \geqslant 0.$$

Adding the assumption $U(\infty) > 0$ and replacing $\widetilde{U}$ (resp. U) with $\widetilde{U}^+$ (resp. U^+) in (2.7) (resp. (2.9)–(2.10)), we do not require the assumption $U(0) \geqslant 0$ in order to obtain the results in this section. However the concavity of $V(t,\cdot)$, which is proved for the case $U(0) = 0$ in the next section, may be no longer the truth if $U(0) < 0$. Although there is room for argument on this point, we may leave the details to future studies.

3. Classical solutions and optimal consumption

In this section, we study the $C^{1,2}$-regularity of the continuous viscosity solution $V(t,x)$ of (2.17), and we present a synthesis of the optimal consumption $c^* \in \mathcal{A}_T$ for the optimization problem (1.1).

Lemma 3.1. *The function $x \mapsto V(t,x)$ is concave for each $t \geqslant 0$, and*

$$\lambda(t_0,t_1,x) := \inf_{t_0 \leqslant t \leqslant t_1} D_x^+ V(t,x) > 0, \quad 0 < t_0 \leqslant t_1,\ x > 0, \tag{3.1}$$

where $D_x^+ V$ denotes the right-hand derivative of the concave function $V(t,\cdot)$.

Proof. Fix arbitrary $p \in [0,1]$, $t, x_1, x_2 > 0$ and $c_1, c_2 \in \mathcal{C}$, and define

$$\widehat{c}_i(s) = c_i(s)\mathbb{1}_{\{s \leqslant \tau^{x_i,c_i}\}}, \quad i = 1,2, \qquad \widetilde{X}(s) = pX^{x_1,\widehat{c}_1}(s) + (1-p)X^{x_2,\widehat{c}_2}(s).$$

Set $x = px_1 + (1-p)x_2$ and $c(s) = p\widehat{c}_1(s) + (1-p)\widehat{c}_2(s)$. Thanks to the comparison results, we then have $\widetilde{X} \leqslant X^{x,c}$ a.e. and $\tau^{x_1,c_1} \vee \tau^{x_2,c_2} \leqslant \tau^{x,c}$ a.s. by means of the concavity of f. Since U and g are concave and non-negative, we observe

$$V(t,x) \geqslant J_t(c:x) \geqslant pJ_t(c_1:x_1) + (1-p)J_t(c_2:x_2),$$

and thus $V(t,\cdot)$ is concave.

To prove (3.1), we will assume that $\lambda(t_0,t_1,x_0) = 0$ for certain $0 < t_0 \leqslant t_1$ and $x_0 > 0$, and then contradict this assumption. Then for all $\varepsilon > 0$ there is an $s_\varepsilon \in [t_0,t_1]$ such that $D_x^+ V(s_\varepsilon,x_0) \leqslant \varepsilon$. Since $x \mapsto D_x^+ V(s_\varepsilon,x)$ is non-increasing, we see that $D_x^+ V(s_\varepsilon,x) \leqslant \varepsilon$ for all $x \geqslant x_0$, and hence

$$V(s_\varepsilon,x) \leqslant V(s_\varepsilon,x_0) + \varepsilon(x - x_0), \quad x \geqslant x_0.$$

Extracting a subsequence, $s_\varepsilon \xrightarrow{\varepsilon \downarrow 0} s_0 \in [t_0,t_1]$. Letting $\varepsilon \downarrow 0$ in the above inequality, by Proposition 2.1, we have

$$\mathbb{E}\left[\int_0^{s_0} e^{-\alpha s} U\big(xe^{(l-1)s}\mathcal{E}_\sigma(s)\big)\,ds + e^{-\alpha s_0} g\big(xe^{(l-1)s_0}\mathcal{E}_\sigma(s_0)\big)\right] \leqslant \widetilde{U}(y)\int_0^{s_0} e^{-\alpha s}\,ds + x_0 e^{\beta s_0} y + e^{-\alpha s_0} g\big(x_0 e^{(\alpha+\beta)s_0}\big)$$

for all $x \geqslant x_0$ and $y > 0$. Letting $x \to \infty$, Fatou's lemma gives

$$U(\infty) \leqslant \inf_{y>0}\big[\widetilde{U}(y) + x_0\xi y\big] + \xi e^{-(\alpha+\beta)s_0}\big[g\big(x_0 e^{(\alpha+\beta)s_0}\big) - g(\infty)\big] \leqslant U(x_0\xi),$$

where $\xi^{-1} := e^{-\beta s_0}\int_0^{s_0} e^{-\alpha v}\,dv$. This is in contradiction with (1.3). □

Define

$$u(t,x) = V(T-t,x), \quad t \in [0,T],\ x \geqslant 0. \tag{3.2}$$

Then we have the first main result.

Theorem 3.2. *Under* (1.3)*, u satisfies the following properties*:

- (i) *The function $x \mapsto u(t,x)$ is strictly increasing and concave on $\mathbf{R}_+$ for each $t \in [0,T)$.*
- (ii) *$u \in C([0,T] \times \mathbf{R}_+) \cap C^{1,2}([0,T) \times (0,\infty))$ is a solution to the HJB equation* (1.4)*.*

Proof. We have already proved the assertion (i) in the previous lemma. We can also deduce the assertion (ii) from the following:

$$V \in C\big(\mathbf{R}_+^2\big) \cap C^{1,2}\big((0,\infty)^2\big) \quad \text{is a solution to (2.17)}. \tag{3.3}$$

To show this claim, let $Q := (t_0,t_1) \times (x_0,x_1)$ be any bounded open subset of $\mathbf{R}_+^2$. We consider the parabolic equation

$$v_t - \mathcal{L}v - \widetilde{U}(v_x \vee \lambda) = 0 \quad \text{in } Q \tag{3.4a}$$

with the boundary condition

$$v(t_i,x) = V(t_i,x), \qquad v(t,x_i) = V(t,x_i), \quad (t,x) \in \overline{Q},\ i = 0,1, \tag{3.4b}$$

where the constant $\lambda = \lambda(t_0,t_1,x_1) > 0$ is as in (3.1).

