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Modular representations of profinite groups☆

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ABSTRACT

Our aim is to transfer several foundational results from the modular representation theory of finite groups to the wider context of profinite groups. We are thus interested in profinite modules over the completed group algebra $k[[G]]$ of a profinite group G, where k is a finite field of characteristic p.

We define the concept of relative projectivity for a profinite $k[[G]]$-module. We prove a characterization of finitely generated relatively projective modules analogous to the finite case with additions of interest to the profinite theory. We introduce vertices and sources for indecomposable finitely generated $k[[G]]$-modules and show that the expected conjugacy properties hold—for sources this requires additional assumptions. Finally we prove a direct analogue of Green's Indecomposability Theorem for finitely generated modules over a virtually pro-p group.

1. Introduction

The modular representation theory of a finite group G attempts to describe the modules over the group algebra kG, where k is a field of characteristic p dividing the order of G. Under these circumstances kG is not semisimple and the vast majority of kG-modules are not completely reducible. Towards an understanding of these modules the important concept of relative projectivity has been considered in some depth.

Modular representation theory seems very well suited for consideration in the wider context of profinite groups. If G is a profinite group and k is a finite field, then there is a very natural profinite analogue of the group algebra for G, and hence of the corresponding profinite modules. There is also a well-defined Sylow theory of profinite groups that in particular allows us to consider analogues for p-subgroups. The close connection between a profinite object and its finite quotients allows us to generalize several foundational results of modular representation theory to a much wider universe of groups.

We give here an indication of our approach and the main results. In Section 3 we define the concept of relative projectivity for a $k[[G]]$-module and prove a characterization of finitely generated relatively H-projective modules, where H is a closed subgroup of G (Theorem 3.7). Of particular note in this characterization is the fact that a $k[[G]]$-module is relatively H-projective if and only if it is relatively HN-projective for every open normal subgroup N of G. In Section 4 we introduce the vertex of an indecomposable finitely generated module, proving existence (Corollary 4.3) and uniqueness up to conjugation in G (Theorem 4.6). Crucial in the proof of 4.6, and elsewhere, is the helpful fact that a finitely generated indecomposable $k[[G]]$-module has local endomorphism ring (Proposition 4.4). In Section 5 we introduce the concept of source, but note that this object seems less natural in the profinite category than it does in the finite case. We prove under additional hypotheses that finitely generated sources are unique up to conjugation (Theorem 5.5). In the last section we prove an analogue of Green's indecomposability theorem for modules over the completed group algebra of a virtually pro-p group (Theorem 6.7). To do this, we first show that an important characterization of absolutely indecomposable modules, known to hold for

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finite groups, also holds for virtually pro-p groups (Theorem 6.6). Finally, we answer the question of what happens when the module in question is not necessarily absolutely indecomposable, showing that the induced summands are isomorphic (Theorem 6.10). In many proofs we utilize a class of quotient modules known as coinvariants. These give a natural inverse system for a module with some very useful properties, many of which are elucidated in Section 2.

It is hoped that in the future, results in the area will have number theoretic applications (to Iwasawa algebras or to Galois theory, for instance) as well as being of interest from a purely algebraic perspective.

There are excellent books available covering the prerequisite material of this paper. For a detailed introduction to profinite objects see [13], or for an explicitly functorial approach well suited to our needs see [9]. For the modular representation theory of finite groups see [1,2] or the encyclopedic [4]. Our discussion will for the most part follow the path laid out in the seminal paper [6] of Green, published almost exactly 50 years ago.

2. Preliminaries

Throughout our discussion let k be a finite field of characteristic p and let G be a profinite group. We define well-known profinite analogues of the natural objects of modular representation theory. Denote by $k[[G]]$ the *completed group algebra* of G — that is, the completion of the abstract group algebra kG with respect to the open normal subgroups of G. Since k is finite and G is profinite, the completed group algebra $k[[G]]$ is profinite. A profinite $k[[G]]$-module is a profinite additive abelian group U together with a continuous map $k[[G]] \times U \to U$ satisfying the usual module axioms. It follows from [9, 5.1.1] that U is the inverse limit of an inverse system of finite quotient modules of U. If not explicitly stated, our modules are *profinite left* modules.

Let H be a closed subgroup of G. If W is a right $k[[H]]$-module and V is a left $k[[H]]$-module, then we denote by $W\widehat{\otimes}_{k[[H]]}V$ the *completed tensor product* of W and V over $k[[H]]$ [9, 5.5]. This is the natural profinite analogue of the abstract tensor product and satisfies most of the properties one would expect. If either W or V is finitely generated as a $k[[H]]$-module then the completed tensor product and abstract tensor product coincide [9, 5.5.3(d)]. Now let V be a profinite $k[[H]]$-module and define the *induced* $k[[G]]$-module $V\uparrow^G$ as $k[[G]]\widehat{\otimes}_{k[[H]]}V$ with action from G on the left factor. If U is a $k[[G]]$-module then the *restricted* $k[[H]]$-module $U\downarrow_H$ is the module U with coefficients restricted to $k[[H]]$.

A profinite $k[[G]]$-module U is said to be *finitely generated* if there is a finite subset $\{u_1, \dots, u_n\}$ of U with every element of U a $k[[G]]$-linear combination of the elements $u_1, \dots, u_n$. Thus U is the module abstractly generated by the given finite subset, but by [13, 7.2.2] this module is in fact profinite.

Whenever U, W are profinite $k[[G]]$-modules, denote by $\mathrm{Hom}_{k[[G]]}(U, W)$ the k-module of continuous $k[[G]]$-module homomorphisms from U to W. We sketch proofs for some properties of this object that do not seem to be explicitly mentioned in the literature.

Lemma 2.1. *Let U and $W = \varprojlim_I W_i$ be profinite $k[[G]]$-modules. then there is a topological isomorphism*

$$\mathrm{Hom}_{k[[G]]}(U, W) \cong \varprojlim_{i\in I} \mathrm{Hom}_{k[[G]]}(U, W_i),$$

where each set of maps is given the compact-open topology.

Proof. Abstractly this is essentially the definition of inverse limit. Using basic properties of the compact-open topology it is easily verified that the obvious isomorphism is a homeomorphism. □

Corollary 2.2. *If U is a finitely generated profinite $k[[G]]$-module and W is a profinite $k[[G]]$-module, then $\mathrm{Hom}_{k[[G]]}(U, W)$ is profinite.*

If H is a closed subgroup of G ($H \leq_C G$) the functor $(-)\uparrow_H^G$ is left adjoint to $(-)\downarrow_H^G$. The unit $\eta : 1 \to (-)\uparrow^G\downarrow_H$ is given by $\eta_V(v) = 1\widehat{\otimes}v$ and the counit $\varepsilon : (-)\downarrow_H\uparrow^G\to 1$ by $\varepsilon_U(g\widehat{\otimes}u) = gu$. In particular we have the following:

Lemma 2.3. *Let $H \leq_C G$ and V a $k[[H]]$-module. Having identified V with $1\widehat{\otimes}_{k[[H]]}V \subseteq V\uparrow^G$, every continuous $k[[H]]$-module homomorphism $V \to U\downarrow_H$ extends uniquely to a continuous $k[[G]]$-module homomorphism $V\uparrow^G\to U$.*

The following result will also be of use. For the definition of a filter base see [13, 1.2].

