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Representing filtration consistent nonlinear expectations as g-expectations in general probability spaces

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Abstract

We consider filtration consistent nonlinear expectations in probability spaces satisfying only the usual conditions and separability. Under a domination assumption, we demonstrate that these nonlinear expectations can be expressed as the solutions to Backward Stochastic Differential Equations with Lipschitz continuous drivers, where both the martingale and the driver terms are permitted to jump, and the martingale representation is infinite dimensional. To establish this result, we show that this domination condition is sufficient to guarantee that the comparison theorem for BSDEs will hold, and we generalise the nonlinear Doob–Meyer decomposition of Peng to a general context.

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Keywords: BSDE; Nonlinear expectation; Doob–Meyer decomposition

1. Introduction

Much work has been done regarding risk-averse decision making in various contexts. One approach to this has been to assume that agents make decisions based on the 'expectation' of a random outcome, but to allow this expectation to be nonlinear. This allows resolution of the famous Allais and Ellsberg paradoxes (see [11]), while still retaining much of the flavour of classical approaches.

A significant problem in this context is to guarantee that these nonlinear expectations are *time consistent*, that is, that they can be consistently updated using new observations. As many

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nonlinear expectations are not time-consistent, it is useful to give representations for those which are.

In [17] (see also [19]), Peng gives an axiomatic approach to these nonlinear expectations. In [17], of particular interest are the 'g-expectations', which arise from the solutions to Backward Stochastic Differential Equations (BSDEs). In [7], it is shown that every nonlinear expectation satisfying a certain domination property must solve a BSDE. At the end of that paper [7, Remark 7.1], the following comment is made.

> "In this paper we have limited ourselves to treat the situation where the filtration is generated by a Brownian motion. A natural question is whether our nonlinear supermartingale decomposition approach can be applied to more general situations. A general positive answer seems unlikely, due to the lack of comparison theorem for BSDEs driven by discontinuous processes."

In this paper, we answer this question in the affirmative, using the BSDEs and comparison theorem in [4]. We show that all nonlinear expectations satisfying a domination property similar to that in [7] can be represented by solutions to BSDEs. The domination property which we use is sufficient to guarantee that a comparison theorem holds, and so this extension of [7] is possible. We do this making no substantive assumptions on the probability space (we only assume the usual conditions and that $L^2(\mathcal{F}_T)$ is separable). Furthermore, even in the context of a Brownian filtration, our results extend [7] to allow a countable number of independent Brownian motions.

Various other extensions of [7] are known. For the case of a Lévy filtration, a corresponding result was obtained by Royer [21]. In discrete time, a stronger representation is also known (see [3,5]). A more general result, restricted to the context of a Brownian filtration, is given by Hu et al. [12]. This result uses a weaker domination property, which corresponds to considering solutions to quadratic BSDEs. As no existence results for quadratic BSDEs are available in the general context considered in [4], we are not yet able to encompass these cases.

Alternative approaches to the representation of nonlinear expectations exist, for example, Bion-Nadal [1,2] has a representation for the penalty term of time-consistent convex risk measures (which, up to a change of sign, can be seen to be equivalent to the nonlinear expectations considered here). Similarly, in the Brownian filtration, Delbaen et al. [9] represent these penalty terms using g-expectations. The approach of this paper is instead to give a representation of the nonlinear expectation directly, which allows us to avoid any assumption of convexity.

In this paper, we begin by summarising and generalising the results and approach of [4] to BSDEs in general probability spaces. We then also reproduce the key results on filtration-consistent expectations (without proof where the result is exactly as in [7]). We proceed to generalise a result of [18], giving a Doob–Meyer type decomposition for g-expectations in general probability spaces, and furthermore, for general nonlinear expectations satisfying our domination property. Finally, using the previous results, we show that any nonlinear expectation satisfying our domination property must equal a g-expectation.

2. BSDEs in general spaces

2.1. Existence of BSDE solutions

We here give the key results regarding BSDEs in general probability spaces. These are taken without proof from [4]. For simplicity, we shall restrict our attention to the scalar case. As usual, unless otherwise indicated, all (in-)equalities should be read as 'up to evanescence'.

Assumption 1. We shall henceforth assume that

(i) the usual conditions hold on our filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\in[0,T]}, \mathbb{P})$, and $\mathcal{F}_0$ is the $\mathbb{P}$-completion of the trivial σ-algebra $\{\Omega, \emptyset\}$,
(ii) $L^2(\mathcal{F}_T)$ is separable, and
(iii) we have some (arbitrary) deterministic, strictly increasing process μ with $\mu_T < \infty$.

Remark 1. The process μ will be used in the place of Lebesgue measure in our BSDE. The assumption that $\mathcal{F}_0$ is trivial is not strictly necessary, but is used to simplify notation (as it implies no martingale has a jump at zero).

Definition 1. For any nondecreasing process of finite variation μ, we define the measure induced by μ to be the measure over $\Omega \times [0, T]$ given by

$$A \mapsto E\left[\int_{[0,T]} I_A(\omega, t)d\mu\right].$$

Here $A \in \mathcal{P}$, the predictable σ-algebra, I_A is the indicator function of A, and the integral is taken pathwise in a Stieltjes sense.

The following version of the martingale representation theorem (from [8], see also [14,16]) is fundamental to our approach.

Theorem 1 (*Martingale Representation Theorem; [8]*). *Suppose $L^2(\mathcal{F}_T)$ is a separable Hilbert space, with an inner product $(X, Y) = E[XY]$. Then there exists a finite or countable sequence of square-integrable $\{\mathcal{F}_t\}$-martingales $M^1, M^2, \ldots$ such that every square integrable $\{\mathcal{F}_t\}$-martingale N has a representation*

$$N_t = N_0 + \sum_{i=1}^{\infty} \int_{]0,t]} Z_u^i dM_u^i$$

for some sequence of predictable processes Z^i. This sequence satisfies

$$E\left[\sum_{i=0}^{\infty} \int_{]0,T]} (Z_u^i)^2 d\langle M^i\rangle_u\right] < +\infty. \tag{1}$$

These martingales are orthogonal (that is, $E[M_T^i M_T^j] = 0$ for all $i \neq j$), and the predictable quadratic variation processes $\langle M^i\rangle$ satisfy

$$\langle M^1\rangle \succ \langle M^2\rangle \succ \cdots,$$

where $\succ$ denotes absolute continuity of the induced measures (Definition 1). Furthermore, these martingales are unique, in that if N^i is another such sequence, then $\langle N^i\rangle \sim \langle M^i\rangle$, where $\sim$ denotes equivalence of the induced measures.

We shall denote by $\mathbb{R}^\infty$ the set of countable sequences of real values.

Definition 2 (*See [4]*). We define the stochastic seminorm $\|\cdot\|_{M_t}$ on $\mathbb{R}^\infty$ as follows. For each $i \in \mathbb{N}$, consider $\langle M^i\rangle$ as a measure on the predictable σ-algebra. Let $\langle M^i\rangle$ have the Lebesgue-decomposition

$$\langle M^i\rangle_t = m_t^{i,1} + m_t^{i,2},$$

where $m_t^{i,1}$ is absolutely continuous with respect to $\mu \times \mathbb{P}$ and $m_t^{i,2}$ is orthogonal to $\mu \times \mathbb{P}$. As they represent bounded measures on the predictable σ-algebra, both $m_t^{i,1}$ and $m_t^{i,2}$ will be nondecreasing predictable processes. We define, for $z_t \in \mathbb{R}^\infty$,

$$\|z_t\|_{M_t}^2 := \sum_i \left[(z_t^i)^2 \frac{dm^{i,1}}{d(\mu \times \mathbb{P})} \right]$$

where $z_t^i \in \mathbb{R}$ is the i'th element in z_t and $\frac{dm^{i,1}}{d(\mu\times\mathbb{P})}$ is a version of the Radon–Nikodym derivative which is zero $m^{i,2}$-a.e.

