Chromatic number of prime distance graphs

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Abstract

For any set D of positive integers, the distance graph G(D) = G(V, E) is the graph with vertex set V(G) = ℤ and edge set E(G) = { (u, v): |u - v| ∈ D }. In Research Problem 77 (Discrete Math. 69 (1988) 105–106) Eggleton, Erdős and Skilton propose the problem to determine all minimal subsets D of the prime numbers such that graph G(D) is 4-chromatic. In the present paper this problem is solved for 4-element prime sets D.

1. Introduction

Let D = {d₁, d₂, d₃, ..., dₙ} ⊆ ℤ be a finite nonempty subset of the set ℤ of all positive integers.

The graph G(D) = G(V, E) with vertex set V(G) := ℤ (ℤ is the set of all integers) and edge set E(G) := { (u, v): |u - v| ∈ D } is called the distance graph of the distance set D. χ(D) denotes the chromatic number of G(D).

It is known [1, 5, 8] that:
- χ(D) = 4 if D is the set ℙ of all primes,
- χ(D) = 2 if D ⊆ ℙ \ {2},
- χ(D) ≤ 3 if D ⊆ ℙ \ {3},
- χ(D) ∈ {3, 4} if {2, 3} ⊆ D ⊆ ℙ.

Therefore we investigate sets D ⊆ ℙ with {2, 3} ⊆ D only.

Eggleton et al. [3] gave the following conjecture: Let D be a subset of the set ℙ ({2, 3} ⊆ D) of primes. Then χ(D) = 4 if and only if D contains a twin of primes.

It is easy to see that χ(D) = 4 if D contains a twin of primes. The second part of this conjecture is disproved in [2] and in [8] by counterexamples.
In 1988, an update on the above conjecture was published (see [Z]). Eggleton et al. now formulated the problem in the following way: characterize all sets of primes with $\chi(D) = 3$.

We prove the following theorem: There are exactly eight pairs $p, q$ of primes ($p \geq 7, q > p + 2$) with $\chi(2, 3, p, q) = 4$.

2. Definitions

We investigate colourings of $\mathbb{Z}$ with three colours. These three colours are denoted by $a$, $b$ and $c$.

**Definition 2.1.** A colouring $f: \mathbb{Z} \rightarrow \{a, b, c\}$ is called $d_i$-consistent, if $f(v) \neq f(v + d_i)$ holds for all $v \in \mathbb{Z}$. A colouring $f$ is $D$-consistent if $f$ is $d_i$-consistent for each $i \in \{1, 2, \ldots, r\}$.

**Definition 2.2.** A colouring $f: \mathbb{Z} \rightarrow \{a, b, c\}$ is called periodically with period $\lambda$ if $f(v) = f(v + \lambda)$ for all $v \in \mathbb{Z}$. Such a colouring is denoted by $P_\lambda$.

In what follows we obtain certain “sections of colours” which can occur in $(2, 3)$-consistent 3-colourings. We choose without loss of generality (w.l.o.g.) always the colour $a$ as the initial colour of such finite colour-sections or briefly sections for our description.

Let $\mathbb{Z}^a := \{v \mid v \in \mathbb{Z} \text{ with } f(v) = a \text{ and } f(v - 1) \neq a\}$. In Fig. 1 we have: $\{-14, -9, -5, 0, 5, 10, 14\} \subseteq \mathbb{Z}^a$.

**Definition 2.3.** A sequence $S_l := (f^1, \ldots, f^l)$ of colours $f^i$ with $l > 1$, $f^i \in \{a, b, c\}$, $i \in \{1, \ldots, l\}$, $f^1 = a, f^i \neq a$, is called a (colour-)section of length $l$.

A colour-section $S_l$ ($l \geq 3$) with $f^i \neq a \ \forall i = 3, \ldots, l$ is called an elementary (colour-)section of length $l$.

For instance, $(a, a, b, c, a, b, b, c)$ is a section of length 9, but this section is not elementary, whereas $(a, a, b, c, c, a, b, b, c)$ is an elementary section of length 5.