Lemma 2.4. *Let U be a $k[[G]]$-module and let $\{W_i \mid i \in I\}$ be a filter base of open subgroups of G. Then $U\downarrow_W\uparrow^G\cong \varprojlim_i U\downarrow_{W_i}\uparrow^G$, where $W = \bigcap W_i$.*

Proof. This follows from [9, 5.2.2, 5.5.2, 5.8.1].

If U is a finitely generated $k[[G]]$-module then we can give a reasonably explicit inverse system for U using *coinvariant* quotient modules. If N is a closed normal subgroup of G, then the coinvariant module U_N is defined as $k\widehat{\otimes}_{k[[N]]}U$, where the left factor k is the trivial $k[[G]]$-module. The action of G on U_N is given by $g(\lambda\widehat{\otimes}u) = \lambda\widehat{\otimes}gu$. In tensor product notation we usually denote U_N by $k\widehat{\otimes}_N U$. The module U_N can usefully be described as follows:

Lemma 2.5. *U_N together with the canonical projection map $\varphi_N : U \to U_N$ is (up to isomorphism) the unique $k[[G]]$-module on which N acts trivially and satisfying the following universal property:*

Every continuous $k[[G]]$-module homomorphism ρ from U to a profinite $k[[G]]$-module X on which N acts trivially factors uniquely through φ_N. That is, there is a unique continuous homomorphism $\rho' : U_N \to X$ such that $\rho'\varphi_N = \rho$.

Note that N is in the kernel of the action of $k[[G]]$ on U_N, so that U_N can be considered as a $k[[G/N]]$-module. It follows that if N is open and U is finitely generated then U_N is finite. From properties of the completed tensor product (which in this case is the same as the abstract tensor product) it is also easy to check that the operation $(-)_N$ is a right exact functor from the category of (finitely generated) $k[[G]]$-modules to the category of (finitely generated) $k[[G/N]]$-modules.

We collect here several properties of coinvariant modules. First a list of important technical details:

Lemma 2.6. *Let G be a profinite group, N, M closed normal subgroups of G with $N \leq M$ and H a closed subgroup of G. Let U, W be $k[[G]]$-modules and let V be a $k[[H]]$-module. Then*

1. *$(U_N)_M$ is naturally isomorphic to U_M.*
2. *$(U \oplus W)_N \cong U_N \oplus W_N$.*
3. *$V_{H\cap N}$ is naturally a $k[[HN/N]]$-module.*
4. *$(V\uparrow^G)_N \cong V_{H\cap N}\uparrow^{G/N}$.*
5. *$U_N \downarrow_{HN/N} \cong (U\downarrow_{HN})_N$.*

Proof. The maps required for *1.* are obtained by repeated use of the universal property 2.5. The remaining isomorphisms are easily verified. □

We are primarily interested in coinvariant modules for the following reason:

Proposition 2.7. *If U is a profinite $k[[G]]$-module, then $\{U_N \mid N \lhd_O G\}$ together with the set of canonical quotient maps forms a surjective inverse system with inverse limit U.*

Proof. It is clear that the maps $\varphi_{MN} : U_N \to U_M$ given by $1\widehat{\otimes}_N u \mapsto 1\widehat{\otimes}_M u$ whenever $N \leq M$ are well defined and give an inverse system of the $k[[G]]$-modules U_N. It is also clear that we have a compatible set of maps $\varphi_N : U \to U_N$ given by $u \mapsto 1\widehat{\otimes}_N u$. We need only show that U is in fact the inverse limit. The maps φ_N are the components of a surjective map of inverse systems, giving a continuous surjection $u \mapsto (1\widehat{\otimes}_N u)$ onto the limit by [9, 1.1.5], so we need only check that this map is injective.

To do this we use the universal property 2.5. By definition U is profinite, so is the inverse limit of some inverse system of finite quotient modules. Fix $u \neq 0$ in U and some finite quotient module U/W in which the image of u is non-zero. Then since U/W is finite some $N \lhd_O G$ must act trivially on U/W, so that the quotient map $U \twoheadrightarrow U/W$ factors through U_N via φ_N. But if the image of u under the composition is non-zero then certainly the image of u under φ_N is non-zero, and so the image of u in $\varprojlim U_N$ is non-zero. Thus, our map is injective and $U \cong \varprojlim_N U_N$, as required. □

Lemma 2.8. *Let G be a profinite group and U a non-zero profinite $k[[G]]$-module. Let N be a closed pro-p subgroup of G. Then $U_N \neq 0$.*

Proof. First suppose that G is a pro-p group. Since U is profinite it has a proper open submodule of finite index and hence a maximal submodule U', so the module U/U' is simple. But U/U' is finite, so can be regarded as a module for the finite p-group G/N_0 for some $N_0 \lhd_O G$. The only simple module over a finite p-group is k, so that $U/U' \cong k$ and we have a surjection $\beta : U \twoheadrightarrow k$. But every $N \lhd_C G$ acts trivially on k, so that β factors through every U_N, and thus $U_N \neq 0$ for each $N \lhd_C G$.

Now let G be a general profinite group. Since $U\downarrow_N$ is a non-zero module for the pro-p subgroup N, by the previous paragraph $0 \neq (U\downarrow_N)_N \cong U_N\downarrow_N$. Hence $U_N \neq 0$. □

The above result is particularly useful when G is a virtually pro-p group so that G has a basis of open normal pro-p subgroups. In this case the following result will be of the utmost importance:

Proposition 2.9. *Let G be a virtually pro-p group and let U be an indecomposable finitely generated $k[[G]]$-module. Then there exists some $N_0 \lhd_O G$ such that U_N is indecomposable for every $N \leq N_0$.*

Proof. We work within the cofinal (see [9, 1.1.9]) inverse system $\{U_N, \varphi_{MN}\}$ of coinvariant modules for which N is a pro-p group. Since φ_{MN} is functorial and summands cannot have zero image by 2.8, we see that as we move up our system the number of direct summands of the U_N cannot increase. It follows that for some $N_0 \lhd_O G$ and any $N \leq N_0$ the number n of indecomposable summands of U_N is equal to the number of indecomposable summands of U_{N_0}. We take the cofinal inverse system of those N contained in N_0

For each N, let s_N be a set $\{X_{N,1}, \ldots, X_{N,n}\}$ of n indecomposable submodules of U_N intersecting pairwise in 0 and having the property that U_N is equal to the (direct) sum $X_{N,1} \oplus \cdots \oplus X_{N,n}$. Denote by S_N the set of all possible s_N — a non-empty finite set. We form a new inverse system of the finite sets S_N via the maps $\psi_{MN} : S_N \to S_M$ given by

$$\psi_{MN}(s_N) = \psi_{MN}(\{X_{N,1}, \ldots, X_{N,n}\}) = \{\varphi_{MN}(X_{N,1}), \ldots, \varphi_{MN}(X_{N,n})\}.$$

Since each S_N is finite and non-empty the inverse limit of this system is non-empty by [9, 1.1.4]. We fix once and for all some element (s_N) of $\varprojlim S_N$, and for each N we choose our direct sum decomposition of U_N to be the one given to us by s_N.

