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The Legendre wavelet method for solving initial value problems of Bratu-type

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A B S T R A C T

The aim of this work is to study the Legendre wavelets for the solution of initial value problems of Bratu-type, which is widely applicable in fuel ignition of the combustion theory and heat transfer. The properties of Legendre wavelets together with the Gaussian integration method are used to reduce the problem to the solution of nonlinear algebraic equations. Also a reliable approach for convergence of the Legendre wavelet method when applied to a class of nonlinear Volterra equations is discussed and an error estimation for the proposed method is also introduced. Illustrative examples have been discussed to demonstrate the validity and applicability of the technique and the results have been compared with the exact solution. We finally show the high accuracy and efficiency of the proposed method.

1. Introduction

In recent years, the studies of initial value problems in the second order ordinary differential equations (ODEs) have attracted many researchers. One of the equations describing this type is the Bratu-type equations formulated as

$$u'' + \lambda e^u = 0, \quad 0 < x < 1, \qquad u(0) = u'(0) = 0, \quad \lambda \text{ is a constant.} \tag{1}$$

The standard Bratu problem [1] was used to model a combustion problem in a numerical slab. Bratu's problem [2–6] is also used in a large variety of applications such as the fuel ignition model of the thermal combustion theory, the model of the thermal reaction process, the Chandrasekhar model [7] of the expansion of the universe, questions in geometry and relativity concerning the Chandrasekhar model, chemical reaction theory, radiative heat transfer and nanotechnology [8–13].

A substantial amount of research work has been done for the study of the Bratu problem [1,14–17]. Boyd [14,15] employed Chebyshev polynomial expansions and the Gegenbauer as base functions. Syam and Hamdan [18] presented the Laplace Adomian decomposition method for solving Bratu's problem. Also Yigit Aksoy and Mehmet Pakdemirli had solved new perturbation iteration solutions for Bratu-type equations [19]. Wazwaz [20] presented the Adomian Decomposition method for solving Bratu's problem.

In recent years, wavelets have found their way in to many different fields of science and engineering. Many researchers started using various wavelets [21–23] for analyzing problems of greater computational complexity and proved wavelets to be powerful tools to explore a new direction in solving differential equations. Legendre wavelet based approximate solution of lane-Emden type was studied by Yousefi [24] recently.

In the present article, we apply the Legendre wavelet method (LWM) to find the approximate solution of Bratu-type equations. The Legendre wavelet method is based on conversion of Bratu-type equations to integral equations and expanding

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the solution by Legendre wavelets with unknown coefficients. The properties of Legendre wavelets together with the Gaussian integration formula are utilized to evaluate the unknown coefficients and then an approximate solution to (1) will be identified.

The article is summarized as follows. In Section 2, we describe the basic formulation of wavelets and Legendre wavelets required for our subsequent development. Section 3 is devoted to the solution of (1) by using integral operator and Legendre wavelets. Uniqueness theorem and Convergence theorem have been studied in Section 4. Also the error estimation for the proposed LWM has been discussed in Section 5. In Section 6, we report our numerical finding and demonstrate the accuracy of the proposed scheme by considering numerical examples. Concluding remarks are given in Section 7.

2. Properties of Legendre wavelets

2.1. Wavelets and Legendre wavelets

Wavelets constitute a family of functions constructed from dilation and translation of a single function called the mother wavelet. When the dilation parameter 'a' and the translation parameter 'b' vary continuously, we have the following family of continuous wavelets as:

$$\psi_{a,b}(t) = |a|^{-\frac{1}{2}}\,\psi\left(\frac{t-b}{a}\right), \quad a, b \in R,\ a \neq 0.$$

If we restrict the parameters a and b to discrete values as $a = a_0^{-k}$, $b = nb_0a_0^{-k}$, $a_0 > 1$, $b_0 > 0$ and n, k positive integers, we have the following family of discrete wavelets:

$$\psi_{k,n}(t) = |a|^{-\frac{1}{2}}\,\psi\left(a_0^k t - nb_0\right)$$

where $\psi_{k,n}(t)$ form a basis of $L^2(R)$. In particular, when $a_0 = 2$ and $b_0 = 1$ then $\psi_{k,n}(t)$ forms an orthonormal basis.