According to [8], there exists a solution $v \in C(\overline{Q}) \cap C^{1,2}(Q)$ to (3.4) since $\widetilde{U}(\cdot \vee \lambda)$ is Lipschitz. Clearly, v is a viscosity solution to (3.4). On the other hand, we can deduce by the concavity of V that $\varphi_x(s_0, y_0) \geqslant D_x^+ V(s_0, y_0)$ for any $\varphi \in C^{1,2}((0,\infty)^2)$ and $(s_0, y_0) \in (0,\infty)$ which gives an extremal value of $(V - \varphi)$. Therefore we know by Theorem 2.8 that V is a continuous viscosity solution to (3.4). By the standard arguments as in Theorem 8.2 of [1], we can prove the comparison results on viscosity solutions of (3.4). Hence $V = v \in C(\overline{Q}) \cap C^{1,2}(Q)$. Thus we have (3.3). □

Let us define

$$I(x) = \operatorname*{argmax}_{c \geqslant 0}\big[U(c) - cx\big] = (U')^{-1}(x) \cdot \mathbb{1}_{\{x < U'(0+)\}}, \quad x > 0,$$

$$G(t,x) = I\big(u_x(t,x)\big) \cdot \mathbb{1}_{\{t<T\}} \cdot \mathbb{1}_{\{x>0\}}, \quad t \in [0,T],\ x \in \mathbf{R},$$

and consider the SDE

$$dX_t^* = \big[f\big(X_t^*\big) - G\big(t, X_t^*\big)\big]\,dt + \sigma X_t^*\,dW_t, \quad t \in [0,T], \qquad X_0^* = x \geqslant 0. \tag{3.5}$$

Lemma 3.3. *The SDE* (3.5) *has a unique non-negative solution* X^*.

Proof. 1. In view of (3.2) and (2.6), by L'Hospital's rule and Fatou's lemma, we have

$$\begin{aligned} u_x(t,0+) &\geqslant \mathbb{E}\left[\int_0^{T-t} e^{-\alpha s} \lim_{x \downarrow 0} \frac{U(xye^{(l-y)s}\mathcal{E}_\sigma(s))}{x}\,ds\right] \\ &= U'(0+)\frac{y}{\alpha - l + y}\big[1 - e^{-(\alpha - l + y)(T-t)}\big] \xrightarrow{y \to \infty} U'(0+), \end{aligned} \tag{3.6}$$

which implies $I(u_x(t,0+)) = 0$. Hence G is continuous on $[0,T) \times \mathbf{R}$ and $x \mapsto G(t,x)$ is non-decreasing.

2. For $m \geqslant 1$, $t \in [0,T]$ and $x \in \mathbf{R}$, let us define $g_m(t,x) := G(t, x \vee m^{-1}) - G(t, m^{-1})$. Then we note that $g_m(t,\cdot)$ is locally Lipschitz on $\mathbf{R}$ and $0 \leqslant g_m \leqslant g_{m+1}$. Since f is Lipschitz on $\mathbf{R}_+$, Theorem V.1.1 of [7] guarantees that there is a unique solution x_m of the SDE

$$dx_m(t) = \big[f\big(x_m(t)^+\big) - g_m\big(t, x_m(t)\big)\big]\,dt + \sigma x_m(t)\,dW_t, \quad t \in [0,T], \tag{3.7}$$

with an initial data $x_m(0) = x \geqslant 0$. The comparison results show

$$x_{m+1}(t) \leqslant x_m(t) \leqslant xe^{f'(0+)t} \cdot \mathcal{E}_\sigma(t) \quad \text{a.s.},\ t \in [0,T].$$

Since $f(x^+) - g_m(t,x) = 0$ for $x \leqslant 0$, by applying Itô's formula to $\phi(x) := (x^-)^3 \in C^2(\mathbf{R})$, we have $d\phi(x_m(t)) = \phi(x_m(t))[3\sigma^2\,dt + 3\sigma\,dW_t]$, i.e.,

$$\big(x_m(t)^-\big)^3 = \big(x^-\big)^3 \cdot e^{3\sigma^2 t} \cdot \mathcal{E}_{3\sigma}(t) = 0 \quad \text{a.s.},\ t \in [0,T],$$

which means $x_m \geqslant 0$. Now we define $X^*(t) := \lim_{m\to\infty} x_m(t)$, $t \in [0,T]$. Sending m to infinity in (3.7), we know that X^* satisfies (3.5) because $g_m \nearrow G$ as $m \to \infty$.

3. Let X^0 and X^1 be two non-negative solutions of (3.5) and set $Z_t := X_t^0 - X_t^1$. Since $G(t,\cdot)$ is non-increasing, by Itô's formula, we have

$$\begin{aligned} dZ_t^2 &= 2Z_t\big[f\big(X_t^0\big) - f\big(X_t^1\big) - \big\{G\big(t,X_t^0\big) - G\big(t,X_t^1\big)\big\}\big]\,dt + \sigma^2 Z_t^2\,dt + 2\sigma Z_t^2\,dW_t \\ &\leqslant Z_t^2[C_0\,dt + 2\sigma\,dW_t], \end{aligned}$$

where $C_0 = 2(|f'(0+)| + |f'(\infty)|) + \sigma^2$. Hence the comparison results prove that

$$Z_t^2 \leqslant Z_0^2 \cdot e^{C_0 t} \cdot \mathcal{E}_{2\sigma}(t) = 0, \quad t \in [0,T],$$

which implies the uniqueness of X^*. □

Now we obtain another main result.