Recall that we are only considering those $N \lhd_O G$ contained in N_0, so that each U_N maps onto U_{N_0}. Fix some indecomposable summand X_{N_0} of U_{N_0} (an element of s_{N_0}) and for each N in our system define X_N to be the unique element of s_N with $\varphi_{N_0N}(X_N) = X_{N_0}$. It is now easy to check that $\{X_N, \varphi_{MN}|_{X_N}\}$ is an inverse system of submodules of the U_N. Denote the inverse limit of this system by X — a submodule of U.

We want to show that X is a summand of U. For each N, we have a canonical inclusion map $X_N \hookrightarrow U_N$, and these maps give a map of inverse systems $\{X_N\} \to \{U_N\}$ in which each component splits. This corresponds to an injection $\iota : X \hookrightarrow U$. For each N, let P_N denote the finite, non-empty set of projection maps $U_N \twoheadrightarrow X_N$ splitting the corresponding component of ι. The functoriality of $(-)_N$ gives us an inverse system of the P_N, and an element of the limit is a map of inverse systems corresponding to a splitting $\pi : U \to X$ of ι. Thus X is a direct summand of U. But $X \neq 0$ since $X_N \neq 0$ and the maps $\varphi_N|_X$ are surjective, so that since U is indecomposable we must have $X = U$. But now $X_N = \varphi_N(X) = \varphi_N(U) = U_N$ for each N, and thus each U_N is indecomposable, as required. $\square$

3. Relative projectivity

Our main definition is completely analogous to the equivalent definition for finite groups.

Definition 3.1. Let G be a profinite group and let $H \leq_C G$. Then a profinite $k[[G]]$-module U is *relatively H-projective* if whenever we are given a diagram

$$\begin{tikzcd} & U \arrow[d, "\varphi"] \\ V \arrow[r, two heads, "\beta"'] & W \end{tikzcd}$$

of profinite $k[[G]]$-modules and continuous $k[[G]]$-module homomorphisms, then there exists a continuous $k[[G]]$-module homomorphism $\rho : U \to V$ with $\beta\rho = \varphi$ provided there is a $k[[H]]$-module homomorphism with this property.

As in the finite case, a projective module is precisely a 1-projective module in the definition above. Our goal for this section is to obtain a characterization of relatively H-projective finitely generated $k[[G]]$-modules analogous to D.G. Higman's characterization in the finite case, for which see [2, 3.6.4]. We will also demonstrate two new characterizations that are trivial in the finite case but of great use in our more general setting.

Lemma 3.2. *Let G be a profinite group and H a closed subgroup of G. If U is a profinite $k[[G]]$-module then the following are equivalent:*

1. *U is relatively H-projective.*
2. *If ever a continuous $k[[G]]$-epimorphism $V \twoheadrightarrow U$ splits as a $k[[H]]$-module homomorphism, then it splits as a $k[[G]]$-module homomorphism.*
3. *U is a direct summand of $U\downarrow_H\uparrow^G$.*
4. *U is a direct summand of a module induced from some profinite $k[[H]]$-module.*

Proof. This is proved just as for finite groups so the details are omitted. At several points we require 2.3. $\square$

We give now two very useful characterizations of finitely generated profinite $k[[G]]$-modules. As is standard, we write $U \,\big|\, W$ to mean that the profinite module U is isomorphic to a direct summand of the profinite module W — of course we insist that the splitting maps are continuous.

Proposition 3.3. *Let U be a finitely generated profinite $k[[G]]$-module, and $H \leq_C G$. Then U is relatively H-projective if and only if U is relatively HN-projective for every $N \lhd_O G$.*

Proof. The 'only if' statement is clear. We need only show that if U is relatively HN-projective for each N, then U is relatively H-projective.

By 2.4 we have $U\downarrow_H\uparrow^G \cong \varprojlim_{N \lhd_O G}\{U\downarrow_{HN}\uparrow^G, \psi_{MN}\}$. We will form the required splitting homomorphisms as limits of maps of inverse systems.

For each $N \lhd_O G$ the identity map $U\downarrow_{HN} \to U\downarrow_{HN}$ extends uniquely to a surjection $\pi_N : U\downarrow_{HN}\uparrow^G \twoheadrightarrow U$ by 2.3. By checking commutativity of the relevant diagrams on $U\downarrow_{HN}$ it follows that $\{\pi_N \mid N \lhd_O G\}$ is a surjective map of inverse systems. This map yields a continuous surjective homomorphism $\pi : U\downarrow_H\uparrow^G \to U$.

We note that the map $U \to U\downarrow_{HN}\uparrow^G$ given by $u \mapsto 1\widehat{\otimes}u$ is a $k[[HN]]$-homomorphism, and that it splits π_N. Hence, since U is HN-projective, we have that π_N splits as a $k[[G]]$-homomorphism. Let I_N denote the non-empty set of $k[[G]]$-splittings of the map π_N.

Since U is finitely generated, we have that $\mathrm{Hom}_{k[[G]]}(U, U\downarrow_{HN}\uparrow^G)$ is compact by 2.2. Since the map from $\mathrm{Hom}_{k[[G]]}(U, U\downarrow_{HN}\uparrow^G)$ to $\mathrm{End}_{k[[G]]}(U)$ given by $\alpha \mapsto \pi_N\alpha$ is continuous, the inverse image of id_U, which is I_N, is closed and hence compact.

The maps $I_N \to I_M$ given by $\iota_N \mapsto \psi_{MN}\iota_N$ whenever $N \leq M$ make the I_N into an inverse system of non-empty compact sets, and this system has a non-empty inverse limit by [9, 1.1.4]. By definition an element of this limit is a compatible map of inverse systems $\{\iota_N\} : U \to \{U\downarrow_{HN}\uparrow^G\}$. This map of systems yields a unique $k[[G]]$-homomorphism $\iota : U \to U\downarrow_H\uparrow^G$.