We note that, for any predictable, progressively measurable process Z taking values in $\mathbb{R}^\infty$, and in particular for processes satisfying (1), we have the inequality

$$\begin{aligned} E\left[\int_A \|Z_t\|_{M_t}^2 d\mu_t\right] &\le E\left[\sum_i \int_A (Z_t^i)^2 d\langle M^i\rangle_t\right] \\ &= E\left[\sum_i \left(\int_A Z_t^i dM_t^i\right)^2\right] = E\left[\left(\sum_i \int_A Z_t^i dM_t^i\right)^2\right] \end{aligned} \tag{2}$$

for any predictable set $A \subseteq \Omega \times [0, T]$. (Note the latter equalities are simply the standard isometry used in the construction of the stochastic integral, by the orthogonality of the M^i.)

Definition 3. For any predictable process Z taking values in $\mathbb{R}^\infty$ with (2) finite, any predictable set A, for notational simplicity we shall write

$$\begin{aligned} &\int_A Z_t \cdot dM_t := \sum_i \int_A Z_t^i dM_t^i, \\ &Z_t \cdot \Delta M_t := \sum_i Z_t^i \Delta M_t^i, \\ &\int_A Z_t^2 \cdot d\langle M\rangle_t := \sum_i \int_A (Z_t^i)^2 d\langle M^i\rangle_t. \end{aligned}$$

Definition 4. We define the following spaces

$$\begin{aligned} H_M^2 &= \left\{ Z : \Omega \times [0,T] \to \mathbb{R}^\infty,\ \text{predictable},\ E\left[\int_{]0,T]} Z_t^2 \cdot d\langle M\rangle_t\right] < +\infty \right\}, \\ S^2 &= \left\{ Y : \Omega \times [0,T] \to \mathbb{R},\ \text{adapted},\ E\left[\sup_{t\in[0,T]} (Y_t)^2\right] < +\infty \right\}, \\ H_\mu^2 &= \left\{ Y : \Omega \times [0,T] \to \mathbb{R},\ \text{progressive},\ \int_{]0,T]} E[(Y_t)^2] d\mu_t < +\infty \right\}, \end{aligned}$$

where two elements $Z, \bar{Z}$ of H_M^2 are deemed equivalent if

$$E\left[\int_{[0,T]} (Z_t - \bar{Z}_t)^2 \cdot d\langle M\rangle_t\right] = 0,$$

two elements of S^2 are deemed equivalent if they are indistinguishable, and two elements of H_μ^2 are equivalent if they are equal $\mu \times \mathbb{P}$-a.s.

Remark 2. We note that H^2_M is itself a complete metric space, with norm given by $Z \mapsto E\left[\int_{]0,T]} Z_t^2 \cdot d\langle M\rangle_t\right]$. Similarly for H^2_μ. Note also that the martingale representations constructed in Theorem 1 are unique in H^2_M.

Theorem 2 (*See [4]*). *Let $g : \Omega \times [0,T] \times \mathbb{R} \times \mathbb{R}^\infty \to \mathbb{R}$ be a predictable function such that*

- $E\left[\int_{]0,T]} |g(\omega,t,0,\mathbf{0})|^2 d\mu_t\right] < +\infty.$
- *There exists a quadratic firm Lipschitz bound on F, that is, a measurable deterministic function c_t uniformly bounded by some $c \in \mathbb{R}$, such that, for all $y, y' \in \mathbb{R}, z, z' \in \mathbb{R}^\infty$, all $t > 0$*

$$|g(\omega,t,y,z) - g(\omega,t,y',z')|^2 \le c_t|y-y'|^2 + c\|z-z'\|^2_{M_t}\ d\mu \times d\mathbb{P}\text{—a.s.}$$

and

$$c_t(\Delta\mu_t)^2 < 1 \quad \textit{for all } t > 0.$$

Note that the variable bound c_t need only apply to the behaviour of F with respect to y.

A function satisfying these conditions will be called standard. Then for any $Q \in L^2(\mathcal{F}_T)$, the BSDE with driver g

$$Y_t - \int_{]t,T]} g(\omega,u,Y_{u-},Z_u)d\mu_u + \int_{]t,T]} Z_u \cdot dM_u = Q \tag{3}$$

has a unique solution $(Y,Z) \in S^2 \times H^2_M$.

From this point onwards, for notational simplicity, we shall regard ω as implicit in the function g, whenever this does not lead to confusion.

Remark 3. Note that the behaviour of g at $t = 0$ is irrelevant to the solution of the BSDE; however we still obtain a solution with values (Y_t, Z_t) for $t \in [0,T]$. Note also that for any $y \in \mathbb{R}, z, z' \in \mathbb{R}^\infty$, we know $g(t,y,z) = g(t,y,z')$ $m^{i,2}$-a.e. for all i, by the definition of the norm $\|\cdot\|_{M_t}$.

2.2. The comparison theorem

Theorem 3 (*Comparison Theorem, See [4]*). *Suppose we have two BSDEs corresponding to standard coefficients and terminal values (g,Q) and (g',Q'). Let (Y,Z) and (Y',Z') be the associated solutions. Suppose that for some s, the following conditions hold:*

(i) $Q \ge Q'$ *$\mathbb{P}$-a.s.*
(ii) $\mu \times \mathbb{P}$*-a.s. on* $[s,T] \times \Omega$,

$$g(u,Y'_{u-},Z'_u) \ge g'(u,Y'_{u-},Z'_u).$$

(iii) *There exists a measure $\tilde{\mathbb{P}}$ equivalent to $\mathbb{P}$ such that*

$$X_r := -\int_{]s,r]} (g(u,Y'_{u-},Z_u) - g(u,Y'_{u-},Z'_u))d\mu_u + \int_{]s,r]} (Z_u - Z'_u)\cdot dM_u$$

is a $\tilde{\mathbb{P}}$ supermartingale on $[s,T]$.

It is then true that $Y \ge Y'$ on $[s,T] \times \Omega$, except possibly on some evanescent set. Furthermore, this comparison is strict, that is, for any s and any $A \in \mathcal{F}_s$ such that $Y_s = Y'_s$ $\mathbb{P}$-a.s. on A, we have $Y_u = Y'_u$ on $[s,T] \times A$, up to evanescence.

In light of this, we make the following definition (which is a strengthening of that in [4]).

Definition 5 (*Balanced Drivers*)**.** If g is such that condition (iii) of Theorem 3 holds for any special semimartingales $Y, Y' \in S^2$, (where Z and Z' are from the martingale representation theorem applied to the martingale parts of Y and Y') then g shall be called *balanced*.

Using this definition, we can give a novel condition under which the comparison theorem will hold.

Lemma 1. *If*

$$\frac{|g(t,y,z)-g(t,y,z')|}{\|z-z'\|^2_{M_t}}|(z-z')\cdot\Delta M_t| < 1$$

up to evanescence for all $y \in \mathbb{R}$, all $z, z' \in \mathbb{R}^\infty$, then g is balanced.

To prove this, we first need the following lemma, based on results of Lepingle and Mémin [15], (see also [20]).