A colouring $f: \mathbb{Z} \rightarrow \{a, b, c\}$ contains the section $S_l = (f^1, \ldots, f^l)$ if there is a vertex $v \in \mathbb{Z}^a$ with $f(v) = f^1 = a, f(v + 1) = f^2, \ldots, f(v + l - 1) = f^l \neq a, f(v + l) = a$. We denote this also by $S_l \in f$ ($S_l$ is a section of $f$). In Fig. 1 we have for instance: $(a, a, b, c, c, a, b, b, c) \in f$.

![Fig. 1.](image-url)
3. (2, 3)-consistency of 3-colourings

Elementary sections are investigated which may occur in a (2, 3)-consistent 3-colouring \( f \).

We fix w.l.o.g. the ordering of colours for any elementary section: \( a, b, c \). We get the same ordering of colours for all elementary sections of \( f \) and we obtain that there exist exactly five elementary sections, namely:

\[
A_4 := (a, b, b, c), \quad A_5 := (a, b, b, c, c), \quad B_5 := (a, a, b, c, c), \quad C_5 := (a, a, b, b, c) \text{ and } A_6 := (a, a, b, b, c, c).
\]

**Theorem 3.1** (Voigt [6]). A 3-colouring \( f \) is (2, 3)-consistent iff the elementary sections of \( f \) are none other than \( A_4, A_5, B_5, C_5 \) or \( A_6 \) and no section \( S_1 \) of \( f \) contains: \( A_4 A_4, C_5 A_4, C_5 A_5 \) or \( A_4 A_5 \).

**Corollary 3.2.** Let \( f \) be a (2, 3)-consistent 3-colouring of \( Z \). Then there exists at least one \( v^0 \) with \( v^0 \in Z^* \) in an arbitrary interval of length six.

4. \( p \)-consistency of (2, 3)-consistent 3-colourings

**Specifications.** Let \( S_1 \) be a section of a 3-colouring \( f \) of \( Z \). The first elementary section of \( S_1 \) is denoted by \( E_F \), the last by \( E_L \), the last before \( S_1 \) by \( E_{L-1} \) and the next after \( S_1 \) by \( E_{L+1} \) (see Fig. 2).

**Theorem 4.1.** A (2, 3)-consistent 3-colouring is \( n \)-consistent (\( n \in \mathbb{N}, n \geq 4 \)) iff the following five conditions are fulfilled:

(i) \( \forall S_1 \in f \Rightarrow l \neq n \),

(ii) \( \forall S_{n+2} \in f \Rightarrow E_F \neq C_5 \) or \( E_{L+1} \neq A_5 \),

(iii) \( \forall S_{n-2} \in f \Rightarrow E_F \neq A_5 \) or \( E_{L+1} \neq C_5 \),

(iv) \( \forall S_{n-1} \in f \Rightarrow E_{F-1} \in \{A_4, C_5\} \text{ and } E_{L+1} \in \{A_4, A_5\} \),

(v) \( \forall S_{n+1} \in f \Rightarrow E_F \in \{A_4, A_5\} \text{ and } E_L \in \{A_4, C_5\} \).
Main idea of the proof. (1) (i), ..., (v) follow from \( n \)-consistency. Assume the contrary for any of the five conditions. The contradiction follows immediately.

(2) \( n \)-consistency follows from (i), ..., (v). We have to show: \( \forall v_1, v_2 \in \mathbb{Z} \) with \( v_2 - v_1 = n \), it holds: \( f(v_1) \neq f(v_2) \). We look for integers \( v^0_1, v^0_2 \in \mathbb{Z}^n \) "near" \( v_1 \) and \( v_2 \), respectively. Such an integer \( v^0_i \in \mathbb{Z}^n \) exists in each interval of length 6 (see Corollary 3.2). Therefore we have:

\[ \exists v^0_1 \in \mathbb{Z}^n, x_1 \in \{ -3, \ldots, +2 \} \text{ with } v_1 = v^0_1 + x_1. \]

It is favourable for the proof to choose the interval for \( x_2 \) as a function of the value of \( x_1 \).

\[ \exists v^0_2 \in \mathbb{Z}^n, x_2 \in \{ x_2, 1, \ldots, x_2, 2 \} \text{ with } x_2, 2 - x_2, 1 = 5 \text{ and } v_2 = v^0_2 + x_2. \]

It follows that \( v^0_2 - v^0_1 = v_2 - v_1 + x_1 - x_2 = n + x_1 - x_2 \) and \( v^0_2 = v^0_1 + n + x_1 - x_2 \). From (i) we immediately obtain \( x_2 \neq x_1 \).