Legendre wavelets $\psi_{nm}(t) = \psi\left(k, \hat{n}, m, t\right)$ have four arguments: $\hat{n} = 2n - 1$, $n = 1, 2, 3, \ldots, 2^{k-1}$, k can assume any positive integer, m is the order of Legendre polynomials and t is the normalized time. They are defined in [23] on the interval $[0, 1)$ as

$$\psi_{nm}(t) = \begin{cases} \sqrt{m+\dfrac{1}{2}}\,2^{\frac{k}{2}}\,P_m\left(2^k t - \hat{n}\right), & \text{for } \dfrac{\hat{n}-1}{2^k} \le t \le \dfrac{\hat{n}+1}{2^k}, \\ 0, & \text{otherwise} \end{cases} \tag{2}$$

where $m = 0, 1, 2, \ldots, M-1$, $n = 1, 2, 3, \ldots, 2^{k-1}$. The coefficient $\sqrt{m+\frac{1}{2}}$ is for orthonormality, the dilation parameter is $a = 2^{-k}$ and translation parameter is $b = \hat{n}2^{-k}$.

Here $P_m(t)$ are well-known Legendre polynomials of order m which are defined on the interval $[-1, 1]$, and can be determined with the aid of the following recurrence formulas:

$$P_0(t) = 1, \qquad P_1(t) = t$$
$$P_{m+1}(t) = \left(\frac{2m+1}{m+1}\right) t\,P_m(t) - \left(\frac{m}{m+1}\right) P_{m-1}(t), \quad m = 1, 2, 3, \ldots.$$

2.2. Function approximation

A function $f(t)$ defined over $[0, 1)$ may be expanded as

$$f(t) = \sum_{n=1}^{\infty}\sum_{m=0}^{\infty} c_{nm}\,\psi_{nm}(t) \tag{3}$$

where $c_{nm} = \langle f(t), \psi_{nm}(t)\rangle$, in which $\langle\cdot,\cdot\rangle$ denotes the inner product. If the infinite series in Eq. (3) is truncated, then Eq. (3) can be written as

$$f(t) \cong \sum_{n=1}^{2^{k-1}}\sum_{m=0}^{M-1} c_{nm}\,\psi_{nm}(t) = C^T\,\psi(t),$$

where C and $\psi(t)$ are $2^{k-1}M \times 1$ matrices given by

$$C = \left[c_{10}, c_{11}, \ldots, c_{1M-1}, c_{20}, \ldots, c_{2M-1}, \ldots, c_{2^{k-1}0}, \ldots, c_{2^{k-1}M-1}\right]^T, \tag{4}$$

$$\psi(t) = \left[\psi_{10}(t), \psi_{11}(t), \ldots, \psi_{1M-1}(t), \psi_{20}(t), \ldots, \psi_{2M-1}(t), \ldots, \psi_{2^{k-1}0}(t), \ldots, \psi_{2^{k-1}M-1}(t)\right]^T. \tag{5}$$

3. Solution of Bratu-type equations

Consider the Bratu-type equation given in Eq. (1).
Define the integral operator $L(\cdot) = \int_0^x \int_0^x (\cdot)\ dx\, dx$.
Applying L to both sides of Eq. (1) yields

$$u(x) + \lambda \int_0^x \int_0^x e^u\, dx\, dx = 0. \tag{6}$$

Let $F(x, u(x)) = \int_0^x e^{u(x)}\, dx$
Eq. (6) implies $u(x) + \lambda \int_0^x F(t, u(t))\ dt = 0$.
Thus we have

$$u(x) = -\lambda \int_0^x F(t, u(t))\ dt. \tag{7}$$

Let $u(x) = C^T\, \psi(x)$.
Therefore we have

$$C^T\, \psi(x) = -\lambda \int_0^x F\left(t, C^T\, \psi(t)\right)\ dt. \tag{8}$$

We now collocate Eq. (8) at $2^{k-1}M$ points at x_i as

$$C^T\, \psi\, (x_i) = -\lambda \int_0^{x_i} F\left(t,\ C^T\, \psi(t)\right)\ dt. \tag{9}$$