Definition 6 (*Doléans-Dade Exponential*)**.** Let N be a local martingale. Then we shall write

$$\mathfrak{E}(N;t) := \exp(N_t - \langle N^c\rangle_t/2)\prod_{0<s\le t}(1+\Delta N_s)\exp(-\Delta N_s),$$

which is the solution $\mathfrak{E}(N;t) = \Lambda_t$ of the equation

$$\Lambda_t = 1 + \int_{]0,t]} \Lambda_{s-}dN_s.$$

Lemma 2. *Let N be a square-integrable martingale, with $\langle N\rangle$ bounded. Then $\mathfrak{E}(N;\cdot)$ is a martingale, and for any $p > 0$, $E[|\mathfrak{E}(N;T)|^p] < \infty$.*

Proof. It is clear that $\mathfrak{E}(N;\cdot)$ is a local martingale, by Lepingle and Mémin [15, Theorem II.2] it is a square integrable martingale. It is easy to verify that

$$\mathfrak{E}^2(N;t) = 1 + \int_{]0,t]}\mathfrak{E}^2(N;s-)d(2N+[N])_s = \mathfrak{E}(2N+[N];t).$$

As $\langle N\rangle \le k$ for some k, we can write

$$\mathfrak{E}^2(N;t) = \mathfrak{E}(2N+[N]-\langle N\rangle+\langle N\rangle;t) \le e^k\mathfrak{E}(2N+[N]-\langle N\rangle;t). \tag{4}$$

We now see that $\tilde{N} := 2N+[N]-\langle N\rangle = 2N+[N^d]-\langle N^d\rangle$ and this is a local martingale, hence

$$\langle \tilde{N}^c\rangle = 2\langle N^c\rangle \le 2k$$

and

$$(\Delta\tilde{N})^2 = (3\Delta N - \Delta\langle N^d\rangle)^2 \le 18(\Delta N)^2 + 2(\Delta\langle N^d\rangle)^2.$$

As N is square-integrable and $\langle N^d\rangle \le \langle N\rangle$ is bounded, $\tilde{N}$ is a square-integrable martingale. Furthermore,

$$\langle \tilde{N}^d\rangle \le 18\langle N^d\rangle + 2\sum_{0<u\le t}((\Delta\langle N^d\rangle)^2) \le 18\langle N^d\rangle + 2\langle N^d\rangle^2 \le 18k+2k^2,$$

and we see that $\langle \tilde{N} \rangle \leq 20k + 2k^2$, in particular, that this is a finite bound. Hence $\tilde{N}$ is a square-integrable martingale with $\langle \tilde{N} \rangle$ bounded.

From [15, Theorem II.2], we see that $\mathfrak{E}(\tilde{N}; t)$ is a square integrable martingale, and from (4)

$$E[(\mathfrak{E}(N; T))^4] \leq e^{2k} E[(\mathfrak{E}(\tilde{N}; T))^2] < \infty.$$

We now iterate this process, noticing that $\tilde{N}$ satisfies the requirements of the lemma, and hence if $\tilde{\tilde{N}} = 2\tilde{N} + [\tilde{N}] - \langle \tilde{N} \rangle$ (which is, by the same logic, a square integrable martingale with $\langle \tilde{\tilde{N}} \rangle$ bounded)

$$E[(\mathfrak{E}(N; T))^8] = E[(\mathfrak{E}(\tilde{N}; T))^4] \leq e^{2(20k+2k^2)} E[(\mathfrak{E}(\tilde{\tilde{N}}; T))^2] < \infty.$$

Hence we obtain, after n iterations,

$$E[(\mathfrak{E}(N; T))^{2^n}] < \infty$$

and by Jensen's inequality, the result is proven for any finite $p > 0$. $\square$

Proof of Lemma 1. Define

$$N_t = \int_{]0,t]} \left(\frac{g(u, Y'_u, Z_u) - g(u, Y'_u, Z'_u)}{\|Z_u - Z'_u\|^2_{M_u}} \right) (Z_u - Z'_u) \cdot dM_u.$$

Let Λ be the process defined by the Doléans-Dade exponential

$$\Lambda_t = 1 + \int_{]0,t]} \Lambda_{u-} dN_u = \mathfrak{E}(N; t).$$

By the assumption of the lemma, we see that $|\Delta N_t| < 1$, and so Λ_t is a strictly positive local martingale. Furthermore, we know that N has predictable quadratic variation

$$\begin{aligned}
\langle N \rangle_t &= \int_{]0,t]} \left(\frac{g(u, Y'_u, Z_u) - g(u, Y'_u, Z'_u)}{\|Z_u - Z'_u\|^2_{M_u}} \right)^2 (Z_u - Z'_u)^2 \cdot d\langle M \rangle_u \\
&= \int_{]0,t]} \frac{(g(u, Y'_u, Z_u) - g(u, Y'_u, Z'_u))^2}{\|Z_u - Z'_u\|^2_{M_u}} d\mu_u \\
&\leq c\mu_t
\end{aligned}$$

where c is the Lipschitz constant of g, and the second equality is by the construction of $\|\cdot\|_{M_t}$ and Remark 3. By Lemma 2, this shows that Λ has moments of all orders, and is a true martingale on $[0, T]$. We can therefore define the measure $\tilde{\mathbb{P}}$ by $d\tilde{\mathbb{P}}/d\mathbb{P} = \Lambda_T$.

By Girsanov's theorem (see [13, Theorem 3.11]), we see that

$$\tilde{M}^i_t = M^i_t - \int_{]0,t]} \frac{g(u, Y'_u, Z_u) - g(u, Y'_u, Z'_u)}{\|Z_u - Z'_u\|^2_{M_u}} (Z_u - Z'_u)^i d\langle M^i \rangle_u$$

is a $\tilde{\mathbb{P}}$-local martingale. Hence

$$\begin{aligned}
X_t &= \int_{]0,t]} (Z_u - Z'_u) \cdot d\tilde{M}_u \\
&= -\int_{]0,t]} (g(u, Y'_u, Z_u) - g(u, Y'_u, Z'_u)) d\mu_u + \int_{]0,t]} (Z_u - Z'_u) \cdot dM_u
\end{aligned}$$

is a $\tilde{\mathbb{P}}$-local martingale.

Finally, by Hölder's inequality, for any stopping time τ, any $\epsilon \in]0, 2]$

$$E_{\tilde{\mathbb{P}}}[X_\tau^{2-\epsilon}] = E_{\mathbb{P}}[\Lambda_T X_\tau^{2-\epsilon}] \leq E_{\mathbb{P}}[\Lambda_T^{2/\epsilon}]^{(\epsilon/2)} E_{\mathbb{P}}[X_\tau^2]^{1-\epsilon/2},$$

which is uniformly bounded, by Lemma 2 and the fact X is $\mathbb{P}$-square-integrable. It follows that X is a true $\tilde{\mathbb{P}}$-martingale. □

2.3. A scalar existence extension

As we are considering the case of scalar-valued BSDEs, it is useful to extend our existence result beyond the firmly Lipschitz assumptions of [4], as this will enable us to use various penalisation methods. The following theorem gives us such an extension, for the case of scalar BSDEs.

Theorem 4. *Let $g : \Omega \times [0, T] \times \mathbb{R} \times \mathbb{R}^\infty$ be a predictable function such that*

1. $E\left[\int_{]0,T]} g(t, 0, \mathbf{0})^2 d\mu_t\right] < +\infty$
2. *g is Lipschitz, that is, there exists $c \in \mathbb{R}$ such that for any $y, y' \in \mathbb{R}$, any $z, z' \in \mathbb{R}^\infty$*

$$|g(t, y, z) - g(t, y', z')|^2 \leq c(|y - y'|^2 + \|z - z'\|^2_{M_t})\, d\mu \times d\mathbb{P}\text{—a.s.}$$

and furthermore, for all $y \neq y'$, g satisfies

$$\left(\frac{g(t, y, z) - g(t, y', z)}{y - y'}\right) \Delta\mu_t \leq 1 - (1 + c)^{-1}.$$

Then for any $Q \in L^2(\mathcal{F}_T)$, the BSDE with driver g (3) has a unique solution $(Y, Z) \in S^2 \times H^2_M$. Furthermore, if g is balanced (that is, condition (iii) *of Theorem* 3 *is satisfied), then the comparison theorem holds.*

Proof. As g is Lipschitz with constant c and μ is a finite valued increasing function, there are at most finitely many times $t_1, t_2, \ldots, t_k$ such that $c(\Delta\mu_t)^2 \geq 1$ (and these times are deterministic). Hence, between these times, we have a standard BSDE. We shall show that we can paste together solutions at and between these times, specifically the following.