- Let \( x_1 = -3 \). Put \( x_2 \in \{ -5, -4, -2, -1, 0 \} \).
- Let \( x_1 = -2 \). Put \( x_2 \in \{ -3, -1, 0, 1, 2 \} \).
- Let \( x_1 = -1 \). Put \( x_2 \in \{ -3, -2, 0, 1, 2 \} \).
- Let \( x_1 = 0 \). Put \( x_2 \in \{ -3, -2, -1, 1, 2 \} \).
- Let \( x_1 = 1 \). Put \( x_2 \in \{ -1, 0, 2, 3, 4 \} \).
- Let \( x_1 = 2 \). Put \( x_2 \in \{ -1, 0, 1, 3, 4 \} \).

We have to consider \( f(v_1) \) and \( f(v_2) \) for all 30 cases. Taking the conditions (i), ..., (v) into consideration we always obtain the inequality \( f(v_1) \neq f(v_2) \).

In what follows, we discuss the cases for \( x_1 = -2 \) to illustrate the argumentation.

Hence, suppose \( x_1 = -2 \). It follows that

\[ f(v_1) = \begin{cases} 
    b & \text{if } A_4 \text{ or } C_5 \text{ ends in } v^0_1 - 1, \\
    c & \text{otherwise}. 
\end{cases} \]

Case 1: \( x_2 = -3, f(v_2) = b \). It follows that \( v^0_2 = v^0_1 + n + 1 \) and \( A_4 \) or \( A_5 \) begin in \( v^0_1 \) (see (v)). Therefore we have \( f(v^0_1 + 1) = b \) and \( f(v^0_1 - 2) = f(v_1) \neq b \) (3-consistency).

Case 2: \( x_2 = -1, f(v_2) = c \). It follows that \( v^0_2 = v^0_1 + n - 1 \) and \( A_4 \) or \( C_5 \) end in \( v^0_1 - 1 \) (see (iv)). Thus we have \( f(v_1) = b \neq c \).

Case 3: \( x_2 = 0, f(v_2) = a \neq f(v_1) \).

Case 4: \( x_2 = 1, f(v_2) = \begin{cases} 
    b & \text{if } A_4 \text{ or } A_5 \text{ begins in } v^0_2, \\
    a & \text{otherwise}. 
\end{cases} \]

We show: \( f(v_1) \) and \( f(v_2) \) cannot be equal to "b" simultaneously. Let \( f(v_1) = b \).

(1) \( A_4 \) ends in \( v^0_1 - 1 \). We have \( v^1_1 := v^0_1 - 4 \in \mathbb{Z}^n \) and \( v^0_2 - v^1_1 = n + 1 \). It follows from (v) that \( A_4 \) or \( C_5 \) ends in \( v^0_2 - 1 \) and consequently \( f(v^0_2 - 2) = b \). Because of 3-consistency we have \( f(v^0_2 + 1) = f(v_2) \neq b \).
(2) $C_5$ ends in $v_1^0 - 1$. We have $v_1^1 := v_1^0 - 5 \in \mathbb{Z}^*$ and $v_2^0 - v_1^1 = n + 2$. It follows from (ii) that $A_5$ does not begin with $v_2^0$. Let us assume $A_4$ begins in $v_2^0$. We have $v_1^1 := v_2^0 + 4 \in \mathbb{Z}^*$ and $v_2^1 - v_1^0 = n + 1$. It follows from (v) that $f(v_1^0 + 1) = b$ and $f(v_1^0 - 2) = f(v_1) \neq b$ because of 3-consistency. This contradiction implies: $A_4$ does not begin in $v_2^0$. Therefore we have $f(v_1) \neq b$.

Case 5: $x_2 = 2$, $f(v_2) = b$. Now $v_2^0 - v_1^0 = n - 4$. We have to show: $A_4$ or $C_5$ does not end in $v_1^0 - 1$. First assume $A_4$ ends in $v_1^0 - 1$. It follows that $v_1^1 := v_1^0 - 4 \in \mathbb{Z}^*$ and $v_2^1 - v_1^0 = n$. Then assume $C_5$ ends in $v_1^0 - 1$. It now follows that $v_1^1 := v_1^0 - 5 \in \mathbb{Z}^*$ and $v_2^0 - v_1^1 = n + 1$. Thus $A_4$ or $A_5$ begins with $v_1^1$ (see (v)) — contradiction to assumption.