Now by the functoriality of $\varprojlim$ we have

$$\pi\iota = \varprojlim \pi_N \varprojlim \iota_N = \varprojlim \pi_N \iota_N = \varprojlim \mathrm{id}_U = \mathrm{id}_U$$

so that $U \mid U\downarrow_H\uparrow^G$, as required. □

We can refine this further into a condition relying only on the finite quotients U_N. The following lemma will help us here and elsewhere:

Lemma 3.4. *Let U, W be finitely generated profinite $k[[G]]$-modules and let $\mathcal{N}$ be a cofinal inverse system of open normal subgroups of G.*

- *If $U_N \mid W_N$ for each $N \in \mathcal{N}$, then $U \mid W$.*
- *If $U_N \cong W_N$ for each $N \in \mathcal{N}$, then $U \cong W$.*

Proof. For each $N \in \mathcal{N}$, let P_N denote the non-empty finite set of surjections $\pi_N : W_N \twoheadrightarrow U_N$ that split. Whenever $N \leq M$ define $\gamma_{MN} : P_N \to P_M$ by $\pi_N \mapsto (\pi_N)_M$. This gives an inverse system of finite non-empty sets. Thus we have a non-empty inverse limit, and we fix an element (π_N) of this limit.

For each N we have a non-empty finite set I_N of injections $\iota_N : U_N \to W_N$ splitting π_N. As above we have a map $I_N \to I_M$ since

$$\pi_M(\iota_N)_M = (\pi_N)_M(\iota_N)_M = (\pi_N\iota_N)_M = \mathrm{id}_{U_M}$$

and again we have an inverse system. An element (ι_N) of the limit of this system is a splitting of (π_N), and it follows that $U \mid W$.

The second claim follows from the first by noting (for instance) that if each map π_N is injective, then so is the limit map π. □

Proposition 3.5. *Let U be a finitely generated profinite $k[[G]]$-module, and $H \leq_C G$. Then U is relatively H-projective if and only if U_N is relatively HN-projective for every $N \lhd_O G$.*

Proof. Fix $N \lhd_O G$. If U is H-projective then U is HN-projective. Now the functoriality of $(-)_N$ ensures that

$$U \mid U\downarrow_{HN}\uparrow^G \implies U_N \mid (U\downarrow_{HN}\uparrow^G)_N \implies U_N \mid U_N\downarrow_{HN}\uparrow^G$$

so that U_N is HN-projective.

To show the converse, fix some $M \lhd_O G$. We take the cofinal inverse system of U_N for $N \lhd_O G$ and $N \leq M$, noting that each U_N is relatively HM-projective. We will show that $\varprojlim U_N = U$ is relatively HM-projective.

By assumption we have $U_N \mid U_N\downarrow_{HM/N}\uparrow^{G/N}$ for each N in our inverse system. But $U_N\downarrow_{HM/N}\uparrow^{G/N} \cong (U\downarrow_{HM}\uparrow^G)_N$ by 2.6, so that for each N we have

$$U_N \mid (U\downarrow_{HM}\uparrow^G)_N$$

and the claim now follows from 3.4. Thus U is HM-projective for each M, and the result follows from 3.3. □

Definition 3.6. If $H \leq_O G$ and U, W are $k[[G]]$-modules, then the *trace map*

$$\mathrm{Tr}_{H,G} : \mathrm{Hom}_{k[[H]]}(U\downarrow_H, W\downarrow_H) \to \mathrm{Hom}_{k[[G]]}(U, W)$$

is defined by

$$\alpha \mapsto \sum_{s\in G/H} s\alpha s^{-1}.$$

For open H the properties of the trace map given in [2, 3.6.3] carry through just as for finite groups. We now complete our characterization of finitely generated relatively H-projective $k[[G]]$-modules:

Theorem 3.7. *Let G be a profinite group, let $H \leq_C G$, and let U be a finitely generated profinite $k[[G]]$-module. Then the following are equivalent:*

1. *U is relatively H-projective.*
2. *If ever a continuous $k[[G]]$-epimorphism $V \twoheadrightarrow U$ splits as a $k[[H]]$-module homomorphism, then it splits as a $k[[G]]$-module homomorphism.*
3. *U is a direct summand of $U\downarrow_H\uparrow^G$.*
4. *U is relatively HN-projective for every $N \lhd_O G$.*
5. *U_N is relatively HN-projective for every $N \lhd_O G$.*
6. *U is a direct summand of a module induced from some profinite $k[[H]]$-module.*
7. *For every $N \lhd_O G$ there exists a continuous $k[[HN]]$-endomorphism α_N of U such that* $\mathrm{id}_U = \mathrm{Tr}_{HN,G}(\alpha_N)$.

Proof. The equivalence of statements 1, 2, 3, 4, 5 and 6 follows from results above. That 6 implies 7 is shown as with the finite proof [2, 3.6.4] after using the transitivity property $X\uparrow^G \cong X\uparrow^{HN}\uparrow^G$, where X is the $k[[H]]$-module coming from 6. The proof that 7 implies 4 also mimics the finite case [2, 3.6.4]. □

4. Vertices

Our definition for vertex is again in direct analogy with the corresponding definition when the group G is finite:

Definition 4.1. Let U be a finitely generated indecomposable profinite $k[[G]]$-module. A vertex Q of U is a closed subgroup of G with respect to which U is relatively projective, but such that U is not projective relative to any proper closed subgroup of Q.

Unlike in the finite case, we must check that a vertex of U exists. We do this using the following lemma, which is useful in other situations:

Lemma 4.2. *Let G be a profinite group and let $\mathcal{W} = \{W_i \mid i \in I\}$ be a filter base of closed subgroups of G. Let U be a finitely generated profinite $k[[G]]$-module that is projective relative to each of the W_i. Then U is projective relative to $W = \bigcap_{i\in I} W_i$.*

Proof. By 3.3 it suffices to show that U is relatively WN-projective for arbitrary $N \lhd_O G$, so fix some such N. From [13, 0.3.1(h)] we have

$$WN = \left(\bigcap W_i\right) N = \bigcap W_i N.$$

The set $\{W_iN \mid i \in I\}$ is finite and thus for some $W_{i1}, \dots, W_{in} \in \mathcal{W}$ we have

$$WN = W_{i1}N \cap \cdots \cap W_{in}N = (W_{i1} \cap \cdots \cap W_{in})N.$$

But now by hypothesis there is some $W_j \in \mathcal{W}$ with $W_j \subseteq W_{i1} \cap \cdots \cap W_{in}$ so that $WN = W_jN$ for some $j \in I$. The result follows. □

Corollary 4.3. *If U is an indecomposable finitely generated profinite $k[[G]]$-module, then a vertex of U exists.*

Proof. Demonstrating the existence of a vertex amounts to showing that the set $\mathcal{I}$ of closed subgroups of G with respect to which U is relatively projective has a minimal element.