Theorem 3.4. *Under* (1.3), $c^* = \{c^*(t) := G(t, X^*(t))\}_{t \geqslant 0} \in \mathcal{A}_T$ *is an optimal consumption process for the optimization problem* (1.1).

Proof. The previous lemma implies $c^* \in \mathcal{A}_T$. Since u is a classical solution of the HJB equation (1.4), we can use Itô's formula to get

$$u(0,x) = \mathbb{E}\left[\int_0^{\theta_n \wedge T \wedge \tau} e^{-\alpha t} U\big(c_t^*\big)\,dt\right] + \mathbb{E}\big[e^{-\alpha(\theta_n \wedge T \wedge \tau)} u\big(\theta_n \wedge T \wedge \tau, X^*_{\theta_n \wedge T \wedge \tau}\big)\big]$$
$$\leqslant J_T\big(c^* : x\big) + \mathbb{E}\big[u\big(\theta_n, X^*_{\theta_n}\big)\mathbb{1}_{\{\theta_n < T \wedge \tau\}}\big], \tag{3.8}$$

where $\tau = \tau^{x,c^*}$ and $\theta_n = \inf\{t \geqslant 0\colon\ X^*(t) \geqslant n\}$, $n \geqslant 1$. Since $X^*(t) \leqslant xe^{f'(0+)t} \cdot \mathcal{E}_\sigma(t)$, by (2.7), we have

$$\mathbb{E}\big[u(\theta_n, n)\mathbb{1}_{\{\theta_n < T \wedge \tau\}}\big] \leqslant \boldsymbol{c}_0(1+n)\mathbb{P}\Big\{\sup_{0 \leqslant t \leqslant T} xe^{f'(0+)t} \cdot \mathcal{E}_\sigma(t) \geqslant n\Big\}$$
$$\leqslant \boldsymbol{c}_0 x^2 e^{2(\alpha+\beta)T}\mathbb{E}\Big[\sup_{0 \leqslant t \leqslant T} \mathcal{E}_\sigma(t)^2\Big]\frac{1+n}{n^2} \xrightarrow{n \to \infty} 0,$$

where $\boldsymbol{c}_0 = \boldsymbol{c}_0(x,T)$ is a positive constant. Hence we obtain $u(0,x) \leqslant J_T(c^* : x)$, i.e. c^* is optimal. □

By the definition of u in (3.2), we observe that

$$e^{-\alpha t} u(t,x) = \sup_{c \in \mathcal{A}_T} \mathbb{E}\left[\int_t^{T \wedge \theta_t^{x,c}} e^{-\alpha s} U(c_s)\,ds + e^{-\alpha T} g\big(Y_T^{t,x,c}\big)\mathbb{1}_{\{\theta_t^{x,c} > T\}}\right], \quad t \in [0,T],\ x \geqslant 0,$$

where a controlled process $Y^{t,x,c}$ is given as a solution to the SDE

$$dY_s = \big[f(Y_s) - c_s\big]\,ds + \sigma Y_s\,dW_s, \quad s \in [t,T], \qquad Y_t = x \geqslant 0$$

and $\theta_t^{x,c} = \inf\{s \geqslant t\colon\ Y^{t,x,c}(s) = 0\}$.

Theorem 3.5. *Assume* (1.3). *Let* $w \in C([0,T] \times \mathbf{R}_+) \cap C^{1,2}([0,T) \times (0,\infty))$ *be a solution to the HJB equation* (1.4) *with polynomial growth. Then* $w = u$ *on* $[0,T] \times \mathbf{R}_+$.

Proof. Since $w(t,0) = 0$ and $\widetilde{U}(w_x(t,x)) < \infty$, we first note $w(t,x) \geqslant 0$. Let $(t,x) \in [0,T) \times (0,\infty)$ and $c \in \mathcal{A}_T$ be arbitrary.

1. By Itô's formula and (1.4), we have

$$e^{-\alpha t} w(t,x) + \sigma \int_t^{T \wedge \theta_t^{x,c}} e^{-\alpha s} Y_s w_x(s, Y_s)\,dW_s \geqslant \int_t^{T \wedge \theta_t^{x,c}} e^{-\alpha s} U(c_s)\,ds + e^{-\alpha(T \wedge \theta_t^{x,c})} w\big(T \wedge \theta_t^{x,c}, Y_{T \wedge \theta_t^{x,c}}\big) \geqslant 0$$

and thereby

$$e^{-\alpha t} w(t,x) \geqslant \mathbb{E}\left[\int_t^{T \wedge \theta_t^{x,c}} e^{-\alpha s} U(c_s)\,ds + e^{-\alpha T} g(Y_T)\mathbb{1}_{\{\theta_t^{x,c} > T\}}\right].$$

This implies $w \geqslant u$.