Suitable collocation points are zeros of Chebyshev polynomials

$$x_i = \cos\left((2i+1)\,\pi/2^k M\right), \quad i = 1, 2, \ldots, 2^{k-1}M.$$

In order to use the Gaussian integration formula for Eq. (9), we transfer the intervals $[0, x_i]$ into the interval $[-1, 1]$ by means of the transformation

$$\tau = \frac{2}{x_i} t - 1.$$

Eq. (9) may then be written as

$$C^T\, \psi\, (x_i) = -\lambda \frac{x_i}{2} \int_{-1}^{1} F\left(\frac{x_i}{2}(\tau+1),\ C^T\, \psi\left(\frac{x_i}{2}(\tau+1)\right)\right) d\tau.$$

By using the Gaussian integration formula, we get

$$C^T\, \psi\, (x_i) \approx \frac{-\lambda\, x_i}{2} \sum_{j=1}^{s} w_j\, F\left(\frac{x_i}{2}\left(\tau_j+1\right),\ C^T\, \psi\left(\frac{x_i}{2}\left(\tau_j+1\right)\right)\right), \quad i = 1, 2, \ldots, 2^{k-1}M, \tag{10}$$

where τ_j's are s zeros of Legendre polynomials, and P_{s+1} and w_j are its corresponding weights. The idea behind the above approximation is the exactness of the Gaussian integration formula for polynomials of degree not exceeding $2s+1$. Here the weight w_j can be identified with the help of the formula $w_j = \int_{-1}^{1} \prod_{j=0, j\neq i}^{s} \left(\frac{\tau-\tau_j}{\tau_i-\tau_j}\right) d\tau$.

Eq. (10) gives $2^{k-1}M$ nonlinear equations which can be solved for the elements of C in Eq. (7) by the Newton iterative method.

4. Existence of uniqueness and convergence analysis.

In this section, we discuss the theoretical analysis of uniqueness and convergence of our approach.

Theorem 4.1. *Uniqueness theorem*
Eq. (1) has a unique solution whenever $0 < \alpha < 1$, *where* $\alpha = L\lambda x$.

Proof. Eq. (1) can be written in the form $u(x) = -\lambda \int_0^x F(t,\ u(t))\ dt$ such that the nonlinear term $F(u)$ is Lipschitz continuous with $|F(u) - F(v)| \le L\ |u - v|$.

Let u and u^* be two different solutions for Eq. (1).

$$\left|u - u^*\right| = \left|-\lambda \int_0^x F\,(t,\ u(t))\ dt - \lambda \int_0^x F\,\left(t,\ u^*(t)\right)\ dt\right|$$

$$\begin{aligned}\left|u - u^*\right| &= \left|-\lambda \int_0^x \left[F(u) - F\left(u^*\right)\right] dt\right| \\ &= \lambda \left|\int_0^x \left[F(u) - F\left(u^*\right)\right] dt\right|\end{aligned}$$

$$\begin{aligned}\left|u - u^*\right| &\le \lambda \int_0^x \left|F(u) - F\left(u^*\right)\right| dt \ \le L\lambda \int_0^x \left|u - u^*\right|\ dt \\ &\le L\lambda \left|u - u^*\right| x.\end{aligned}$$

This implies that $|u - u^*|\ \ (1 - L\lambda x)\ \le\ 0$

i.e. $\left|u - u^*\right| (1 - \alpha) \le\ 0$ where $\alpha = L\lambda x$.

As $0 < \alpha < 1$, $|u - u^*| = 0$, implies $u = u^*$ and this completes the proof. □

Theorem 4.2. *Convergence theorem*
The series solution (3) of problem (1) using LWM converges towards $u(x)$.

Proof. Let $L^2(R)$ be the Hilbert space and let $\psi_{k,n}(t) = |a|^{-\frac{1}{2}}\,\psi\left(a_0^k t - nb_0\right)$ where $\psi_{k,n}(t)$ form a basis of $L^2(R)$. In particular, when $a_0 = 2$ and $b_0 = 1$, $\psi_{k,n}(t)$ forms an orthonormal basis.