The set $\mathcal{I}$ is a partially ordered set when ordered by inclusion. We need only show that any chain $\mathcal{J}$ in $\mathcal{I}$ has a lower bound in $\mathcal{I}$, and then Zorn's lemma gives us that $\mathcal{I}$ has a minimal element Q. But from 4.2 it follows that U is projective relative to $R = \bigcap\{H \mid H \in \mathcal{J}\}$. Thus R is a lower bound for $\mathcal{J}$ and the result follows. □

Our main result for this section is that two vertices of a finitely generated indecomposable $k[[G]]$-module U are conjugate by an element of G. To prove this we require that U have local endomorphism ring. This is known when G is virtually pro-p [10, 2.1] but by observing that profinite modules are pure injective, we easily obtain the result for general G:

Proposition 4.4. *Let G be a profinite group and let U be an indecomposable finitely generated $k[[G]]$-module. Then U has local endomorphism ring.*

Proof. Let $E = \mathrm{End}_{k[[G]]}(U)$ be the ring of continuous $k[[G]]$-endomorphisms of U, and note that by [13, 7.2.2] this ring coincides with the ring of abstract $k[[G]]$-endomorphisms of U. If W were an abstract summand of U then W would be finitely generated and hence profinite. It follows that U is indecomposable as an abstract $k[[G]]$-module. A profinite module is compact in the sense of [12] and so it follows from [12, Theorem 2] that U is pure-injective.

Now [5, 2.27] tells us that the abstract endomorphism ring of an abstract indecomposable pure-injective module is a local ring. In particular, E is a local ring. □

The relevance of this proposition is the following well-known general result. We include a short proof for the reader's convenience.

Lemma 4.5. *Let R be a ring with 1 and let U, V, W be R-modules, where U has local endomorphism ring. If $U \,\big|\, (V \oplus W)$, then $U \,\big|\, V$ or $U \,\big|\, W$.*

Proof. Whenever X is isomorphic to a summand of $V \oplus W$, let π_X, ι_X denote splitting maps in the obvious way. We have

$$\mathrm{id}_U = \pi_U(\iota_V\pi_V + \iota_W\pi_W)\iota_U = \pi_U\iota_V\pi_V\iota_U + \pi_U\iota_W\pi_W\iota_U$$

and since U has local endomorphism ring (so in particular the non-units form an additive group), one of the summands on the right hand side (the first, say) is invertible. Thus $\mathrm{id}_U = \pi_U\iota_V\pi_V\iota_U\gamma$ for some $\gamma \in \mathrm{End}(U)$, and now parenthesizing as $\mathrm{id}_U = (\pi_U\iota_V)(\pi_V\iota_U\gamma)$ demonstrates that $U \,\big|\, V$. □

Theorem 4.6. *Let G be a profinite group, U an indecomposable finitely generated $k[[G]]$-module, and let Q, R be vertices of U. Then there exists $x \in G$ such that $Q = xRx^{-1}$.*

Proof. The module U is relatively R-projective so is relatively RN-projective for every open normal subgroup N of G. Fix some such N. Since $U \,\big|\, U\downarrow_{RN}\uparrow^G$ and $U \,\big|\, U\downarrow_Q\uparrow^G$ we have that U is a direct summand of

$$U\downarrow_{RN}\uparrow^G\downarrow_Q\uparrow^G \cong \bigoplus_{s\in Q\backslash G/RN} s(U\downarrow_{RN})\downarrow_{Q\cap sRNs^{-1}}\uparrow^G$$

where the above sum (coming from the Mackey decomposition formula [11, 2.2]) makes sense since RN is open so the set of double coset representatives is finite. But now since U has local endomorphism ring, 4.5 shows that

$$U \,\big|\, s(U\downarrow_{RN})\downarrow_{Q\cap sRNs^{-1}}\uparrow^G$$

for some $s \in G$. Thus U is relatively $Q \cap sRNs^{-1}$-projective. But Q is minimal, so we must have $Q \subseteq sRNs^{-1}$.

Denote by C_N the set of all $s \in G$ such that $Q \subseteq sRNs^{-1}$. Since C_N is a union of sets of the form $QgRN$ for appropriate $g \in G$, it follows that each C_N is closed in G. We thus have a collection of closed, non-empty sets $\{C_N \mid N \lhd_O G\}$ and we wish to show that their intersection is non-empty. Let $N_1, \dots, N_n$ be open normal subgroups of G. Then $N_1 \cap \dots \cap N_n \lhd_O G$ and so by the previous argument $C_{N_1\cap\dots\cap N_n} \neq \emptyset$. This means that there exists $s \in G$ such that $Q \subseteq sR(N_1 \cap \dots \cap N_n)s^{-1}$ so that certainly for each $i \in \{1, 2, \dots, n\}$ we have $Q \subseteq sRN_is^{-1}$. So $C_{N_1\cap\dots\cap N_n} \subseteq C_{N_1} \cap \dots \cap C_{N_n}$ and thus $C_{N_1} \cap \dots \cap C_{N_n} \neq \emptyset$. By compactness we now have $\bigcap_N C_N$ is non-empty.

It follows that there is some $x \in G$ such that

$$\begin{aligned} &Q \subseteq xRNx^{-1} && \forall N \lhd_O G \\ &Q \subseteq \bigcap\{xRx^{-1}N \mid N \lhd_O G\} && \\ &Q \subseteq xRx^{-1} && \text{by [13, 0.3.3]}. \end{aligned}$$

Repeating the same argument with Q and R interchanged, we find $y \in G$ such that $R \subseteq yQy^{-1}$.

But now $Q \subseteq xRx^{-1} \subseteq (xy)Q(xy)^{-1}$. Since profinite groups are well behaved under conjugation it follows that $Q = (xy)Q(xy)^{-1}$, and so $Q = xRx^{-1}$ as required. □

For the background Sylow theory we require for the following results see [13, Chapter 2].

Proposition 4.7. *If H is a closed subgroup of a profinite group G containing a p-Sylow subgroup of G, then any finitely generated profinite $k[[G]]$-module U is relatively H-projective.*

Proof. Since U is finitely generated, by 3.3 we need only show that U is relatively HN-projective for any given $N \lhd_O G$. Suppose we have a diagram as in 3.1 and a continuous $k[[HN]]$-module homomorphism $\rho' : U \to V$ making the diagram commute. Since the supernatural number $|G : H|$ is coprime to p, the finite number $|G : HN|$ is non-zero in the field k. Hence the continuous map

$$\rho = 1/|G : HN| \sum_{s\in G/HN} s\rho' s^{-1}$$

is well defined, and as in the finite case we check that ρ is a $k[[G]]$-module homomorphism such that $\beta\rho = \varphi$. □

Corollary 4.8. *If U is a finitely generated indecomposable $k[[G]]$-module, then any vertex of U is a pro-p group.*

Proof. By 4.7, U has a pro-p vertex, and now since the set of pro-p subgroups of G is closed under conjugation the result follows from 4.6. □

5. Sources

For an indecomposable finitely generated module U over a finite group, there is attached to any vertex Q of U a finitely generated indecomposable kQ-module S with the property that $U \,\big|\, S\uparrow^G$. This object is easily seen to be unique up to conjugation by elements of $\mathrm{N}_G(Q)$. If G is a profinite or even a pro-p group, the corresponding notion of source seems less natural, and even existence is not clear in general. None-the-less, we prove that if G is virtually pro-p and U is an indecomposable finitely generated $k[[G]]$-module with vertex Q and finitely generated sources S and T, then S and T are conjugate in $\mathrm{N}_G(Q)$.