2. Since $w \geqslant u$ and $w(s,0) = u(s,0) = 0$, we see that $w_x(s,0+) \geqslant u_x(s,0+) \geqslant U'(0+)$. Thus, by the same line as Lemma 3.3, we know that there exists a unique non-negative solution Y^* of the SDE

$$dY_s^* = \big[f\big(Y_s^*\big) - G^w\big(s, Y_s^*\big)\big]\,dt + \sigma Y_s^*\,dW_s, \quad s \in [t,T], \qquad Y_t^* = x \geqslant 0,$$

where $G^w(s,y) = I(w_x(s,y)) \cdot \mathbb{1}_{\{t \leqslant s < T\}} \cdot \mathbb{1}_{\{y > 0\}}$. Since $w(s,y) \leqslant C_0(1+y^k)$ for some constants $C_0, k > 0$, the same arguments as in Theorem 3.4 give

$$e^{-\alpha t} w(t,x) \leqslant \mathbb{E}\left[\int_t^{T \wedge \theta^*} e^{-\alpha s} U\big(c_s^w\big)\,ds + e^{-\alpha T} g\big(Y_T^*\big)\mathbb{1}_{\{\theta^* > T\}}\right] \leqslant e^{-\alpha t} u(t,x),$$

where $c^w = \{c^w(s) := G^w(s, Y^*(s))\}_{s \geqslant 0} \in \mathcal{A}_T$ and $\theta^* = \theta_t^{x,c^w}$. Hence the proof is complete. □

4. Infinite time horizon Ramsey problem

In this section, we give an application of the above results to the infinite time horizon problem so as to maximize the discounted expected utilities with a discount rate $\alpha > 0$:

$$J_\infty(c:x) = \mathbb{E}\left[\int_0^{\tau^{x,c}} e^{-\alpha t} U\big(c(t)\big)\,dt\right] \tag{4.1}$$

over the class $\mathcal{C}$. In order to present the similar results to the theorems in Section 3, let us define

$$\widehat{V}(x) = \sup_{c\in\mathcal{C}} J_\infty(c:x), \quad x \geqslant 0,$$

and we make the following hypotheses: There exist $\beta, \gamma \in \mathbf{R}$ and $\xi_0 > 0$ such that

$$\alpha \geqslant f'(0+) - \beta + \sigma\gamma, \tag{4.2a}$$

$$\widetilde{F}(\xi_0) := \mathbb{E}\left[\int_0^\infty e^{-\alpha t}\widetilde{U}\big(\xi_0 e^{-\beta t}\mathcal{E}_\gamma(t)\big)\,dt\right] < \infty. \tag{4.2b}$$

Theorem 4.1. *Let the conditions* (1.3) *and* (4.2) *hold true. Then the following assertions are valid*:

(i) $\widehat{V}$ *is strictly increasing and concave on* $\mathbf{R}_+$.
(ii) $\widehat{V} \in C(\mathbf{R}_+) \cap C^2(0,\infty)$ *is a solution to the HJB equation*

$$-\mathcal{L}\widehat{V}(x) - \widetilde{U}\big(\widehat{V}'(x)\big) = 0, \quad x > 0, \qquad V(0) = 0. \tag{4.3}$$

Proof. 1. Let us begin with showing

$$0 \leqslant \widehat{V}(x) \leqslant \widetilde{F}(\xi_0) + x\xi_0, \quad x \geqslant 0. \tag{4.4}$$

Let $x \geqslant 0$ and $c \in \mathcal{C}$ be arbitrary. By (2.4), we have

$$\mathbb{E}\left[\int_0^{\tau^{x,c}\wedge T} e^{-\alpha t} U(c_t)\,dt\right] = \mathbb{E}\left[\int_0^{\tau^{x,c}\wedge T} e^{-\alpha t}\big[U(c_t) - \xi e^{-\beta t}\mathcal{E}_\gamma(t)c_t\big]\,dt\right] + \xi\mathbb{E}\left[\int_0^{\tau^{x,c}\wedge T} e^{-(\alpha+\beta)t}\mathcal{E}_\gamma(t)c_t\,dt\right]$$
$$\leqslant \mathbb{E}\left[\int_0^\infty e^{-\alpha t}\widetilde{U}\big(\xi e^{-\beta t}\mathcal{E}_\gamma(t)\big)\,dt\right] + \xi x.$$

Letting $T \to \infty$, Fatou's lemma shows (4.4).

Also we can easily deduce the following analogue of (2.6):

$$\widehat{V}(x) \geqslant \mathbb{E}\left[\int_0^\infty e^{-\alpha t} U\big(xze^{(l-z)t}\mathcal{E}_\sigma(t)\big)\,dt\right], \quad x, z \geqslant 0. \tag{4.5}$$

2. The proof of Lemma 3.1 guarantees the concavity of $\widehat{V}$. We denote by $\widehat{V}'_+$ the right-hand derivative of $\widehat{V}$. We shall prove that $\widehat{V}'_+(x) > 0$ for all $x > 0$ by the reduction to absurdity. Suppose that $\widehat{V}'_+(x_0) = 0$ for some $x_0 > 0$. By (4.4)–(4.5) and the same line as Lemma 3.1, we then have

$$\frac{U(\infty)}{\alpha} \leqslant \widetilde{F}(\xi_0) + \xi_0 x_0 < \infty.$$

Hence we may assume $\widetilde{U}(0) = U(\infty) < \infty$. Choosing $\gamma = 0$ and $\beta = (f'(0+) - \alpha)^+$,

$$\widetilde{F}(\xi_0) = \int_0^T e^{-\alpha t}\widetilde{U}\big(\xi_0 e^{-\beta t}\big)\,dt + \int_T^\infty e^{-\alpha t}\widetilde{U}\big(\xi_0 e^{-\beta t}\big)\,dt \leqslant \frac{1-e^{-\alpha T}}{\alpha}\widetilde{U}\big(\xi_0 e^{-\beta T}\big) + \frac{e^{-\alpha T}}{\alpha}\widetilde{U}(0)$$

for all $\xi_0 > 0$. Combining two inequalities above, we obtain

$$U(\infty) \leqslant \inf_{\xi_0 > 0}\left[\widetilde{U}\big(\xi_0 e^{-\beta T}\big) + \frac{\alpha}{1-e^{-\alpha T}}\xi_0 x_0\right] = U\left(\frac{\alpha e^{\beta T}}{1-e^{-\alpha T}}x_0\right).$$

This is in contradiction with (1.3). Thus the assertion (i) holds true.