Let $u(x) = \sum_{i=1}^{M-1} C_{1i}\,\psi_{1i}(x)$ where $C_{1i} = \langle u(x),\ \psi_{1i}(x)\rangle$ for $k = 1$ and $\langle .,.\rangle$ represents an inner product.

$$u(x) = \sum_{i=1}^{n} \langle u(x), \psi_{1i}(x)\rangle\ \ \psi_{1i}(x)\,.$$

Let us denote $\psi_{1i}(x)$ as $\psi(x)$.
Let $\alpha_j = \langle u(x),\ \psi(x)\rangle$.
Define the sequence of partial sums $\{S_n\}$ of $\left(\alpha_j\,\psi\left(x_j\right)\right)$; let S_n and S_m be arbitrary partial sums with $n \ge m$. We are going to prove that $\{S_n\}$ is a Cauchy sequence in Hilbert space.
Let $S_n = \sum_{j=1}^{n} \alpha_j\,\psi\left(x_j\right)$

$$\begin{aligned}\langle u(x),\ S_n\rangle &= \left\langle u(x),\ \sum_{j=1}^{n} \alpha_j\,\psi\left(x_j\right)\right\rangle \\ &= \sum_{j=1}^{n} \overline{\alpha_j}\,\left\langle u(x), \psi\left(x_j\right)\right\rangle \\ &= \sum_{j=1}^{n} \overline{\alpha_j}\,\alpha_j \\ &= \sum_{j=1}^{n} \left|\alpha_j\right|^2.\end{aligned}$$

We will claim that $\|S_n - S_m\|^2 = \sum_{j=m+1}^{n} \left|\alpha_j\right|^2$ for $n > m$.
Now

$$\begin{aligned}\left\|\sum_{j=m+1}^{n} \alpha_j\,\psi\left(x_j\right)\right\|^2 &= \left\langle \sum_{i=m+1}^{n} \alpha_i\,\psi\,(x_i)\,,\ \sum_{j=m+1}^{n} \alpha_j\,\psi\left(x_j\right)\right\rangle \\ &= \sum_{i=m+1}^{n}\sum_{j=m+1}^{n} \alpha_i\,\overline{\alpha_j}\ \ \left\langle \psi\,(x_i)\,,\ \psi\left(x_j\right)\right\rangle \\ &= \sum_{j=m+1}^{n} \alpha_j\,\overline{\alpha_j} \\ &= \sum_{j=m+1}^{n} \left|\alpha_j\right|^2\end{aligned}$$

i.e. $\|S_n - S_m\|^2 = \sum_{j=m+1}^{n} \left|\alpha_j\right|^2$ for $n > m$.

From Bessel's inequality, we have $\sum_{j=1}^{\infty} \left|\alpha_j\right|^2$, is convergent and hence

$$\|S_n - S_m\|^2 \to 0 \quad \text{as } m, n \to \infty.$$

i.e. $\|S_n - S_m\| \to 0$ and $\{S_n\}$ is a Cauchy sequence and it converges to say 's'.
We assert that $u(x) = s$.
Infact,

$$\begin{aligned} \left\langle S - u(x), \psi\left(x_j\right)\right\rangle &= \left\langle S, \psi\left(x_j\right)\right\rangle - \left\langle u(x), \psi\left(x_j\right)\right\rangle \\ &= \left\langle \underset{n\to\infty}{Lt} S_n, \psi\left(x_j\right)\right\rangle - \alpha_j \\ &= \underset{n\to\infty}{Lt} \left\langle S_n, \psi\left(x_j\right)\right\rangle - \alpha_j \\ &= \alpha_j - \alpha_j \end{aligned}$$

$$\Rightarrow \left\langle S - u(x), \psi\left(x_j\right)\right\rangle = 0.$$

Hence $u(x) = S$ and $\sum_{j=1}^{n} \alpha_j\, \psi\left(x_j\right)$ converges to $u(x)$ and this completes the proof. □

As the convergence has been proved, consistency and stability are ensured automatically.