(i) For each t_i, we can take any $Y_{t_i} \in L^2(\mathcal{F}_{t_i})$, and obtain a unique pair (Y_{t_i*}, Z_{t_i}) solving

$$Y_{t_i} = Y_{t_i*} - g(t_i, Y_{t_i*}, Z_{t_i})\Delta\mu_{t_i} + Z_{t_i} \cdot \Delta M_{t_i}, \tag{5}$$

where $Y_{t_i*} \in L^2(\mathcal{F}_{t_i-})$, Z_{t_i} is $\mathcal{F}_{t_i-}$-measurable and $Z_{t_i} \cdot \Delta M_{t_i} \in L^2(\mathcal{F}_{t_i})$.

(ii) We can then use this value Y_{t_i*} as the terminal value for a BSDE on the interval $[t_{i-1}, t_i[$, which has a unique solution, as our driver is standard (recalling that the behaviour of the driver at the left-endpoint is unimportant for the BSDE solution).

(iii) The BSDEs we construct on $[t_{i-1}, t_i[$ satisfy $\lim_{t\uparrow t_i} Y_t = Y_{t_i*}$ almost surely, so our solutions satisfy $Y_{t_i*} = Y_{t_i-}$ up to evanescence.

Backward induction then yields that we have a solution to the BSDE on $[0, T]$. Note that, as $\{t_1, \ldots, t_k\}$ is finite, the processes we construct are appropriately predictable.

We first show that (i) our solution can be constructed at each problematic jump-time t_i. At t_i, we have Eq. (5), where (Y_{t_i*}, Z_{t_i}) are to be determined. Taking an expectation and difference, we see that $Z_{t_i} \cdot \Delta M_{t_i} = Y_{t_i} - E[Y_{t_i}|\mathcal{F}_{t_i-}]$. As this is a martingale difference, by the martingale representation theorem, we obtain a solution Z_{t_i}. Fixing Z_{t_i} at this solution, we then see that

$$E[Y_{t_i}|\mathcal{F}_{t_i-}] = Y_{t_i*} - g(t_i, Y_{t_i*}, Z_{t_i})\Delta\mu_{t_i}.$$

Writing $\phi(y) := y - g(t_i, y, Z_{t_i})\Delta\mu_{t_i}$, our assumptions on g show that ϕ is bi-Lipschitz with constant $(1 + c)$ and strictly increasing. Hence it has a strictly increasing bi-Lipschitz inverse, also with constant $(1+c)$. We therefore define $Y_{t_i *} = \phi^{-1}(E[Y_{t_i}|\mathcal{F}_{t_i -}])$. By Lipschitz continuity and Jensen's inequality, $Y_{t_i *} \in L^2(\mathcal{F}_{t_i -})$.

We now consider (ii), our BSDE on an interval $]t_{i-1}, t_i[$. As g is standard on this interval, $g' := g(t, y, z) I_{t \neq t_i}$ is standard on $]t_{i-1}, t_i]$. Hence it has a solution (Y', Z') on $[t_{i-1}, t_i]$, with $Y'_{t_i} = Y_{t_i *}$. As we have a terminal value which is $\mathcal{F}_{t_i -}$-measurable, it is easy to verify that our solution will satisfy $Z'_{t_i} \equiv 0$. We see that this is identical to the BSDE with driver g written on the interval $]t_{i-1}, t_i[$, and so we can define our solution $(Y_t, Z_t) = (Y'_t, Z'_t)$ for all $t \in [t_{i-1}, t_i[$. Note that as $Z'_{t_i} \equiv 0$ and $g'(t_i, \cdot, \cdot) \equiv 0$, we also have (iii), $Y'_{t_i -} = Y'_{t_i} = Y_{t_i *}$.

For the comparison theorem, we immediately see that it holds on each interval $[t_{i-1}, t_i[$. At t_i, we have an essentially identical argument as that given in discrete time in [5, Theorems 3.2 and 3.5]. □

Remark 4. Note that, if g is Lipschitz continuous and nonincreasing in y, then it is easy to verify that condition (2) holds.

2.4. *Grönwall's inequality*

In [4], we also derive a version of Grönwall's inequality, which shall be useful here.

Definition 7. Let ν be a càdlàg function of finite variation with $\Delta\nu_t < 1$ for all t. The *right-jump-inversion* of ν is defined by

$$\tilde{\nu}_t := \nu_t + \sum_{0 \leq s \leq t} \frac{(\Delta\nu_s)^2}{1 - \Delta\nu_s},$$

and satisfies $\mathfrak{E}(-\nu; t) = \mathfrak{E}(\tilde{\nu}; t)^{-1}$.

Definition 8. Let u, v be two measures on a σ-algebra $\mathcal{A}$. We write $du \leq dv$ if $u(A) \leq v(A)$ for all $A \in \mathcal{A}$.

Lemma 3 (*Backward Grönwall Inequality, See [4]*)**.** *Let u be a process such that, for ν a nonnegative Stieltjes measure with $\Delta\nu_t < 1$ and α a $\tilde{\nu}$-integrable process, u is ν-integrable and*

$$u_t \leq \alpha_t + \int_{]t,T]} u_s d\nu_s,$$

then

$$u_t \leq \alpha_t + \mathfrak{E}(-\nu; t) \int_{]t,T]} \mathfrak{E}(\tilde{\nu}; s) \alpha_s d\tilde{\nu}_s.$$

If $\alpha_t = \alpha$ is constant, this simplifies to

$$u_t \leq \alpha \mathfrak{E}(\tilde{\nu}; T) \mathfrak{E}(\tilde{\nu}; t)^{-1} = \alpha \mathfrak{E}(-\nu; t) \mathfrak{E}(-\nu; T)^{-1}.$$

3. Filtration consistent expectations

3.1. *General nonlinear expectations*

We now reproduce, for completeness, relevant definitions and results for filtration consistent nonlinear expectations. These are given without proof where the argument of Coquet et al. [7] carries over without change, or is standard.

Definition 9. A nonlinear expectation is a functional $\mathcal{E} : L^2(\mathcal{F}_T) \to \mathbb{R}$ which satisfies *strict monotonicity*:

$$\text{if } Q \geq Q' \text{ then } \mathcal{E}(Q) \geq \mathcal{E}(Q'), \quad \text{and}$$
$$\text{if } Q \geq Q' \text{ and } \mathcal{E}(Q) = \mathcal{E}(Q') \text{ then } Q = Q'$$

and *preserves constants*: $\mathcal{E}(c) = c$ for all constants c.

Definition 10. A nonlinear expectation is *filtration consistent* (or $\mathcal{F}$-consistent) if for each $Q \in L^2(\mathcal{F}_T)$ and each $t \in [0, T]$ there exists a random variable $Q^t \in L^2(\mathcal{F}_t)$ such that $\mathcal{E}(I_A Q) = \mathcal{E}(I_A Q^t)$ for all $A \in \mathcal{F}_t$. Such a nonlinear expectation is called an *$\mathcal{F}$-expectation*.

The following lemma proves that Q^t is unique. We will write $\mathcal{E}(Q|\mathcal{F}_t) := Q^t$, and call this the $\mathcal{F}_t$-conditional $\mathcal{F}$-expectation of Q.