3. Since $\widetilde{F}(\xi) \searrow 0$ as $\xi \to \infty$, we see $\widehat{V}(0+) = 0$ by (4.4). Hence the concave function $\widehat{V}$ is continuous on $\mathbf{R}_+$.

4. Let us introduce the auxiliary control problems

$$\widehat{V}_n(x) := \sup_{c \in \mathcal{C}_n} J_\infty(c : x), \quad x \geqslant 0,\ n \geqslant 1,$$

and we will establish a dynamic programming principle for $\widehat{V}_n$:

For every $\mathbb{F}$-stopping time θ with $\theta < \infty$ a.s., V_n satisfies the following property:

$$\widehat{V}_n(x) = \sup_{c \in \mathcal{C}_n} \mathbb{E}\left[\int_0^{\theta \wedge \tau^{x,c}} e^{-\alpha s} U\big(c(s)\big)\, ds + e^{-\alpha(\theta \wedge \tau^{x,c})} \widehat{V}_n\big(X^{x,c}\big(\theta \wedge \tau^{x,c}\big)\big) \right], \quad x \geqslant 0, \tag{4.6}$$

where θ may depend on a control $c \in \mathcal{C}_n$.

By the same calculation as in (2.14), it is clear that the part "$\leqslant$" in (4.6) holds. Let $0 < t < T$ and $c \in \mathcal{C}_n$ be arbitrary. Theorem 2.5 provides

$$V_n(T, x) \geqslant \mathbb{E}\left[\int_0^{\theta \wedge \tau_t^{x,c}} e^{-\alpha s} U\big(c(s)\big)\, ds + e^{-\alpha(\theta \wedge \tau_t^{x,c})} V_n\big(T - \theta \wedge \tau_t^{x,c}, X^{x,c}\big(\theta \wedge \tau_t^{x,c}\big)\big) \right],$$

where $\tau_t^{x,c} = \tau^{x,c} \wedge t$ and $g \equiv 0$. Since $V_n(T, x) \xrightarrow{T \to \infty} \widehat{V}_n(x)$, Fatou's lemma gives

$$\widehat{V}_n(x) \geqslant \mathbb{E}\left[\int_0^{\theta \wedge \tau_t^{x,c}} e^{-\alpha s} U\big(c(s)\big)\, ds + e^{-\alpha(\theta \wedge \tau_t^{x,c})} \widehat{V}_n\big(X^{x,c}\big(\theta \wedge \tau_t^{x,c}\big)\big) \right].$$

Letting $t \to \infty$, we obtain the part "$\geqslant$" in (4.6).

5. By the similar arguments to Lemma 2.7, we can prove that $\widehat{V}_n$ is a continuous viscosity solution to the associated HJB equation

$$-\mathcal{L}\widehat{V}_n(x) - \widetilde{U}_n\big(\widehat{V}_n'(x)\big) = 0, \quad x > 0.$$

Also the same arguments as in Proposition 2.3 ensure that $\widehat{V}_n(x) \nearrow \widehat{V}(x)$ as $n \to \infty$ for each $x \geqslant 0$. Then Dini's theorem implies that $\widehat{V}_n$ converges to $\widehat{V}$ locally uniformly on $\mathbf{R}_+$. Thus we can deduce from the standard stability results [1, §6] that $\widehat{V}$ is a viscosity solution to (4.3).

6. Let $Q := (x_0, x_1)$ be any bounded open subset of $\mathbf{R}_+$ and set $\lambda := V_+'(x_1) > 0$. It is known in [3] that the following boundary value problem has a smooth solution v:

$$-\mathcal{L}v(x) - \widetilde{U}\big(v'(x) \vee \lambda\big) = 0, \quad x \in Q, \qquad v(x_i) = \widehat{V}(x_i), \quad i = 0, 1. \tag{4.7}$$

Clearly, v is a viscosity solution to (4.7). On the other hand, since $\widehat{V}$ is the concave viscosity solution to (4.3), we see easily that $\widehat{V}$ is a continuous viscosity solution to (4.7). Thus, by the comparison results [1, Theorem 3.3] for viscosity solutions, we have $\widehat{V} = v \in C(\overline{Q}) \cap C^2(Q)$. Hence the proof is complete. □

To obtain an optimal consumption process for the infinite time horizon problem, we make the following condition which is slightly stronger than (4.2): There exist $\rho \in (0, \alpha)$, $\beta, \gamma \in \mathbf{R}$ and $\xi_0 > 0$ such that

$$\alpha - \rho \geqslant f'(0+) - \beta + \sigma\gamma \quad \text{and} \quad \mathbb{E}\left[\int_0^\infty e^{-(\alpha-\rho)t} \widetilde{U}\big(\xi_0 e^{-\beta t} \mathcal{E}_\gamma(t)\big)\, dt \right] < \infty. \tag{4.8}$$