5. Error estimation

In this part, an error estimation for the approximate solution of Eq. (7) is discussed. Let us consider $e_n(x) = u(x) - \overline{u}(x)$ as the error function of the approximate solution $\overline{u}(x)$ for $u(x)$, where $u(x)$ is the exact solution of Eq. (7).
$\overline{u}(x) = -\lambda \int_0^x F(t, u(t))\, dt + H_n(x)$ where $H_n(x)$ is the perturbation term.

$$H_n(x) = \overline{u}(x) + \lambda \int_0^x F(t, u(t))\, dt. \tag{11}$$

We proceed to find an approximation $\overline{e_n}(x)$ to the error function $e_n(x)$ in the same way as we did before for the solution of the problem. Subtracting Eq. (11) from Eq. (7), the error function $e_n(t)$ satisfies the problem.

$$e_n(x) + \lambda \int_0^x F\left(t, e_n(t)\right) dt = -H_n(x). \tag{12}$$

It should be noted that in order to construct the approximate $\overline{e_n}(x)$ to $e_n(x)$, only Eq. (12) needs to be recalculated in the same way as we did before for the solution of Eq. (7).

6. Illustrative examples

Using the method presented in this paper, we solve three examples and the results have been compared with the exact solution.

Example 1. Consider the initial value problem

$$\begin{aligned} &u'' - 2e^u = 0, \quad 0 < x < 1 \\ &u(0) = u'(0) = 0. \end{aligned} \tag{13}$$

We solve Eq. (13) by the method discussed in this paper with $k = 1$, $M = 7$ and $s = 6$.
Integrating (13), we get

$$\int_0^x \int_0^x u''\, dx = 2 \int_0^x \int_0^x e^u\, dx$$

$$u = 2 \int_0^x \int_0^x e^u\, dx$$

$$u = 2 \int_0^x (t - x)\, e^u\, dt.$$

Let $u(x) = C^T \psi(x) = c_{10}\psi_{10} + c_{11}\psi_{11} + c_{12}\psi_{12} + \cdots + c_{16}\psi_{16}$; then the above equation is transformed into the following form

$$C^T \psi(x_i) = x_i \int_{-1}^{1} F_1\left(\frac{x_i}{2}(\tau + 1), C^T \psi\left(\frac{x_i}{2}(\tau + 1)\right)\right) d\tau - x_i^2 \int_{-1}^{1} F_2\left(\frac{x_i}{2}(\tau + 1), C^T \psi\left(\frac{x_i}{2}(\tau + 1)\right)\right) d\tau.$$

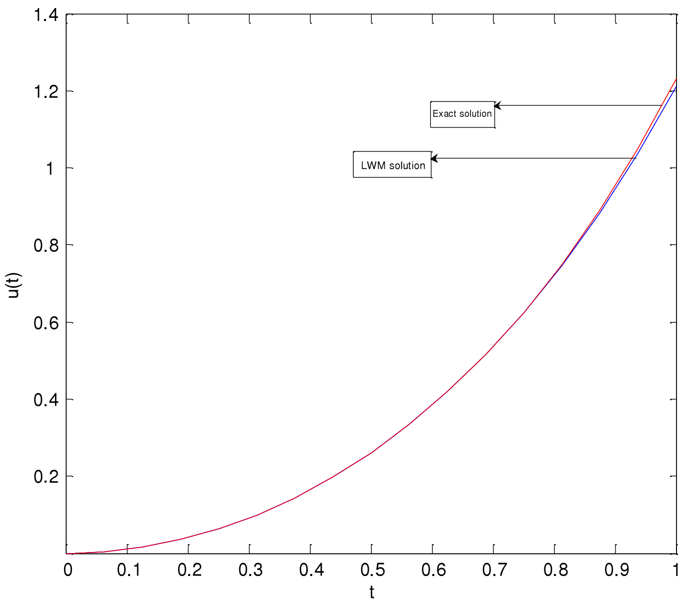

Fig. 1. The exact and LWM solutions of Example 1.

By using the Gaussian integration formula, we get

$$C^T\,\psi\,(x_i) \approx x_i \sum_{j=1}^{s} w_j\, F_1\left(\frac{x_i}{2}\,(\tau_j+1)\,,\;C^T\,\psi\left(\frac{x_i}{2}\,(\tau_j+1)\right)\right) - x_i^2 \sum_{j=1}^{s} w_j\, F_2\left(\frac{x_i}{2}\,(\tau_j+1)\,,\;C^T\,\psi\left(\frac{x_i}{2}\,(\tau_j+1)\right)\right)$$

where $F_1\,(t,u(t)) = t\,e^u$; $F_2\,(t,u(t)) = e^u$.