Definition 11. An $\mathcal{F}$-expectation $\mathcal{E}$ will be called *translation invariant* if

$$\mathcal{E}(Q + R|\mathcal{F}_t) = \mathcal{E}(Q|\mathcal{F}_t) + R \quad \text{for all } R \in L^2(\mathcal{F}_t), \text{ all } Q \in L^2(\mathcal{F}_T).$$

It is called *convex* if, for any $Q, Q' \in L^2(\mathcal{F}_T)$, any $\lambda \in [0, 1]$,

$$\mathcal{E}(\lambda Q + (1 - \lambda)Q') \leq \lambda\mathcal{E}(Q) + (1 - \lambda)\mathcal{E}(Q').$$

It is called *positively homogenous* if, for any $Q \in L^2(\mathcal{F}_T)$, any $\lambda \geq 0$,

$$\mathcal{E}(\lambda Q) = \lambda\mathcal{E}(Q).$$

It is said to have the *monotone convergence property* if

$$\mathcal{E}(\lim_n Q_n) = \lim_n \mathcal{E}(Q_n)$$

for any increasing nonnegative sequence $\{Q_n\} \subseteq L^2(\mathcal{F}_T)$ with $\lim_n Q_n \in L^2(\mathcal{F}_T)$.

Lemma 4 (*See [7]*)**.** *Let $t \leq T$ and $Q_1, Q_2 \in L^2(\mathcal{F}_t)$. If $\mathcal{E}(Q_1 I_A) = \mathcal{E}(Q_2 I_A)$ for all $A \in \mathcal{F}_t$, then $Q_1 = Q_2$.*

Lemma 5 (*See [7]*)**.** *Let $\mathcal{E}$ be an $\mathcal{F}$-expectation. Then the following properties hold for all $Q, Q' \in L^2(\mathcal{F}_T)$.*

(i) *For each $0 \leq s \leq t \leq T, \mathcal{E}(\mathcal{E}(Q|F_t)|F_s) = \mathcal{E}(Q|F_s)$, and in particular, $\mathcal{E}(\mathcal{E}(Q|\mathcal{F}_t)) = \mathcal{E}(Q)$.*
(ii) *For any t, for all $A \in \mathcal{F}_t, \mathcal{E}(QI_A|\mathcal{F}_t) = I_A\mathcal{E}(Q|\mathcal{F}_t)$.*
(iii) *For any t, for all $A \in \mathcal{F}_t, \mathcal{E}(QI_A + Q'I_{A^c}|\mathcal{F}_t) = \mathcal{E}(QI_A|\mathcal{F}_t) + \mathcal{E}(Q'I_{A^c}|\mathcal{F}_t)$.*
(iv) *For any t, if $Q \geq Q'$, then $\mathcal{E}(Q|\mathcal{F}_t) \geq \mathcal{E}(Q'|\mathcal{F}_t)$. If moreover $\mathcal{E}(Q|\mathcal{F}_t) \geq \mathcal{E}(Q'|\mathcal{F}_t)$ for some t, then $Q = Q'$.*

Definition 12. For a given $\mathcal{F}$-expectation $\mathcal{E}$, a process $Y \in S^2$ is called an $\mathcal{E}$-supermartingale if $Y_s \geq \mathcal{E}(Y_t|\mathcal{F}_s)$ a.s. for all $s \leq t$. Similarly, Y is an $\mathcal{E}$-submartingale if $Y_s \leq \mathcal{E}(Y_t|\mathcal{F}_s)$, and an $\mathcal{E}$-martingale if $Y_s = \mathcal{E}(Y_t|\mathcal{F}_s)$.

Lemma 6 (*See [7]*)**.** *If $\mathcal{E}$ is convex and Y is an $\mathcal{E}$-supermartingale, then $-Y$ is an $\mathcal{E}$-submartingale. If $\mathcal{E}$ is convex and positively homogenous, then the sum of two $\mathcal{E}$-supermartingales is an $\mathcal{E}$-supermartingale.*

Theorem 5 (*Up/Downcrossing Inequalities, See [6]*)**.** *Let $\mathcal{E}$ be a convex, translation invariant and positively homogenous $\mathcal{F}$-expectation with the monotone convergence property, and Y be an $\mathcal{E}$-submartingale. For any stopping time $S \le T$, let $M(\omega, Y^S; [\alpha, \beta])$ (resp. $D(\omega, Y^S; [\alpha, \beta])$) denote the number of upcrossings (resp. downcrossings) of the interval $[\alpha, \beta]$ by Y on the interval $[0, S]$.*

Then

$$\begin{aligned}\mathcal{E}(M(\omega, Y^S; [\alpha, \beta])) &\le (\beta - \alpha)^{-1}(\mathcal{E}((Y_S - \alpha)^+) - (Y_0 - \alpha)^+)\\ \mathcal{E}(D(\omega, Y^S; [\alpha, \beta])) &\le -(\beta - \alpha)^{-1}\mathcal{E}(-(Y_S - \beta)^+)\\ &\le (\beta - \alpha)^{-1}\mathcal{E}((Y_S - \beta)^+).\end{aligned}$$

We shall use this result to prove the existence of càdlàg modifications to nonlinear martingales; see Theorem 7.

3.2. g-expectations

Theorem 6 (*See [4]*)**.** *Let g be a balanced driver which satisfies*

$$g(\omega, t, y, \mathbf{0}) = 0, \quad \mu \times \mathbb{P}\text{-a.s.} \tag{6}$$

Then the operator defined by

$$\mathcal{E}_g(Q|\mathcal{F}_t) := Y_t$$

where Y is the solution to a BSDE (3) with driver g, is a conditional $\mathcal{F}$-expectation. $\mathcal{E}_g$ is called the g-expectation.

Lemma 7 (*See [4]*)**.** *If a balanced driver g does not depend on y, then the g-expectation is translation invariant. If g is convex (resp. positively homogenous), then $\mathcal{E}_g$ is convex (resp. positively homogenous).*

As in [7], we can now show that g-expectations are bounded operators.

Lemma 8. *Let g be as in Theorem 6. Then for every real $\epsilon > 0$, there exists a constant C_ϵ such that for every $Q \in L^{2\vee(1+\epsilon)}(\mathcal{F}_T)$,*

$$|\mathcal{E}_g(Q)| \le C_\epsilon \|Q\|_{1+\epsilon},$$

where $\|\cdot\|_{1+\epsilon}$ is the standard norm in $L^{1+\epsilon}(\mathcal{F}_T)$.

Proof. Define the measure

$$\frac{d\tilde{\mathbb{P}}}{d\mathbb{P}} = \Lambda_T = \mathfrak{E}\left(\int_{]0,T]} \frac{g(s, Y_{s-}, Z_s)}{\|Z_s\|^2_{M_s}} Z_s \cdot dM_s; T\right).$$

Similarly as in the proof of Lemma 1, this is a stochastic exponential of the form considered in Lemma 2. Hence $\tilde{\mathbb{P}}$ is a probability measure and Λ_T has finite pth moment, for any p. By Girsanov's theorem, $\mathcal{E}_g(Q) = E_{\tilde{\mathbb{P}}}[Q] = E[\Lambda_T Q]$. By Hölder's inequality, we have $|\mathcal{E}_g(Q)| \le \|\Lambda_T\|_{1+\epsilon^{-1}} \|Q\|_{1+\epsilon}$, and the claim follows. □

3.3. $\mathcal{E}^r$ expectations

We now consider a particularly useful class of g-expectations, which we call $\mathcal{E}^r$-expectations. This class is based on that studied in [7]; however we here must generalise their approach to take into account the infinite dimension and presence of jumps in the martingale M.