Theorem 4.2. *Assume* (1.3) *and* (4.8). *Then $c^\star = \{c^\star(t) := \widehat{G}(X^\star(t))\}_{t \geqslant 0} \in \mathcal{C}$ is an optimal policy for the problem* (4.1), *where $\widehat{G}(x) = I(\widehat{V}'(x)) \cdot \mathbb{1}_{\{x>0\}}$ and $X^\star$ is a unique non-negative solution of the SDE*

$$dX_t^\star = \big[f\big(X_t^\star\big) - \widehat{G}\big(X_t^\star\big)\big]\, dt + \sigma X_t^\star\, dW_t, \quad t \geqslant 0, \qquad X_0^\star = x \geqslant 0. \tag{4.9}$$

Proof. 1. Taking account of (4.5), the same calculations as in (3.6) provide $\widehat{V}'(0+) \geqslant U'(0+)$. Hence we can prove the existence of a unique non-negative solution $X^\star$ to (4.9) by the same line as Lemma 3.3, and thereby $c^\star \in \mathcal{C}$.

2. Let $x \geqslant 0$ and $c \in \mathcal{C}$ be arbitrary. We shall show

$$\varliminf_{T \to \infty} \mathbb{E}\big[e^{-\alpha(T \wedge \tau^{x,c})} \widehat{V}\big(X^{x,c}\big(T \wedge \tau^{x,c}\big)\big)\big] = 0. \tag{4.10}$$

To this end, let us define

$$h(x)=\sup_{c\in\mathcal{C}}\mathbb{E}\left[\int_0^{\tau^{x,c}} e^{-(\alpha-\rho)t}U(c_t)\,dt\right],\quad x\geqslant 0.$$

Thanks to Theorem 4.1, we know that $h\in C(\mathbf{R}_+)\cap C^2(0,\infty)$ satisfies

$$0\leqslant \widehat{V}(x)\leqslant h(x),\qquad \mathcal{L}h(x)+\rho h(x)+\widetilde{U}\big(h'(x)\big)=0<h'(x),\quad x>0.$$

Thus, by Itô's formula, we have

$$\begin{aligned} h(x)+\sigma\int_0^{T\wedge\tau^{x,c}} e^{-\alpha s}X_s h'(X_s)\,dW_s &= e^{-\alpha(T\wedge\tau^{x,c})}h(X_{T\wedge\tau^{x,c}})+\int_0^{T\wedge\tau^{x,c}} e^{-\alpha s}\big[c_s h'(X_s)-\mathcal{L}h(X_s)\big]\,ds\\ &\geqslant \rho\int_0^{T\wedge\tau^{x,c}} e^{-\alpha s}h(X_s)\,ds\geqslant 0. \end{aligned}$$

Hence we have

$$\mathbb{E}\left[\int_0^{\infty} e^{-\alpha s}h(X_s)\mathbb{1}_{\{s\leqslant\tau^{x,c}\}}\,ds\right]\leqslant\frac{h(x)}{\rho}<\infty, \tag{4.11}$$

which implies $\underline{\lim}_{T\to\infty}\mathbb{E}[e^{-\alpha T}h(X_T)\mathbb{1}_{\{T\leqslant\tau^{x,c}\}}]=0$ and thereby (4.10).

3. By the definition of $c^\star$ we know that Itô's formula gives

$$\begin{aligned} \widehat{V}(x) &= \mathbb{E}\left[\int_0^{\theta_n\wedge T\wedge\tau} e^{-\alpha t}U\big(c_t^\star\big)\,dt\right]+\mathbb{E}\big[e^{-\alpha(\theta_n\wedge T\wedge\tau)}\widehat{V}\big(X^\star_{\theta_n\wedge T\wedge\tau}\big)\big]\\ &\leqslant J_\infty\big(c^\star\big)+\mathbb{E}\big[e^{-\alpha(\theta_n\wedge T\wedge\tau)}\widehat{V}\big(X^\star_{\theta_n\wedge T\wedge\tau}\big)\big], \end{aligned} \tag{4.12}$$

where $\tau=\tau^{x,c^\star}$ and $\theta_n=\inf\{t\geqslant 0:\ X^\star(t)\geqslant n\}$, $n\geqslant 1$. Since $X^\star(t)\leqslant xe^{f'(0+)t}\cdot\mathcal{E}_\sigma(t)$, by (4.4) and Doob's maximal inequality, we have

$$\mathbb{E}\Big[\sup_{n\geqslant 1}\widehat{V}\big(X^\star_{\theta_n\wedge T\wedge\tau}\big)\Big]\leqslant\widetilde{F}(\xi_0)+\xi_0 xe^{|f'(0+)|T}\,\mathbb{E}\Big[\sup_{0\leqslant t\leqslant T}\mathcal{E}_\sigma(t)\Big]<\infty.$$

Letting $n\to\infty$ and then $T\to\infty$ in (4.12), by the dominated convergence theorem and (4.10), we obtain $\widehat{V}(x)\leqslant J_\infty(c^\star)$, i.e. $c^\star$ is optimal as asserted. □

Theorem 4.3. *Assume* (1.3) *and* (4.2)*. Let $w\in C(\mathbf{R}_+)\cap C^2(0,\infty)$ be a solution to the HJB equation* (4.3) *with polynomial growth. Suppose further that either one of the following conditions holds*:

(a) *There exist $\rho_1>0$, $h_0\in\mathbf{R}$ and $h\in C(\mathbf{R}_+)\cap C^2(0,\infty)$ such that*

$$\overline{\lim_{x\to\infty}}\frac{w(x)}{h(x)}<\infty,\qquad \mathcal{L}h(x)+\rho_1 h(x)\leqslant h_0,\qquad h(\infty)>0,\qquad h'(x)\geqslant 0.$$

(b) $$\underline{\lim}_{x\to\infty}\frac{U(I(w'(x)))}{w(x)}\geqslant 2\rho_2\quad\textit{for some }\rho_2>0.$$

Then $w=\widehat{V}$ on $\mathbf{R}_+$.