The remaining 25 cases can be handled in an analogous way. $\square$

**Corollary 4.2.** A 3-colouring $f$, consisting of elementary sections $A_6$ and $B_5$ only, is $(2, 3, n)$-consistent if it contains no section of the length $n - 1$, $n$ and $n + 1$.

Theorem 4.1 and Corollary 4.2 permit to investigate the $n$-consistency of 3-colourings.

5. Important results

At first, some known results are presented.

**Lemma 5.1** (Walther [8]). Let $p, q \in \mathbb{P}$, $p \geq 7$, $q = p + 8$. Then $\chi(2, 3, p, q) = 4$ holds if and only if $p \in \{11, 23, 29\}$.

**Lemma 5.2** (Voigt [6]). Let $D = \{2, 3, p, q\}$. $\chi(D)$ is equal to 4 in case of the following five pairs of primes $\{p, q\}$:

1. $p = 11$, $q = 23$;
2. $p = 11$, $q = 37$;
3. $p = 11$, $q = 41$;
4. $p = 17$, $q = 29$;
5. $p = 23$, $q = 41$.

Thus, we have eight sets of four primes $D = \{2, 3, p, q\}$ ($p \geq 7$, $q > p + 2$) with $\chi(D) = 4$ and we can show that additional sets with this property do not exist.

**Lemma 5.3** (Voigt and Walther [7]). Let $\Delta \in \mathbb{N}$, $\Delta \geq 10$. We have $\chi(2, 3, u, u + \Delta) = 3$ for all $u \in \mathbb{N}$ and $u \geq \Delta^2 - 6\Delta + 3$.

**Lemma 5.4** (Walther [8]). Each $d$-consistent periodical colouring $P_d$ is also $(\lambda + d)$- and $(\lambda - d)$-consistent.
Consequently, the colourings $P_{q-3}, P_{q-2}, P_{q+2}, P_{q+3}$ consisting of the sections $B_5$ and $A_6$ only, are $(2, 3, q)$-consistent.

**Theorem 5.5.** Let $D = \{2, 3, p, q\}$ be a set of primes with $p \geq 7$ and $q > p + 2$. Then $\chi(D) = 4$ holds if and only if

$$(p, q) \in \{(11, 19), (11, 23), (11, 37), (11, 41), (17, 29), (23, 31), (23, 41), (29, 37)\}.$$

The idea of the proof of this theorem is described in the following sections.

**6. Construction of colourings**

We construct 3-colourings consisting of the elementary sections $B_5$ and $A_6$ only. First we fix the period $\lambda$ and the number $\alpha$ of sections $B_5$ in $P_\alpha$. Note that $\lambda - 5\alpha$ must be divisible by 6. Consequently we obtain the number $\beta$ of sections $A_6$ by

$$\beta = \frac{\lambda - 5\alpha}{6}.$$

We assign the sections $A_6$ to the sections $B_5$ “evenly” (see algorithm (#)).

**Examples.** $\lambda = 111$.

1. $\alpha = 3, \beta = 16$:


2. $\alpha = 9, \beta = 11$:

$$P_{111} := B_5A_6A_6A_6B_5A_6A_6B_5A_6A_6B_5A_6A_6B_5A_6B_5A_6B_5A_6B_5A_6B_5A_6B_5A_6A_6.$$

3. $\alpha = 15, \beta = 6$:

$$P_{111} := B_5B_5B_5A_6B_5A_6B_5A_6B_5B_5A_6B_5B_5A_6B_5B_5A_6B_5B_5A_6B_5B_5A_6B_5B_5A_6.$$

The largest integer less than or equal to $x$ is denoted by $\lfloor x \rfloor$.