Definition 13. Let r be a predictable process taking values in the space of real-valued countable dimensional matrices $\mathbb{R}^{\infty\times\infty}$, that is, $r_t^{ij}(\omega) \in \mathbb{R}$ for all $i, j \in \mathbb{N}$.

We denote by $r_t z_t$ the vector in $\mathbb{R}^\infty$ with values $(r_t z_t)^i = \sum_j r_t^{ij} z_t^j$. (If z were thought of as a column vector, then this would correspond to the classical matrix–vector product.)

The map $z \mapsto rz$ is a linear operator on H_M^2. We suppose that r is uniformly bounded in a modified operator norm, which we denote $\|\cdot\|_{D_t}$, that is, there is $c \in \mathbb{R}$ such that, for all t,

$$\begin{aligned}\|r_t\|^2_{D_t} &:= \operatorname{ess\,sup}_\omega \sup_{z\in H_M^2} \left\{ \frac{\|r_t(\omega) z_t\|^2_{M_t}}{\|z_t\|^2_{M_t}} \right\} \\ &= \operatorname{ess\,sup}_\omega \sup_{\{u\in\mathbb{R}^\infty:\|u\|_{M_t}=1\}} \{\|r_t(\omega)u\|^2_{M_t}\} < c.\end{aligned}$$

The process r will be called uniformly balanced if

$$\|r_t u\|_{M_t} \times |u\cdot \Delta M| < 1$$

for all $u \in \mathbb{R}^\infty$ with $\|u\|_{M_t} = 1$.

The set of all such uniformly balanced, uniformly bounded in $\|\cdot\|_{D_t}$ processes will be denoted $\mathfrak{D}$.

Definition 14. A driver g will be called *uniformly balanced* if there exists a process $r \in \mathfrak{D}$ such that for any t, y, z, z' of appropriate dimension,

$$|g(t, y, z) - g(t, y, z')| \le \|r_t(z - z')\|_{M_t}$$

up to indistinguishability.

Lemma 9. *A uniformly balanced driver is balanced.*

Proof. We can see that, for any $z, z' \in \mathbb{R}^\infty$,

$$\begin{aligned}\frac{|g(t,y,z) - g(t,y,z')|}{\|z-z'\|^2_{M_t}} |(z-z')\cdot\Delta M_t| &\le \frac{\|r_t(z-z')\|_{M_t}}{\|z-z'\|^2_{M_t}} |(z-z')\cdot\Delta M_t| \\ &= \left\| r_t \frac{z-z'}{\|z-z'\|_{M_t}} \right\|_{M_t} \left| \frac{z-z'}{\|z-z'\|_{M_t}} \cdot \Delta M_t \right|.\end{aligned}$$

Writing $u = \frac{z-z'}{\|z-z'\|_{M_t}}$, the result is clear from Lemma 1. □

Definition 15. Let $r \in \mathfrak{D}$. We shall denote by $\mathcal{E}^r$ the nonlinear expectation given by $\mathcal{E}_g$ with $g(t, y, z) = \|r_t z\|_{M_t}$.

Similarly, we define $\mathcal{E}^{-r}$ to be the nonlinear expectation given by $\mathcal{E}_g$ with $g(t, y, z) = -\|r_t z\|_{M_t}$.

Remark 5. As it is easy to show $\|r_t z\|^2_{M_t} \le \sup_s (\|r_s\|_{D_s})^2 \|z\|^2_{M_t}$, the requirements for the existence of solutions to the BSDE are satisfied. As $r \in \mathfrak{D}$, it is easy to show that $g(t, z) = \|r_t z\|_{M_t}$ is a uniformly balanced driver.

Lemma 10. *$\mathcal{E}^r$ is convex, positively homogenous and translation invariant, hence (as we shall see that it has the monotone convergence property, Lemma 16) the up and downcrossing inequalities of Theorem 5 apply.*

Proof. $\mathcal{E}^r$ is a g-expectation with driver $g(t, y, z) = \|r_t z\|_{M_t}$. It is clear that g is positively homogenous, and by the triangle inequality it is subadditive. Hence, g is convex. We also see that g does not depend on y, so the desired properties hold by Lemma 7. □

Similarly as in [7], we can now give a bound on the nonlinear expectation $\mathcal{E}^r$. However, we must be careful to correctly deal with the jumps in the process.

Lemma 11. *For any $Q \in L^2(\mathcal{F}_T)$,*

$$E[\mathcal{E}^r(Q|\mathcal{F}_t)^2] \leq E[Q^2] \exp\left((\sup_s \|r_s\|^2_{D_s})(\mu_T - \mu_t) \right).$$

Proof. Let $Y_t = \mathcal{E}^r(Q|\mathcal{F}_t)$, so (Y, Z) is the solution of the BSDE

$$Y_t - \int_{]t,T]} g(u, Y_{u-}, Z_u) d\mu_u + \int_{]t,T]} Z_u \cdot dM_u = Q.$$

Let $x_s := (\|r_s\|^{-2}_{D_s} + \Delta\mu_s)^{-1} > 0$. From the differentiation rule, and the inequality $ab \leq xa^2 + x^{-1}b^2$ for $x > 0$, we have

$$\begin{aligned} E[Y_t^2] &= E\left[Q^2 + 2\int_{]t,T]} \|r_s Z_s\|_{M_s} Y_{s-} d\mu_s - \int_{]t,T]} Z_s^2 \cdot d\langle M\rangle_s - \sum_{t<s\leq T} \|r_s Z_s\|^2_{M_s} (\Delta\mu_s)^2 \right] \\ &\leq E\left[Q^2 + \int_{]t,T]} x_s Y_{s-}^2 d\mu_s + \int_{]t,T]} (x_s^{-1} - \Delta\mu_s) \|r_s Z_s\|^2_{M_s} d\mu_s - \int_{]t,T]} Z_s^2 \cdot d\langle M\rangle_s \right]. \end{aligned} \tag{7}$$

From (2) and the definition of $\|\cdot\|_{D_t}$, we see that

$$\begin{aligned} &E\left[\int_{]t,T]} (x_s^{-1} - \Delta\mu_s) \|r_s Z_s\|^2_{M_s} d\mu_s - \int_{]t,T]} Z_s^2 \cdot d\langle M\rangle_s \right] \\ &= E\left[\int_{]t,T]} \|r_s\|^{-2}_{D_s} \|r_s Z_s\|^2_{M_s} d\mu_s - \int_{]t,T]} Z_s^2 \cdot d\langle M\rangle_s \right] \\ &\leq E\left[\int_{]t,T]} \|Z_s\|^2_{M_s} d\mu_s - \int_{]t,T]} Z_s^2 \cdot d\langle M\rangle_s \right] \leq 0. \end{aligned}$$

Hence (7) can be weakened to

$$E[Y_t^2] \leq E[Q^2] + \int_{]t,T]} x_s E[Y_{s-}^2] d\mu_s.$$

As $x_s \Delta\mu_s < 1$, an application of the Backward Grönwall inequality (Lemma 3) yields

$$E[\mathcal{E}^r(Q|\mathcal{F}_t)^2] \leq E[Q^2|\mathcal{F}_t] \mathfrak{E}(\tilde{N}; T) \mathfrak{E}(\tilde{N}; t)^{-1}$$

where $N_t = \int_{]0,t]} x_u d\mu_u$. Considering the continuous and discontinuous parts of N, we see that its right-jump-inversion (Definition 7) is $\tilde{N}_t = \int_{]0,t]} \|r_s\|^2_{D_s} d\mu_s$, and hence

$$\begin{aligned} \mathfrak{E}(\tilde{N}; T) &= \mathfrak{E}(\tilde{N}; t) \exp\left(\int_{]t,T]} \|r_s\|^2_{D_s} d\mu_t \right) \prod_{t<s\leq T} (1 + \Delta\tilde{N}_s) e^{-\Delta\tilde{N}_s} \\ &\leq \mathfrak{E}(\tilde{N}; t) \exp((\mu_T - \mu_t)(\sup_s \|r_s\|^2_{D_s})) \end{aligned}$$

yielding the result. □

3.4. $\mathcal{E}^r$-dominated expectations

We now consider those nonlinear expectations $\mathcal{E}$ which are 'dominated' by $\mathcal{E}^r$ for some $r \in \mathfrak{D}$. This property gives many useful results on the behaviour of $\mathcal{E}$. Again, those results which carry over from [7] are presented without proof.