The following simple lemma will prove key:

Lemma 5.1. *Let G be a virtually pro-p group, let H be a closed subgroup of G and let V be a finitely generated indecomposable $k[[H]]$-module. Then there exists a cofinal inverse system of $N \lhd_O G$ for which each $V\uparrow^{HN}$ is indecomposable.*

Proof. For any $N \lhd_O G$ we have by 2.6 that

$$(V\uparrow^{HN})_N \cong V_{H\cap N}\uparrow^{HN/N} \cong V_{H\cap N}.$$

Since $V \cong \varprojlim_{N\lhd_O G} V_{H\cap N}$ it follows by 2.9 that there is a cofinal inverse system of $N \lhd_O G$ for which $V_{H\cap N}$ and thus $(V\uparrow^{HN})_N$ is indecomposable. Now since we can choose our system of N to be pro-p we have by 2.8 that no non-zero summands of $V\uparrow^{HN}$ can become zero on taking coinvariants, and so $V\uparrow^{HN}$ is indecomposable. □

Recall that if V is a profinite $k[[H]]$-module for $H \leq_C G$ and $x \in G$ then we denote by $x(V)$ the $k[[xHx^{-1}]]$-module $x\widehat{\otimes}_{k[[H]]}V$ with action from xHx^{-1} given by

$$xhx^{-1}(x\widehat{\otimes}v) = x\widehat{\otimes}hv.$$

The functor $x(-)$ is exact. We include two technical facts about how conjugation interacts with induction and coinvariants:

Lemma 5.2. *Let $Q \leq_C H \leq_C G$, let T be a $k[[Q]]$-module, and let $x \in G$. Then*

$$x(T)\uparrow^{xHx^{-1}} \cong x(T\uparrow^{H}).$$

Proof. This is easily checked. $\square$

Lemma 5.3. *Let $H \leq_C G$, $N \lhd_O G$, and let T be a $k[[H]]$-module. Then*

$$(x(T))_{xHx^{-1}\cap N} \cong x(T_{H\cap N}).$$

Proof. If K is the kernel of the canonical map $T \twoheadrightarrow T_{H\cap N}$ then the result follows by conjugating the exact sequence $K \to T \to T_{H\cap N}$ by x. $\square$

Definition 5.4. Let G be a profinite group and let U be a finitely generated indecomposable profinite $k[[G]]$-module with vertex Q. A *source* of U is an indecomposable $k[[Q]]$-module S such that $U \mid S\uparrow^G$.

If G is a virtually pro-p group then our primary unanswered question is whether a finitely generated indecomposable $k[[G]]$-module with vertex Q need be a summand of $V\uparrow_Q^G$ for some finitely generated module V. If not then even the existence of a source for U is uncertain. If a finitely generated source exists then we have the following analogue to the well-known result for finite groups:

Theorem 5.5. *Let G be a virtually pro-p group and let U be a finitely generated indecomposable $k[[G]]$-module with vertex Q and finitely generated source. If S, T are finitely generated $k[[Q]]$-modules that act as sources of U, then $S \cong x(T)$ for some $x \in \mathrm{N}_G(Q)$.*

Proof. We work within a cofinal system of $N \lhd_O G$ for which $S_{Q\cap N}$, $T_{Q\cap N}$, $S\uparrow^{QN}$ and $T\uparrow^{QN}$ are indecomposable — this is allowed by 2.9 and 5.1. For any N in this system we have

$$U \mid S\uparrow^G \implies U\downarrow_{QN} \mid S\uparrow^G\downarrow_{QN} \cong \bigoplus_{z\in QN\backslash G/Q} z(S)\downarrow_{zQz^{-1}\cap QN}\uparrow^{QN}.$$

Since $U \mid U\downarrow_{QN}\uparrow^G$ we must have that some indecomposable summand X of $U\downarrow_{QN}$ has vertex conjugate to Q. If $zQz^{-1}\cap QN$ is properly contained in zQz^{-1} the summands of $z(S)\downarrow_{zQz^{-1}\cap QN}$ have vertex strictly smaller than a conjugate of Q, and so it follows that for some $z \in G$ with $zQz^{-1} \subseteq QN$ we have $X \mid z(S)\downarrow_{zQz^{-1}\cap QN}\uparrow^{QN} = z(S)\uparrow^{QN}$. Note also that

$$zQz^{-1} \subseteq QN \implies zQz^{-1}N \subseteq QN \implies zQNz^{-1} = QN$$

so that $z \in \mathrm{N}_G(QN)$.

Since $z(S)\uparrow^{QN} \cong z(S\uparrow^{QN})$ by 5.2 and $S\uparrow^{QN}$ is indecomposable, it follows that $z(S)\uparrow^{QN}$ is indecomposable and so for this z we have

$$z(S)\uparrow^{QN} \mid U\downarrow_{QN}.$$

On the other hand $U \mid T\uparrow^G$, so

$$z(S)\uparrow^{QN} \mid T\uparrow^G\downarrow_{QN} \cong \bigoplus_{y\in QN\backslash G/Q} y(T)\downarrow_{yQy^{-1}\cap QN}\uparrow^{QN}$$

and by the same argument we find that $z(S)\uparrow^{QN} \cong y(T)\uparrow^{QN}$ for some element $y \in \mathrm{N}_G(QN)$.

Note that $zQz^{-1}/(zQz^{-1}\cap N) \cong zQNz^{-1}/N \cong QN/N$. We will use this observation and 5.3 to transfer these results from the setting of induced modules to the setting of coinvariant modules where we have the necessary tools to draw the conclusions we require:

For each $N \lhd_O G$ in our inverse system we have

$$\begin{aligned}
& z(S)\uparrow^{QN} \cong y(T)\uparrow^{QN} \\
\implies\ & (z(S)\uparrow^{QN})_N \cong (y(T)\uparrow^{QN})_N \\
\implies\ & z(S)_{zQz^{-1}\cap N} \cong y(T)_{yQy^{-1}\cap N} \\
\implies\ & z^{-1}(z(S)_{zQz^{-1}\cap N}) \cong z^{-1}(y(T)_{yQy^{-1}\cap N}) \\
\implies\ & S_{Q\cap N} \cong z^{-1}y(T)_{(z^{-1}y)Q(z^{-1}y)^{-1}\cap N} \\
\implies\ & S_{Q\cap N} \cong z^{-1}y(T_{Q\cap N}).
\end{aligned}$$

Denote by C_N the set of $w \in \mathrm{N}_G(QN)$ such that $S_{Q\cap N} \cong w(T_{Q\cap N})$. Since $z^{-1}y$ (which depends on N) satisfies these conditions it follows that C_N is non-empty. Each C_N is also clearly closed in G. The theorem follows easily once we show the intersection $\bigcap_N C_N$ is non-empty.