**Algorithm (#) to assign evenly**

(i) the numbers $z_1, \ldots, z_\alpha$ are defined by the following algorithm:

(0) $\alpha$ – number of sections $B_5$,

$$\beta - \text{number of sections } A_6,$$

$$\gamma = \frac{\beta}{\alpha} - \left\lfloor \frac{\beta}{\alpha} \right\rfloor(<1),$$

$$z = \left\lfloor \frac{\beta}{\alpha} \right\rfloor,$$

$z_i$ – number of sections $A_6$ assigned to the $i$th section $B_5$. 
(1) \( R_0^1 := 0, \quad R_1^1 := \gamma, \)
\( R_1^2 := 0, \)
\( z_1 := z = \left\lfloor \frac{\beta}{\gamma} \right\rfloor, \)
\( i := 2. \)

(2) \( R_1^1 := R_{i-1}^1 + \gamma, \quad R_i^2 := \begin{cases} 0, & \text{if } \bar{R}_1^1 < 1, \\ 1, & \text{otherwise}, \end{cases} \)
\( z_i := z + R_i^2 = \left\lfloor \frac{\beta}{\gamma} \right\rfloor + R_i^2, \)
\( R_i^1 := \bar{R}_i^1 - R_i^2, \)
\( i := i + 1, \)

for \( i \leq \alpha \) go to (2), otherwise go to END.

END:

(ii) the periodical colouring \( P_A \) is defined by
\[
P_A := B_s(A_6)^{z_1}B_s(A_6)^{z_2} \ldots B_s(A_6)^{z_{r-1}}B_s(A_6)^{z_r}.
\]

Let \( r \) and \( j \) be arbitrary natural numbers. We consider a section \( S_L \) consisting of \( r \) successive sections \( B_s \) and the sections \( A_6 \) assigned by (#) to this sections \( B_s \). Let the section \( S_L \) begin with the \((j + 1)\)st section \( B_s \). We obtain:

\[
L = 5r + 6 \sum_{i=j+1}^{j+r} z_i.
\]
\[
Z := \sum_{i=j+1}^{j+r} z_i \text{ is the number of sections } A_6 \text{ from } S_L. \text{ It is easy to see that}
\]
\[
Z = rz + \sum_{i=j+1}^{j+r} R_i^2 = rz + [R_1^1 + r\gamma].
\]

It follows from \( 0 \leq R_1^1 < 1 \) that
\[
 rz + [r\gamma] \leq Z \leq rz + [r\gamma] + 1. \quad (*)
\]

Let \( P_A \) be a periodical 3-colouring constructed by the algorithm given above.

**Definition 6.1.** Let
\[
L^*_{\min} := \min_{j \in \mathbb{N}} \left( 5r + 6 \sum_{i=j+1}^{j+r} z_i \right), \quad L^*_{\max} := \max_{j \in \mathbb{N}} \left( 5r + 6 \sum_{i=j+1}^{j+r} z_i \right)
\]
be the minimal (maximal) length of sections of $P_\alpha$ consisting of $r$ successive sections $B_\gamma$ and their assigned sections $A_\beta$.

It is easy to see that $L_{\min}^r \leq L_{\max}^r \leq L_{\min}^r + 6$. We obtain from (*):

$$L_{\min}^r = 5r + 6\lfloor rz + \lfloor r\gamma \rfloor \rfloor = 6\left\lfloor \frac{\beta}{\alpha} \right\rfloor + 5r$$

and therefore

$$L_{\max}^r \leq 6\left\lfloor \frac{\beta}{\alpha} \right\rfloor + 5r + 6.$$}

Furthermore it follows from (*):

**Lemma 6.2.** Let $A_\beta$ be an arbitrary section of the colouring $P_\alpha$ consisting of $r$ successive sections $B_\gamma$ and their assigned (by (#)) sections $A_\beta$. Then it holds $L_{\min}^r < L < L_{\max}^r$.

7. $p$-consistency of a colouring $P_\alpha$

The periodical colouring $P_\alpha$ is constructed by using the algorithm (#).

Any section $S_L$ of $P_\alpha$ with a length $L > L_{\min}^3$ contains at least one section $B_\gamma$. Any section $S_L$ of $P_\alpha$ with a length $L < L_{\min}^r$ contains at most three sections $B_\gamma$. Consequently, such a colouring is $p$-consistent for all $p \equiv 1(\text{mod } 6)$ and $L_{\min}^r < p < L_{\max}^r$ by Corollary 4.2. It follows that $L_{\max}^r + 2 \leq p \leq L_{\min}^3$ because of $L_{\min}^3 \equiv -1(\text{mod } 6)$, $p \equiv 1(\text{mod } 6)$ and $L_{\min}^3 \equiv 3(\text{mod } 6)$. Furthermore, we obtain that such a colouring $P_\alpha$ is $p$-consistent for all $p \equiv 1(\text{mod } 6)$ with

$$L_{\max}^r + 2 \leq p \leq L_{\min}^r + 2, \quad r = 6j + 1 \text{ and } j = 0, 1, 2, \ldots .$$