Definition 16. For $r \in \mathfrak{D}$, we say that a nonlinear expectation $\mathcal{E}$ is dominated by $\mathcal{E}^r$ if

$$\mathcal{E}(Q + Q') - \mathcal{E}(Q') \leq \mathcal{E}^r(Q)$$

for all $Q, Q' \in L^2(\mathcal{F}_T)$.

Lemma 12. *If $\mathcal{E}$ is dominated by $\mathcal{E}^r$, then*

$$\mathcal{E}^{-r}(Q) \leq \mathcal{E}(Q + Q') - \mathcal{E}(Q') \leq \mathcal{E}^r(Q)$$

for all $Q, Q' \in L^2(\mathcal{F}_T)$.

Proof. As noted in [7], this is a simple consequence of the fact that $\mathcal{E}^{-r}(Q) = -\mathcal{E}^r(-Q)$. $\square$

Lemma 13. *If $\mathcal{E}$ is dominated by $\mathcal{E}^r$ for some $r \in \mathfrak{D}$, then for all $\epsilon > 0, \mathcal{E}$ is a Lipschitz continuous operator on $L^{2\vee(1+\epsilon)}(\mathcal{F}_T)$, in the sense that there exists C_ϵ such that*

$$|\mathcal{E}(Q) - \mathcal{E}(Q')| \leq C_\epsilon \|Q - Q'\|_{1+\epsilon}.$$

Proof. As noted in [7], this is a consequence of Lemmata 8 and 12. $\square$

Remark 6. This lemma is suggestive of the fact that we could extend any such $\mathcal{E}$ to all of $L^p(\mathcal{F}_T)$ for all $p > 1$. As any $Q \in L^p(\mathcal{F}_T)$ can be approximated in $\|\cdot\|_p$-norm by a sequence in $L^2(\mathcal{F}_T)$ (or indeed in $L^\infty(\mathcal{F}_T)$), we can define the expectation of Q simply as the limit of these approximations. These limits are uniquely defined as we have continuity of the expectation in this norm.

Lemma 14 (*See [7]*). *For $\mathcal{E}$ an $\mathcal{E}^r$-dominated, translation invariant $\mathcal{F}$-expectation,*

$$\mathcal{E}^{-r}(Q|\mathcal{F}_t) \leq \mathcal{E}(Q|\mathcal{F}_t) \leq \mathcal{E}^r(Q|\mathcal{F}_t).$$

Lemma 15 (*See [7]*). *Let $\mathcal{E}$ and $\mathcal{E}'$ be two translation invariant $\mathcal{F}$-expectations, both dominated by $\mathcal{E}^r$ for some $r \in \mathfrak{D}$. If*

$$\mathcal{E}(Q) \leq \mathcal{E}'(Q)$$

for all $Q \in L^2(\mathcal{F}_T)$, then

$$\mathcal{E}(Q|\mathcal{F}_t) \leq \mathcal{E}'(Q|\mathcal{F}_t)$$

up to evanescence.

The following lemma and theorem guarantee useful properties of a dominated expectation in our general setting.

Lemma 16. *If $\mathcal{E}$ is dominated by $\mathcal{E}^r$ for some $r \in \mathfrak{D}$, then $\mathcal{E}$ has the monotone convergence property. In particular, $\mathcal{E}^r$ has the monotone convergence property for all $r \in \mathfrak{D}$.*

Proof. Simply take an increasing sequence in $L^2(\mathcal{F}_T)$ with limit in $L^2(\mathcal{F}_T)$, hence by classical dominated convergence we have convergence in $L^2(\mathcal{F}_T)$. The result follows from the continuity established by Lemma 13. $\square$

Theorem 7. *Let $\mathcal{E}$ be an $\mathcal{F}$-expectation dominated by $\mathcal{E}^r$ for some $r \in \mathfrak{D}$. Then an $\mathcal{E}$-martingale $Y \in S^2$ has a càdlàg modification.*

Proof. As Y is an $\mathcal{E}$-martingale, we have that, for any $t \leq T$

$$Y_t = \mathcal{E}(Y_T|\mathcal{F}_t) \leq \mathcal{E}^r(Y_T|\mathcal{F}_t)$$

and so Y is an $\mathcal{E}^r$-submartingale. As $\mathcal{E}^r$ is convex, translation invariant, positively homogeneous and has the monotone convergence property, we can apply Theorem 5 to see that Y almost surely admits left and right limits.

Define the càdlàg process $Y'_t := \lim_{s\downarrow t} Y_s = Y_{t+}$, this limit being almost surely well defined. As we assume the usual conditions, Y' is adapted. For any $t \leq T$, any $A \in \mathcal{F}_t$, we have $Y'_t I_A = \lim_{s\downarrow t} Y_s I_A$, taking the limit in L^2 (which converges as $Y \in S^2$). From Lemma 13, we see that $\mathcal{E}(Y'_t I_A) = \lim_{s\downarrow t} \mathcal{E}(Y_s I_A)$, but also, as Y is an $\mathcal{E}$-martingale,

$$\mathcal{E}(Y_s I_A) = \mathcal{E}(\mathcal{E}(Y_s|\mathcal{F}_t) I_A) = \mathcal{E}(Y_t I_A)$$

and so $Y'_t = Y_t$ almost surely. $\square$

4. Doob–Meyer decomposition for g-expectations

We now show that, for a g-expectation $\mathcal{E}_g$, a Doob–Meyer decomposition holds. The method of proof is based on those in [18] (see also [21]). However, it is complicated by the presence of jumps in μ. We begin with an $\mathcal{E}_g$-supermartingale Y with $E[\sup_t (Y_t)^2] < \infty$. We wish to show that Y can be written in the form

$$Y_t = Y_0 - \int_{]0,t]} g(u, Y_{u-}, Z_u) d\mu_u - A_t + \int_{]t,T]} Z_u \cdot dM_u,$$

for some nondecreasing càdlàg process A with $A_0 = 0$.

Similar to [18], we shall use a sequence of penalised BSDEs. Consider the sequence of BSDEs with terminal values $Y^n_T = Y_T$, and drivers

$$f^n(t, y, z) = g(t, y, z) + n(Y_{t-} - y)^+.$$

The solutions of these BSDEs will be denoted[1] (Y^n, Z^n).

Lemma 17. *The BSDEs with terminal values Y_T and drivers f^n have solutions (Y^n, Z^n), which satisfy*

$$\mathcal{E}(Y_T|\mathcal{F}_t) = Y^0_t \leq Y^n_t \leq Y^{n+1}_t \leq Y_t$$

and $Y^n_t \uparrow Y_t$ pointwise, up to evanescence. Furthermore $\{Y^n\}$ is a uniformly bounded set in S^2, and $Y^n_{\cdot-} \to Y_{\cdot-}$ in H^2_μ, that is,

$$E\left[\int_{]0,T]} \|Y^n_{t-} - Y_{t-}\|^2 d\mu_t\right] \to 0.$$

Proof. As g is firmly Lipschitz continuous, we have solutions for f^0 by Theorem 2. For $n > 0$, we can apply the same measure change argument as in [4, Theorem 6.1] to assume without loss of generality that the Lipschitz constant of g with respect to y satisfies $c_t \Delta\mu_t < 1 - \epsilon$ for some

[1] Note that this is a slight abuse of notation, as Z^n here refers not to the nth component of Z, but the $\mathbb{R}^\infty$ valued process which solves the BSDE with driver f^n. We shall not need to refer to individual components of Z hereafter, and so this should not lead to confusion.