Proof. Since $w(0)=0$ and $\widetilde{U}(w'(x))<\infty$, we first note $w'(x)\geqslant 0$ and $w(x)\geqslant 0$. Let $x>0$ and $c\in\mathcal{C}$ be arbitrary.

1. By the same line as step 1 in the proof of Theorem 3.5, we have

$$w(x)\geqslant\mathbb{E}\left[\int_0^{T\wedge\tau^{x,c}} e^{-\alpha s}U(c_s)\,ds\right].$$

Thus Fatou's lemma shows $w\geqslant\widehat{V}$.

2. Since $w \geqslant \widehat{V}$ and $w(0) = \widehat{V}(0) = 0$, we see that $w'(0+) \geqslant \widehat{V}'(0+) \geqslant U'(0+)$. Thus, by the same line as Lemma 3.3, we know that there exists a unique non-negative solution $Y^\star$ of the SDE

$$dY_t^\star = \left[f\left(Y_t^\star\right) - \widehat{G}^w\left(Y_t^\star\right)\right]dt + \sigma Y_t^\star\, dW_t, \quad t \geqslant 0, \qquad Y_0^\star = x \geqslant 0,$$

where $\widehat{G}^w(y) = I(w'(y)) \cdot \mathbb{1}_{\{y>0\}}$. Denote $\tau = \inf\{t \geqslant 0\colon\ Y^\star(t) = 0\}$ and we shall show

$$\varliminf_{T\to\infty} \mathbb{E}\left[e^{-\alpha T} w\left(Y_T^\star\right)\mathbb{1}_{\{T<\tau\}}\right] = 0. \tag{4.13}$$

2-1. Assume that condition (a) holds. Similarly to (4.11), we have

$$\mathbb{E}\left[\int_0^\infty e^{-\alpha s}\left[h\left(Y_s^\star\right) - h(0)\right]\mathbb{1}_{\{s\leqslant\tau\}}\, ds\right] \leqslant \frac{h(x)}{\rho_1} + \frac{(h_0 - \rho_1 h(0))^+}{\rho_1\alpha} < \infty,$$

which implies $\varliminf_{T\to\infty} \mathbb{E}[e^{-\alpha T}[h(Y_T^\star) - h(0)]\mathbb{1}_{\{T\leqslant\tau\}}] = 0$. We choose a number $N_1 > 0$ so that

$$0 \leqslant \frac{w(x)}{h(x)} \leqslant \eta := 1 + \varlimsup_{x\to\infty} \frac{w(x)}{h(x)}, \quad \forall x \geqslant N_1.$$

Then we get

$$\mathbb{E}\left[e^{-\alpha T} w\left(Y_T^\star\right)\mathbb{1}_{\{T<\tau\}}\right] \leqslant e^{-\alpha T}\left[w(N_1) + \eta h(0)^+\right] + \eta\mathbb{E}\left[e^{-\alpha T}\left[h\left(Y_T^\star\right) - h(0)\right]\mathbb{1}_{\{T\leqslant\tau\}}\right] \xrightarrow{T\to\infty} 0.$$

2-2. Assume that condition (b) holds. We choose a number $N_2 > 0$ so that $U(I(w'(x))) \geqslant \rho_2 w(x)$ for all $x \geqslant N_2$. Then Itô's formula gives

$$\begin{aligned} w(x) + \sigma \int_0^{T\wedge\tau} e^{-\alpha s} Y_s^\star w'\left(Y_s^\star\right) dW_s &= e^{-\alpha(T\wedge\tau)} w\left(Y_{T\wedge\tau}^\star\right) + \int_0^{T\wedge\tau} e^{-\alpha s} U\left(I\left(w'\left(Y_s^\star\right)\right)\right) ds \\ &\geqslant \rho_2 \int_0^{T\wedge\tau} e^{-\alpha s} w\left(Y_s^\star\right)\mathbb{1}_{\{Y_s^\star \geqslant N_2\}}\, ds \geqslant 0, \end{aligned}$$

which yields $\varliminf_{T\to\infty} \mathbb{E}[e^{-\alpha T} w(Y_T^\star)\mathbb{1}_{\{Y_T^\star\geqslant N_2\}} \cdot \mathbb{1}_{\{T<\tau\}}] = 0$, and thereby (4.13).

3. Since $w(y) \leqslant C_0(1 + y^k)$ for some constants $C_0, k > 0$, by the same arguments as step 3 in the proof of Theorem 4.2, we have

$$w(x) \leqslant J_\infty\left(c^{w\star}\right) \leqslant \widehat{V}(x),$$

where $c^{w\star} = \{c^{w\star}(t) := \widehat{G}^w(Y^\star(t))\}_{t\geqslant 0} \in \mathcal{C}$. Thus the verification theorem is established. □

Here are two examples for the hypotheses (4.2) and (4.8) to be fulfilled.