In the case $p \equiv -1(\text{mod } 6)$ we obtain in an analogous way the consistency for all $p$ with

$$L_{\max}^r + 2 \leq p \leq L_{\min}^r + 2, \quad r = 6j + 3 \text{ and } j = 0, 1, 2, \ldots .$$

Now we derive from (**):

**Lemma 7.1.** A periodical 3-colouring $P_\alpha$ constructed by algorithm (#) is $p$-consistent

(1) for all $p \in P$, $p \equiv 1(\text{mod } 6)$, $r = 6j + 1$ and

$$6\left\lfloor (6j + 1)\frac{\beta}{\alpha} \right\rfloor + 5(6j + 1) + 8 \leq p \leq 6\left\lfloor (6j + 3)\frac{\beta}{\alpha} \right\rfloor + 5(6j + 3) - 2$$

with $j = 0, 1, 2, \ldots ;$

(2) for all $p \in P$, $p \equiv -1(\text{mod } 6)$, $r = 6j + 3$ and
8. Sequences of colourings

We fix an integer $\alpha = \alpha_0$ and a period $\lambda$ such that $\lambda - 5\alpha$ is divisible by 6: $\beta_0 := (\lambda - 5\alpha)/6$. Let

$\alpha_i = 6i + \alpha$ for $i = 0, 1, \ldots, i_{\text{max}}$

(for the choice of $i_{\text{max}}$ see below) and

$\beta_i := \frac{\lambda - 5\alpha_i}{6} = \frac{\lambda - 5(6i + \alpha)}{6} = \beta_{i-1} - 5 \quad (i = 1, \ldots, i_{\text{max}})$.

We investigate periodical colourings $P_\lambda = P_\lambda(\alpha_i, \beta_i), i = 0, 1, \ldots, i_{\text{max}}$, constructed by algorithm (\#). We have to ensure that $\beta_{\text{max}} \geq 0$.

It is easy to see (from (**)) that

$L_{\text{max}}^r(i) = 6 \left\lfloor \frac{\beta_i}{\alpha_i} \right\rfloor + 5r + 6,$

$L_{\text{min}}^{r+2}(i) = 6 \left\lceil \frac{(r + 2)\beta_i}{\alpha_i} \right\rceil + 5r + 10.$

For a given $r$ we fix $i_{\text{max}}$ to be the smallest $i$ with $r\beta_i/\alpha_i < 1$.

The following lemma immediately results from Lemma 7.1.

Lemma 8.1. A sequence $P_\lambda$ of colourings contains $p$-consistent colourings

(1) for all $p \in P$, $p \equiv 1 (\text{mod } 6)$, $r = 6j + 1$ and

$6 \left\lfloor (6j + 1)\frac{\beta_i}{\alpha_i} \right\rfloor + 5(6j + 1) + 8 \leq p \leq 6 \left\lfloor (6j + 3)\frac{\beta_i}{\alpha_i} \right\rfloor + 5(6j + 3) - 2$

with $i = 0, 1, 2, \ldots, i_{\text{max}}, j = 0, 1, 2, \ldots$;

(2) for all $p \in P$, $p \equiv -1 (\text{mod } 6)$, $r = 6j + 3$ and

$6 \left\lfloor (6j + 3)\frac{\beta_i}{\alpha_i} \right\rfloor + 5(6j + 3) + 8 \leq p \leq 6 \left\lfloor (6j + 5)\frac{\beta_i}{\alpha_i} \right\rfloor + 5(6j + 5) - 2$

with $i = 0, 1, 2, \ldots, i_{\text{max}}, j = 0, 1, 2, \ldots$.  

\[
6 \left\lfloor (6j + 3)\frac{\beta_i}{\alpha_i} \right\rfloor + 5(6j + 3) + 8 \leq p \leq 6 \left\lfloor (6j + 5)\frac{\beta_i}{\alpha_i} \right\rfloor + 5(6j + 5) - 2
\]
9. Chromatic number