Certainly $C_{N_1\cap\cdots\cap N_n} \neq \emptyset$ for any finite set $N_1, \ldots, N_n$. Let $N_1 \cap \cdots \cap N_n = M$ and fix $w \in C_M$. Now $M \leq N_i$ for each i and so

$$\begin{aligned} & S_{Q\cap M} \cong w(T_{Q\cap M}) \\ \Longrightarrow\ & (S_{Q\cap M})_{QM\cap N_i} \cong w(T_{Q\cap M})_{QM\cap N_i} \\ \Longrightarrow\ & S_{Q\cap N_i} \cong w(T_{Q\cap N_i}) \end{aligned}$$

by 2.6 so that $w \in C_{N_i}$ for each i, and so $w \in C_{N_1} \cap \cdots \cap C_{N_n}$. Thus, by compactness we have $\bigcap_N C_N \neq \emptyset$.

Fix $x \in \bigcap_N C_N$, so that for each N in our system we have

$$S_{Q\cap N} \cong x(T_{Q\cap N}),$$

and since $x \in \bigcap_N \mathrm{N}_G(QN) = \mathrm{N}_G(Q)$, we can rewrite this isomorphism as

$$S_{Q\cap N} \cong x(T)_{Q\cap N}$$

so that by 3.4 we have $S \cong x(T)$, as required. □

6. Green's indecomposability theorem

Green's indecomposability theorem says that if V is a finitely generated absolutely indecomposable module for the group algebra kH, where H is a subnormal subgroup of the finite group F of index a power of p, then the module $V\uparrow^F$ is also absolutely indecomposable. We extend this result to modules over the completed group algebra of a virtually pro-p group G.

Throughout this section let G be a virtually pro-p group and let U be an indecomposable finitely generated profinite $k[[G]]$-module. By 2.8 and 2.9 we can choose a cofinal inverse system of $N \lhd_O G$ with U_N non-zero and indecomposable, and we will work within this system throughout. All rings we consider have a 1. We do not allow 1 to equal 0.

For each N in our system let $E_N = \mathrm{End}_{k[[G]]}(U_N)$, $R_N = \mathrm{rad}(\mathrm{End}_{k[[G]]}(U_N))$ and $\tilde{E}_N = E_N/R_N$. Each E_N is a local ring and thus $\tilde{E}_N$ is a finite division ring [4, 5.21] so is a finite field. It is clear that this field must contain k. Our aim for the next few lemmas is to show that $\mathrm{End}_{k[[G]]}(U)/\mathrm{rad}(\mathrm{End}_{k[[G]]}(U)) \cong \varprojlim \tilde{E}_N$.

Define maps $\rho_{MN} : E_N \to E_M$ whenever $N \leq M$ as follows: If $\alpha_N \in E_N$ then define $\rho_{MN}(\alpha_N) = \alpha_M \in E_M$ by $\alpha_M(1\widehat{\otimes}_M u) = 1\widehat{\otimes}_M \alpha_N(u)$. Each ρ_{MN} is a ring homomorphism.

Lemma 6.1. *The map ρ_{MN} sends the radical R_N of E_N into R_M, and thus induces a map $\tilde{\rho}_{MN} : \tilde{E}_N \to \tilde{E}_M$, which is a ring homomorphism.*

Proof. This is easily checked by noting that elements of the radical R_N are precisely the nilpotent endomorphisms of U_N. □

Observe that $\{E_N, \rho_{MN}\}$ is an inverse system of finite rings and $\{\tilde{E}_N, \tilde{\rho}_{MN}\}$ is an inverse system of finite fields. Since field homomorphisms are injective we can choose a cofinal inverse system of N for which every $\tilde{E}_N = k'$, for some fixed finite extension field k' of k. From now on we will work inside this cofinal inverse system.

Define $E = \mathrm{End}_{k[[G]]}(U)$, $R = \mathrm{rad}(E)$, $\tilde{E} = \tilde{E}(U) = E/R$. Note that using the universal property of $(-)_N$ a simple tweaking of 2.1 shows that $E \cong \varprojlim_N E_N$. Denote by ρ_N the map $E \to E_N$ from the above limit. This is the map given by applying the functor $(-)_N$ to the morphisms in E.

Lemma 6.2. *The radical of E maps into the radical of E_N under ρ_N, for each N.*

Proof. Recall that the radical of E consists of all non-invertible endomorphisms of U. Fix an element α in the radical of E, so that α is not an isomorphism. If α were surjective, then each α_N would also be onto because $(-)_N$ is right exact. But then each α_N would be an isomorphism, and hence so would be α, contrary to assumption. It follows that α is not surjective. If each $\rho_N(\alpha) = \alpha_N$ were onto then so would be α, so we can find some $N_0 \lhd_O G$ with α_{N_0} not onto.

We note that for any $N' \lhd_O G$ contained in N_0, the corresponding $\alpha_{N'}$ is not onto. Fix some arbitrary $N \lhd_O G$, and consider $L = N \cap N_0$. Then α_L is not onto since $L \leq N_0$, so that $\alpha_L \in R_L$. But now by Lemma 6.1 this implies that $\alpha_N \in R_N$ as well, so that the image of R in E_N is contained inside R_N. □

The endomorphism ring of U is local by Proposition 4.4, and thus $\tilde{E}$ is a division ring.

Lemma 6.3. *The division ring $\tilde{E}$ is a finite field and is isomorphic to $\varprojlim \tilde{E}_N$.*

Proof. For each N we have canonical surjections $\gamma_N : E_N \twoheadrightarrow E_N/R_N = \tilde{E}_N$, which give a map of inverse systems since for $N \leq M$ the diagrams

$$\begin{tikzcd} E_N \arrow[r, two heads, "\gamma_N"] \arrow[d, "\rho_{MN}"] & \tilde{E}_N \arrow[d, "\tilde{\rho}_{MN}"] \\ E_M \arrow[r, two heads, "\gamma_M"] & \tilde{E}_M \end{tikzcd}$$

commute. This map of inverse systems gives a surjection of rings γ from E to $\varprojlim \tilde{E}_N$.

We note now since $\rho_N(R) \subseteq R_N$ for each N, that $R \subseteq \ker(\gamma)$. Hence, we can factor out R to obtain a surjection from $E/R = \tilde{E}$ to $\varprojlim \tilde{E}_N$. But this is now a surjection of division rings and hence an isomorphism of fields, as required. □

If F is a finite group, recall that a kF-module W is said to be *absolutely indecomposable* if the $k'F$-module $k' \otimes_k W$ is indecomposable for all field extensions k' of k. By [4, 30.29], W is absolutely indecomposable if and only if $\tilde{E}(W) \cong k$. We thus have the following immediate corollary to 6.3:

Corollary 6.4. *If G is a virtually pro-p group and U is a finitely generated $k[[G]]$-module with corresponding $\tilde{E} \cong k$, then U is the inverse limit of an inverse system of finite absolutely indecomposable modules.*

From [4, 7.14, 3.34, 30.27] we can make several important deductions. Firstly if F is a finite group and W is a finitely generated kF-module, then W is absolutely indecomposable if and only if $k' \otimes_k W$ is indecomposable for all *finite* field extensions k' of k. Secondly, if W is not absolutely indecomposable then the extension l of the field k required for $l \otimes_k W$ to decompose does not depend directly on F or W, but only on the field $\tilde{E}(W)$. These facts ensure that the following definition is appropriate:

Definition 6.5. A finitely generated profinite $k[[G]]$-module U is *absolutely indecomposable* if the $k'[[G]]$-module $k'U = k'\widehat{\otimes}_k U$ is indecomposable for all finite field extensions k' of k.