Example 4.4 *(HARA utilities).* For $a \in (0, 1)$, we consider for the case

$$U(x) = \frac{x^a}{a}, \qquad \widetilde{U}(y) = \frac{1-a}{a} y^{\frac{-a}{1-a}}, \quad x \geqslant 0,\ y > 0.$$

Since $\mathbb{E}[\widetilde{U}(\xi e^{-\beta t}\mathcal{E}_\gamma(t))] = \widetilde{U}(\xi)\exp([\frac{a}{1-a}\beta + \frac{a}{(1-a)^2}\frac{\gamma^2}{2}]t)$, we need

$$\alpha > A := \frac{a}{1-a}\beta + \frac{a}{(1-a)^2}\frac{\gamma^2}{2}$$

to ensure (4.2b). Let $B := f'(0+) - \beta + \sigma\gamma$. Then we get

$$\min_{\beta,\gamma}(A \vee B) = \min_\gamma\left[\frac{a}{2(1-a)}\gamma^2 + a\sigma\gamma\right] + af'(0+) = -\frac{a(1-a)}{2}\sigma^2 + af'(0+),$$

where $\gamma^* = -(1-a)\sigma$ and $\beta^* = (1-a)[f'(0+) - (2-a)\sigma^2/2]$ give the minimum value. Hence (4.2) is equivalent to

$$\alpha > \left[af'(0+) - \frac{a(1-a)}{2}\sigma^2\right]^+ \tag{4.14}$$

because $\alpha > 0$. Clearly, (4.14) also guarantees (4.8).

Example 4.5. In addition to (1.3), we assume that

$$U'(0+) = \infty \quad \text{and} \quad AE(U) := \overline{\lim_{x\to\infty}} \frac{xU'(x)}{U(x)} < 1.$$

By virtue of Lemma 6.3 in [6], we then have

$$\widetilde{U}(\xi) \leqslant \left(\frac{\xi}{\xi_0}\right)^{\frac{-a}{1-a}} \widetilde{U}(\xi_0), \quad 0 < \xi < \xi_0,$$

for certain $\xi_0 > 0$ and $a \in [AE(U), 1)$. Thus we observe

$$\mathbb{E}\big[\widetilde{U}\big(\xi e^{-\beta t}\mathcal{E}_\gamma(t)\big)\big] \leqslant \widetilde{U}(\xi_0)\left(\mathbb{P}\big\{\xi e^{-\beta t}\mathcal{E}_\gamma(t) \geqslant \xi_0\big\} + \mathbb{E}\left[\left(\frac{\xi e^{-\beta t}\mathcal{E}_\gamma(t)}{\xi_0}\right)^{\frac{-a}{1-a}}\right]\right)$$

for all $\xi > 0$. Since $\alpha > 0$, Example 4.4 implies that (4.8) holds true if α satisfies (4.14).

We are now in a position to make the corrections of Theorems 3.3 and 3.4 in the second author's paper [10].

Remark 4.6. For the reference we use the list of references of the paper [10].

(i) We need to add the following assumption:
There exists a concave function $\psi \in \mathcal{B} \cap C^2(0,\infty)$ such that $\psi(0) = 0$ and

$$-\alpha\psi(z) + \frac{\sigma^2}{2}z^2\psi''(z) + \big(f(z) - \mu z\big)\psi'(z) + \widetilde{U}\big(\psi'(z)\big) \leqslant 0 < \psi'(z) \tag{4.15}$$

for $z > 0$. Then, we replace φ in the estimate of J_1 in the proof of Lemma 3.2 by ψ to have

$$J_1 \leqslant C_\rho|z - \widetilde{z}| + \rho\big[\varphi(z) + \varphi(\widetilde{z})\big].$$

(ii) We replace $\tau_z \downarrow \theta$ in the proof of Theorem 3.4 by the following: Choosing sufficiently small $\varepsilon > 0$, we have

$$E\left[\int_0^{\tau_z} e^{-(\beta+1/\varepsilon)t}c(t)\,dt\right] \leqslant z + E\left[\int_0^{\tau_z} e^{-(\beta+1/\varepsilon)t} f\big(z(t)\big)\,dt\right]$$
$$\leqslant z + f'(0+)E\left[\int_0^{\infty} e^{-(\beta+1/\varepsilon)t}q(t)\,dt\right] \to 0 \quad \text{as } z \downarrow 0.$$

Then, by the concavity of U, we get

$$E\left[\int_0^{\tau_z} e^{-(\beta+1/\varepsilon)t}U\big(c(t)\big)\,dt\right] \to 0 \quad \text{as } z \downarrow 0.$$

By (i) and (ii), we can obtain the same conclusions as these theorems.

Remark 4.7. Let $U(x) = x^a/a$ be the power utility as in Example 4.4. Then we notice that $\psi(x) = x^a$ satisfies the condition (4.15) if

$$\alpha > \left[af'(0+) - \frac{a(1-a)}{2}\sigma^2 + (1-a)a^{1/(a-1)}\right]^+,$$

where α and f are of this paper. (We remark that α and $f(z)$ in (4.15) are different from those of this paper. The α in (4.15) is a positive number which is smaller than the discount rate α of this paper, and the function $f(z) - \mu z$ in (4.15) corresponds to $f(z)$ of this paper.) Obviously, the condition (4.14) is weaker than the above requirement. Hence our results in this section are sharp as compared with [10].

Remark 4.8. We observe that (4.15) is analogous with the condition (V.2.11) of [2] for the exit time control problem in the finite horizon. Excepting some simple cases, however, it is hard to verify whether there is a function ψ satisfying the condition (4.15). On the other hand, given parameters α, σ and functions f, U, it is easy to verify whether the conditions (4.2) and (4.8) hold or not. In comparison with [10], the contribution of this section is to have provided the readable conditions (4.2) and (4.8).

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