Here we get a system of equations involving 7 variables and solving them by Newton's iterative formula with the aid of MATLAB, we get the values

$$C_{10} = 4.44306612088, \qquad C_{11} = 5.18825114824, \qquad C_{12} = 1.26840807332,$$
$$C_{13} = 0.13846668820, \qquad C_{14} = -0.01710306349, \qquad C_{15} = 0.00010334260, \qquad C_{16} = 0.00000002605.$$

$$\begin{aligned} u(x) &= 4.44306612088\psi_{10} + 5.18825114824\,\psi_{11} + 1.26840807332\,\psi_{12} + \cdots + 0.00000002605\,\psi_{16} \\ &= 0.0000009x + x^2 + 0.00000004x^3 + 0.16667x^4 + 0.000000032x^5 + 0.044444x^6 + \cdots \\ &\approx -2\ln(\cos x) \end{aligned}$$

which is the exact solution. Fig. 1 shows that the Legendre wavelet solution is very nearest to the exact solution.

Example 2. Consider the initial value problem

$$\begin{aligned} &u'' - \pi^2 e^u = 0, \quad 0 < x < 1 \\ &u(0) = 0; \qquad u'(0) = \pi. \end{aligned} \tag{14}$$

We solve Eq. (14) with $k = 1$, $M = 5$ and $s = 6$.

Integrating (14), we get

$$\int_0^x \int_0^x u''\,dx = \pi^2 \int_0^x \int_0^x e^u\,dx$$

$$u = \pi x + \pi^2 \int_0^x \int_0^x e^u\,dx$$

$$u = \pi x + \pi^2 \int_0^x (t-x)\;e^u\,dt.$$

By using the Gaussian integration formula, we get

$$\begin{aligned} C^T\,\psi\,(x_i) \approx \pi x + \frac{\pi^2 x_i}{2} \sum_{j=1}^{s} w_j\,F_1\left(\frac{x_i}{2}\,(\tau+1)\,,\;C^T\,\psi\left(\frac{x_i}{2}\,(\tau+1)\right)\right) \\ - \frac{\pi^2 x_i^2}{2} \sum_{j=1}^{s} w_j\,F_2\left(\frac{x_i}{2}\,(\tau+1)\,,\;C^T\,\psi\left(\frac{x_i}{2}\,(\tau+1)\right)\right), \quad i = 1, 2, \ldots, 2^{k-1}M, \end{aligned}$$

where $F_1\,(t,u(t)) = te^u$; $F_2\,(t,u(t)) = e^u$.

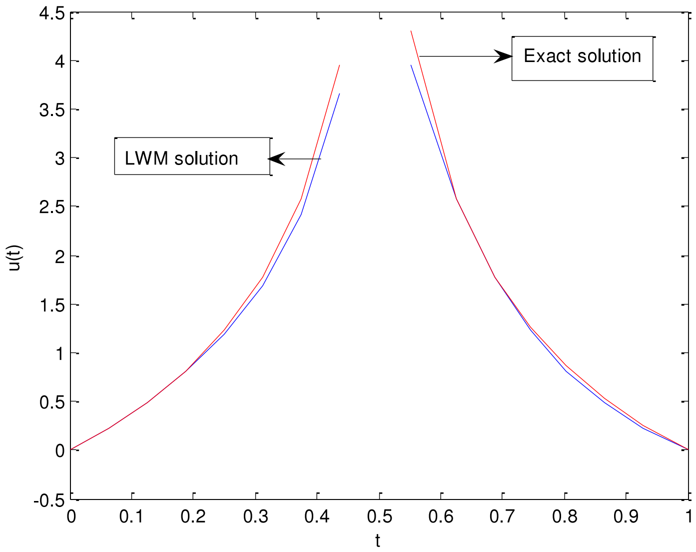

Fig. 2. The exact and LWM solutions of Example 2.

Solving this system of 5 variables by using Newton's iterative formula and with the aid of MATLAB, the approximate solution of (14) is

$$3.141492653\,x - 4.934488044x^2 + 25.83609656x^3 - 16.23278151x^4 + \cdots.$$

which is very close to the exact solution $u(x) = -\ln(1 - \sin \pi x)$. Fig. 2 shows that the Legendre wavelet solution coincides with the exact solution.