Theorem 6.6. *If G is a virtually pro-p group, then a finitely generated $k[[G]]$-module U is absolutely indecomposable if and only if $\tilde{E} \cong k$.*

Proof. If $\tilde{E} \cong k$ then by 6.4, U is the inverse limit of a cofinal inverse system of absolutely indecomposable modules U_N. Suppose that $k' \otimes_k U$ decomposes as $X \oplus Y$ for some finite extension field k' of k and some $X, Y \neq 0$. Then

$$k' \otimes_k U_N \cong (k' \otimes_k U)_N \cong (X \oplus Y)_N \cong X_N \oplus Y_N.$$

But X_N and Y_N are non-zero since N is pro-p, by 2.8, contradicting the absolute indecomposability of U_N.

To show the forward implication, assume that $\tilde{E} = k'$ for k' a finite field extension of k which properly contains k. Since $\tilde{E} \cong \varprojlim \tilde{E}_N$ we have a cofinal inverse system of modules U_N for which $\tilde{E}_N = k'$.

By the discussion prior to 6.5 there is a fixed finite extension field l of k for which each $l\widehat{\otimes}_k U_N$ decomposes. But

$$\varprojlim\,(l\widehat{\otimes}_k U_N) \cong l\widehat{\otimes}_k \varprojlim U_N = l\widehat{\otimes}_k U$$

since by [9, 5.5.2] complete tensoring commutes with $\varprojlim$ and the actions of l and G carry through this isomorphism. Now the contrapositive of 2.9 demonstrates that $l\widehat{\otimes} U$ decomposes, so that U is not absolutely indecomposable. □

We can now prove Green's indecomposability theorem for virtually pro-p groups:

Theorem 6.7. *Let G be a virtually pro-p group, let $H \lhd_C G$ with G/H a pro-p group, and let V be a finitely generated absolutely indecomposable $k[[H]]$-module. Then $V\uparrow^G$ is absolutely indecomposable.*

Proof. Suppose for contradiction that $V\uparrow^G$ decomposes, so that $V\uparrow^G = X \oplus Y$ for $k[[G]]$-modules $X, Y \neq 0$. By 6.6 the module V has corresponding $\tilde{E}(V) = k$, so by 6.4 we can find some open normal pro-p subgroup N of G with $V_{H\cap N}$ absolutely indecomposable. Then

$$V_{H\cap N}\uparrow^{G/N} \cong (V\uparrow^G)_N = (X \oplus Y)_N \cong X_N \oplus Y_N$$

where $X_N, Y_N \neq 0$ by 2.8, so that $V_{H\cap N}\uparrow^{G/N}$ decomposes. But this decomposition contradicts Green's indecomposability theorem for finite groups [4, 19.23], and so $V\uparrow^G$ must be indecomposable.

For absolute indecomposability note that there is a cofinal inverse system of $N \lhd_O G$ for which $\tilde{E}(V_{H\cap N}\uparrow^{G/N}) \cong k$. But $V_{H\cap N}\uparrow^{G/N} \cong (V\uparrow^G)_N$ so that $\tilde{E}((V\uparrow^G)_N) \cong k$ for each N. Now $V\uparrow^G$ is absolutely indecomposable by 6.3 and 6.6. □

As for finite groups we have immediate corollaries:

Corollary 6.8. *Let G be a virtually pro-p group, let $H \leq_C G$ be subnormal in G with $|G : H|$ a (possibly infinite) power of p, and let V be a finitely generated absolutely indecomposable $k[[H]]$-module. Then $V\uparrow^G$ is absolutely indecomposable.*

Corollary 6.9. *Let G be a pro-p group, let $H \leq_C G$, and let V be a finitely generated absolutely indecomposable $k[[H]]$-module. Then $V\uparrow^G$ is absolutely indecomposable.*

Proof. Each G/N is a finite p-group so that HN/N is subnormal in G/N and the result follows as above. □

We include a virtually pro-p version of a variant of Green's indecomposability theorem (for the finite case see [7], [3] or [8]). It seems a pity that this result is not widely known for finite groups.

Theorem 6.10. *Let H be a closed subgroup of a virtually pro-p group G and let V be a finitely generated indecomposable $k[[H]]$-module. If either H is subnormal in G and of index some (possibly infinite) power of p, or G is pro-p, then the indecomposable summands of $V\uparrow^G$ are isomorphic.*

Proof. If the module $V\uparrow^G$ is indecomposable then we are done. Otherwise write $V\uparrow^G = X \oplus Y \oplus Z$ with X, Y non-zero and indecomposable. We will show that $X \cong Y$.

Choose a cofinal inverse system of open normal pro-p subgroups N of G so that V itself and the indecomposable summands of $V\uparrow^G$ remain indecomposable on taking coinvariants. Now for any such N we have

$$V_{H\cap N}\uparrow^{G/N} \cong (V\uparrow^G)_N \cong X_N \oplus Y_N \oplus Z_N.$$

But $V_{H\cap N}$ is a finitely generated indecomposable module over the finite group HN/N, and under either hypothesis given above we have $X_N \cong Y_N$ by [7]. It now follows immediately from 3.4 that $X \cong Y$ and we are done. □

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References

[1] J.L. Alperin, Local Representation Theory, Cambridge University Press, Cambridge, 1986.
[2] D.J. Benson, Representations and Cohomology I, Cambridge University Press, Cambridge, 1995.
[3] M. Broué, Sur l'induction des modules indécomposables et la projectivité relative, Math. Z. 149 (1976) 227–245.
[4] C.W. Curtis, I. Reiner, Methods Of Representation Theory Volume 1, Wiley, New York, 1981.
[5] A. Facchini, Module Theory, Birkhäuser Verlag, Basel, 1998.
[6] J.A. Green, On the indecomposable representations of a finite group, Math. Z. 70 (1959) 430–445.
[7] M.E. Harris, A variant of a theorem of J.A. Green, J. Algebra 104 (1986) 104–107.
[8] L. Puig, Sur un Théorème de Green, Math. Z. 166 (1979) 117–129.
[9] L. Ribes, P. Zalesski, Profinite Groups, Springer, Berlin, 2000.
[10] P.A. Symonds, On the construction of permutation complexes for profinite groups, Geom. Topol. Monogr. 11 (2007) 369–378.
[11] P.A. Symonds, Double coset formulas for profinite groups, Comm. Algebra 36 (2008) 1059–1066.
[12] R.B. Warfield, Purity and algebraic compactness for modules, Pacific J. Math. 28 (1969) 699–719.
[13] J.S. Wilson, Profinite Groups, Oxford University Press, New York, 1998.