**Lemma 9.1.** There exists only a finite number of sets \( D = \{2, 3, p, q\}, \) \( p, q \in \mathbb{P}, p \geq 7 \) and \( q > p + 2, \) with \( \chi(D) = 4. \)

**Idea of the proof.** We construct \((2, 3, p, q)\)-consistent periodical 3-colourings \( P^i_\lambda. \) The parameters of such colourings are listed in Table 1. We have to distinguish four cases:

(i) \( p \equiv 1, q \equiv 1; \) (ii) \( p \equiv 1, q \equiv -1; \) (iii) \( p \equiv -1, q \equiv -1 \) and (iv) \( p \equiv -1, q \equiv 1 \) (see Table 1, columns 1 and 2). Each row of Table 1 represents a sequence of colourings (for \( i = 0, \ldots, i_{\text{max}} \) if \( \alpha_i \) depends on (i)) or a single colouring with period \( \lambda \) given in column 4. The construction of these colourings is meaningful for primes \( q \) greater than or equal to the values in column 3.

The intervals of consistency for \( p \) are derived from Lemmas 7.1 and 8.1 (with fixed \( j \), given in column 7), but the proofs are very sophisticated. We have carried out the case \( p \equiv 1 \mod 6, q \equiv 1 \mod 6 \) only. One can find the remaining three cases in [6].

**Proof of Lemma 9.1.** For \( p \equiv 1 \mod 6 \) and \( q \equiv 1 \mod 6 \), at first, we look at \((+)\) in Table 1.

1. We have to show \( \beta_{i_{\text{max}}} \geq 0. \)
   
   For \( i_{\text{max}} \in \{0, 1\}, \) we have \( \beta_i = (q + 2 - 5(6i + 3))/6 \geq 0, \) because of \( q \geq 43. \)

   For \( i_{\text{max}} \geq 2 \) we have \( \beta_{i_{\text{max}}} \geq 6i_{\text{max}} - 8 \geq 0 \) because of \( i_{\text{max}} \leq (q - 31)/66 + 1. \)

2. **Bounds for \( p.**
   
   The sequence \( P^i_\lambda \) contains \( p \)-consistent colourings for all \( p \) with
   
   \[
   6 \left\lfloor \frac{\beta_i}{\alpha_i} \right\rfloor + 13 \leq p \leq 6 \left\lceil \frac{\beta_i}{\alpha_i} \right\rceil + 13 \quad \text{by Lemma 8.1.}
   
   - **Lower bound for \( p: For i = i_{\text{max}} \) we get
     
     \[
     p \geq 6 \left\lfloor \frac{\beta_{i_{\text{max}}}}{\alpha_{i_{\text{max}}}} \right\rfloor + 13 = 13,
     
     \] because of \( \beta_{i_{\text{max}}} / \alpha_{i_{\text{max}}} < 1. \)

   - **Upper bound for \( p: For i = 0 \) we obtain
     
     \[
     p \leq 6 \left\lceil \frac{\beta_0}{\alpha_0} \right\rceil + 13 = 6 \left\lceil \frac{q - 13}{6} \right\rceil + 13 = q.
     
     \] (3) **Overlapping of the “intervals of consistency”.
       
       We have to show that the difference between the lower bound of the \( i \)th colouring and the upper bound of the \((i + 1)\)th colouring is at most 6 (because \( p \equiv 1 \mod 6 \)).
       
       this means

       \[
       6 \left\lfloor \frac{\beta_i}{6i + 3} \right\rfloor + 13 \leq 6 \left\lceil \frac{\beta_{i+1}}{6i + 9} \right\rceil + 13 + 6 \quad \forall i = 0, 1, \ldots, i_{\text{max}} - 1.
       