Example 3. Consider the initial value problem

$$\begin{aligned} & u'' + \pi^2 e^{-u} = 0, \quad 0 < x < 1 \\ & u(0) = 0; \qquad u'(0) = \pi. \end{aligned} \tag{15}$$

We solve Eq. (15) by the method discussed in this paper with $k = 1$, $M = 5$ and $s = 6$.

Integrating (15), we get

$$\int_0^x \int_0^x u''\,dx = -\pi^2 \int_0^x \int_0^x e^{-u}\,dx$$

$$u = \pi x - \pi^2 \int_0^x \int_0^x e^{-u}\,dx$$

$$u = \pi x - \pi^2 \int_0^x (t - x)\, e^{-u}\,dt$$

By using the Gaussian integration formula, we get

$$C^T\,\psi\,(x_i) \approx \pi x - \frac{\pi^2 x_i}{2} \sum_{j=1}^{s} w_j F_1 \left(\frac{x_i}{2}(\tau+1),\, C^T\,\psi\left(\frac{x_i}{2}(\tau+1)\right)\right)$$

$$+ \frac{\pi^2 x_i^2}{2} \sum_{j=1}^{s} w_j F_2 \left(\frac{x_i}{2}(\tau+1),\, C^T\,\psi\left(\frac{x_i}{2}(\tau+1)\right)\right), \quad i = 1, 2, \ldots, 2^{k-1}M,$$

where $F_1\,(t, u(t)) = t\ e^{-u}$; $F_2\,(t, u(t)) = e^{-u}$

Solving this system of 4 variables by Newton's iterative formula and with the help of MATLAB, the approximate solution of (15) is

$$3.141492653x - 4.934488044\,x^2 + 5.167219313x^3 + 8.116390753x^4 + \cdots.$$

This solution is very close to the exact solution $u(x) = \ln(1 + \sin \pi x)$. Fig. 3 shows that the Legendre wavelet solution coincides with the exact solution.

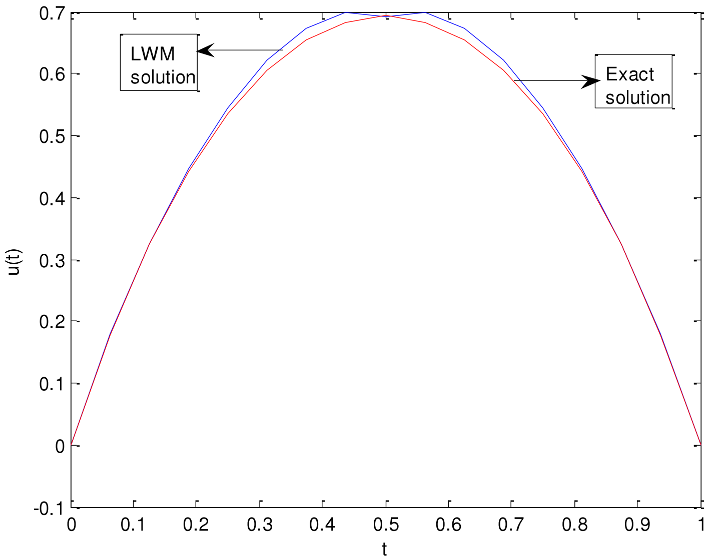

Fig. 3. The exact and LWM solutions of Example 3.

7. Conclusion

In this work, we have presented the Legendre wavelet method (LWM) for solving initial value problems of Bratu-type. The properties of the Legendre wavelets together with the Gaussian integration method are used to reduce the problem to the solution of nonlinear algebraic equations. The sufficient condition that guarantees a unique solution to the given problem is obtained. The convergence study is reliable enough to estimate the error of the Legendre wavelet method solution. As the convergence had been proved, consistency and stability are ensured automatically. Illustrative examples reveal the validity and applicability of the technique. Furthermore, since the basis of Legendre wavelets are polynomial, the values of integrals for the nonlinear integral equations of the form in Eq. (6) are calculated approximately close to the exact solutions.

Acknowledgments

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