       \]
<table>
<thead>
<tr>
<th>p = q = q ≥ λ</th>
<th>α</th>
<th>i_max</th>
<th>j</th>
<th>Interval of consistency of p</th>
</tr>
</thead>
<tbody>
<tr>
<td>1 1 43 q + 2 6i + 3 [ \frac{q - 31}{66} ] + 1 0</td>
<td>[13, q] – { \frac{q - 13}{18} + 7 }</td>
<td>(+)</td>
<td></td>
<td></td>
</tr>
<tr>
<td>q - 2 7 – 0</td>
<td>{ \frac{q - 13}{18} + 7 }</td>
<td>(++)</td>
<td></td>
<td></td>
</tr>
<tr>
<td>1 - 1 47 q - 2 6i + 3 [ \frac{q - 35}{66} ] + 1 0</td>
<td>[13, q - 4] – { \frac{q - 17}{18} + 7 }</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>q + 2 5 – 0</td>
<td>{ \frac{q - 17}{18} + 7 }</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>-1 1 83 q + 2 6i + 5 [ \frac{q - 33}{42} ] + 1 0</td>
<td>[23, q] { \frac{q - 53}{66} + 29, 6 \frac{q - 23}{30} + 17 }</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>q - 2 9 – 0</td>
<td>{ \frac{q - 53}{66} + 29, 6 \frac{q - 47}{54} + 23 }</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>q + 2 17 – 1</td>
<td>{ \frac{q - 47}{54} + 29, 6 \frac{q - 23}{30} + 17 }</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>-1 1 97 q - 2 6i + 7 [ \frac{q - 51}{42} ] + 1 0</td>
<td>[23, 6 \frac{q - 37}{42} + 23 } { \frac{5q - 67}{78} + 29, 6 \frac{q - 37}{42} + 17 }</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>q + 2 9 – 0</td>
<td>{ \frac{q - 67}{78} + 29, 6 \frac{q - 37}{42} + 17 }</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>q - 2 6i + 7 [ \frac{q - 58}{48} ] + 1 i</td>
<td>{ \frac{5q - 37}{42} + 23, q - 20 }</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

*For q = 101 the prime p = 29 is not contained in the corresponding interval of consistency.  
*The primes p = 47 (for q = 83 and q = 89), p = 59 (for q = 107), p = 71 (for q = 131) and p = 107 (for q = 197) are not contained in the corresponding interval of consistency.  
*For q = 103 the prime p = 71 is not contained in the corresponding interval of consistency.

Let i ≠ 0, this means i ∈ \{1, 2, ..., i_{\text{max}} - 1\}. It is sufficient to show

\[
\frac{\beta_i}{6i + 3} \leq \frac{\beta_{i+1}}{2i + 3} + 1 = \frac{\beta_i - 5}{2i + 3} + 1.
\]
This is equivalent to

\[ \beta_i \geq -3i + \frac{3}{2} + \frac{3}{2i}. \]

This inequality is correct for \( i \geq 1 \) because \( \beta_i \geq 0 \).

Let \( i = 0 \). The difference between the lower bound for \( i = 0 \) and the upper bound for \( i = 1 \) is for some \( q \) greater than 6, namely for all \( q \) with \( \lfloor (q - 13)/18 \rfloor = \lfloor (q - 7)/18 \rfloor \). In these cases the integer \( p = 6\lfloor (q - 13)/18 \rfloor + 7 \) is not contained in the intervals of consistency. Therefore, we have to find a \( p \)-consistent colouring for \( p = 6\lfloor (q - 13)/18 \rfloor + 7 \). We look at \((+ +)\) in Table 1 and we obtain by Lemma 7.1 that the colouring is \( p \)-consistent for all \( p \) with \( 6\lfloor (q - 37)/42 \rfloor + 13 \leq p \leq 6\lfloor (q - 37)/42 \rfloor + 13 \). It is easy to see that for \( q \in \mathbb{P} \), \( q \equiv 43 \) and \( q \equiv 1 \pmod{6} \) the integer \( p = 6\lfloor (q - 13)/18 \rfloor + 7 \) is contained in this interval of consistency. \( \square \)

Thus, there is only a finite number of pairs of primes \( p, q \) not contained in Table 1 (see also Lemma 5.3). We can find \( (2, 3, p, q) \)-consistent 3-colourings for all of them except the pairs from Lemmas 5.1 and 5.2. We use only the sections \( B_5 \) and \( A_6 \) or \( A_4 \) and \( B_5 \) for most of such colourings (see [6]). We do not give all these colourings here because most of them are easy to recognize. On the other hand it is not trivial to find the colourings for the three pairs \( p, q \) listed in Table 2.

Thus, the proof of Theorem 5.5 is complete. We have shown that there are exactly eight pairs of primes \( p, q (p \geq 7, q > p + 2) \) with \( \chi(2, 3, p, q) = 4 \).

References