$\epsilon > 0$, and furthermore, $c > \epsilon^{-1} - 1$. Hence we see that f^n satisfies the requirements for Theorem 4. Therefore these equations have solutions (Y^n, Z^n).

By the comparison theorem (noting that f^n is balanced as g is balanced), we can see that Y^n_t is nondecreasing in n for all t, and that $Y^0_t = \mathcal{E}(Y_T|\mathcal{F}_t)$. Also if $Y^n_t > Y_t$, then by right continuity this must hold on some optional interval $]\sigma, \tau]$, with $Y_\tau \geq Y^n_\tau$. However, on $]\sigma, \tau]$, $Y^n_t = \mathcal{E}_g(Y^n_\tau|\mathcal{F}_t) \leq \mathcal{E}(Y_\tau|\mathcal{F}_t) \leq Y_t$ leading to a contradiction. Hence $Y^n_t \leq Y_t$ for all n, and all t. Therefore we have, for all n and all t,

$$\mathcal{E}(Y_T|\mathcal{F}_t) = Y^0_t \leq Y^n_t \leq Y^{n+1}_t \leq Y_t.$$

Furthermore, suppose for some $\epsilon > 0$, on some optional set A nonempty with positive probability, we had $Y^n_t < Y_t - \epsilon$ for all n, all $t \in A$. Then $E[\int_{]0,T]} n(Y_{t-} - Y^n_{t-})^+ d\mu_t] \to \infty$, hence $Y^n_0 \to \infty$, which is a contradiction. Therefore, by continuity, $Y^n_t \uparrow Y_t$ except possibly on an evanescent set. By the dominated convergence theorem, it follows that Y^n is a uniformly bounded set in S^2, and $Y^n_{\cdot-} \to Y_{\cdot-}$ in H^2_μ. $\square$

Lemma 18. *Let $A^n_t = n\int_{]0,t]}(Y_{s-} - Y^n_{s-})^+ d\mu_s$. Then there exists a constant C independent of n such that $E\left[\int_{]0,T]}(Z^n_t)^2 d\langle M\rangle_t\right] < C$ and $E[(A^n_T)^2] < C$.*

Proof. From Ito's formula applied to $(Y^n)^2$, we see that,

$$\begin{aligned}
&E[(Y^n_t)^2] + E\left[\int_{]t,T]}(Z^n_u)^2 \cdot d\langle M\rangle_u\right] + E\left[\sum_{u\in]t,T]}(g(u, Y^n_{u-}, Z^n_u)\Delta\mu_u + \Delta A^n_u)^2\right]\\
&\quad = E[Y_T^2] + 2E\left[\int_{]t,T]} Y^n_{u-}(g(u, Y^n_{u-}, Z^n_u)d\mu_u + dA^n_u)\right]
\end{aligned}$$

and hence,

$$\begin{aligned}
E\left[\int_{]t,T]}(Z^n_u)^2 \cdot d\langle M\rangle_u\right] &\leq E[Y_T^2] + 2E\left[\int_{]t,T]} Y^n_{u-}g(u, Y^n_{u-}, Z^n_u)d\mu_u\right]\\
&\quad + 2E\left[\int_{]t,T]} Y^n_{u-}(dA^n_u)\right]. \qquad (8)
\end{aligned}$$

For c the Lipschitz constant of g, we also have

$$\begin{aligned}
&2E\left[\int_{]t,T]} Y^n_{u-}g(u, Y^n_{u-}, Z^n_u)d\mu_u\right]\\
&\quad \leq 4cE\left[\int_{]t,T]}(Y^n_{u-})^2 d\mu_u\right] + (4c^{-1})E\left[\int_{]t,T]}(g(u, Y^n_{u-}, Z^n_u))^2 d\mu_u\right]\\
&\quad \leq 4cE\left[\int_{]t,T]}(Y^n_{u-})^2 d\mu_u\right]\\
&\qquad + (4c^{-1})E\left[\int_{]t,T]}(c(Y^n_{u-})^2 + c\|Z^n_u\|^2_{M_u} + g(u, 0, \mathbf{0})^2)d\mu_u\right] \qquad (9)
\end{aligned}$$

and

$$\begin{aligned}
2E\left[\int_{]t,T]} Y^n_{u-}(dA^n_u)\right] &\leq 2E[A^n_T(\sup_u |Y^n_{u-}|)] \leq 2E[\sup_u(Y^n_{u-})^2]^{1/2}E[(A^n_T)^2]^{1/2}\\
&\leq (16c\mu_T + 8)E[\sup_u(Y^n_{u-})^2] + (16c\mu_T + 8)^{-1}E[(A^n_T)^2]. \qquad (10)
\end{aligned}$$

As $(Y^n)^2 \leq (Y^0)^2 + Y^2 \in S^2$ and $E\left[\int_{]t,T]} \|Z^n_u\|^2_{M_u} d\mu_u\right] \leq E\left[\int_{]t,T]} (Z^n_u)^2 \cdot d\langle M\rangle_u\right]$, combining (8)–(10), it follows that there is a constant C_1 independent of n such that

$$E\left[\int_{]t,T]} (Z^n_u)^2 \cdot d\langle M\rangle_u\right] \leq C_1 + (8c\mu_T + 4)^{-1} E[(A^n_T)^2]. \tag{11}$$

Furthermore, we also have

$$\begin{aligned} A^n_T &= Y^n_0 - Y^n_T - \int_{]0,T]} g(u, Y^n_{u-}, Z^n_u) d\mu_u + \int_{]0,T]} Z^n_u \cdot dM_u \\ &\leq |Y_0| + |Y_T| + \int_{]0,T]} |g(u, Y^n_{u-}, Z^n_u)| d\mu_u + \left|\int_{]0,T]} Z^n_u \cdot dM_u\right| \end{aligned}$$

from which, expanding $(g(u, Y^n_{u-}, Z^n_u))^2$ as in (9), it follows that there exists a constant C_2 independent of n such that

$$\begin{aligned} &E[(A^n_T)^2] \\ &\quad \leq 4E[(|Y_0| + |Y_T|)^2] + 4\mu_T E\left[\int_{]0,T]} (g(u, Y^n_{u-}, Z^n_u))^2 d\mu_u\right] \\ &\qquad + 2E\left[\left(\int_{]0,T]} Z^n_u \cdot dM_u\right)^2\right] \\ &\quad \leq C_2 + (4c\mu_T + 2) E\left[\int_{]t,T]} (Z^n_u)^2 \cdot d\langle M\rangle_u\right]. \end{aligned} \tag{12}$$

Combining (11) and (12) yields the result. □

We can now prove the convergence of our solutions. Unlike in [18,21], due to the use of left-limits in the BSDE, we are able to prove the strong convergence of Z^n in L^2, rather than only in L^p for $p < 2$.

Theorem 8. *A càdlàg $\mathcal{E}_g$-supermartingale Y has a unique representation of the form*

$$Y_t = Y_0 - \int_{]0,t]} g(u, Y_{u-}, Z_u) d\mu_u - A_t + \int_{]0,t]} Z_u \cdot dM_u,$$

where Z is a process in H^2_M and A is a predictable càdlàg nondecreasing process.

Proof. By Lemma 18, we know that $\{Z^n\}_{n\in\mathbb{N}}$ is weakly compact in H^2_M, and, defining $g^n_t := g(t, Y^n_{t-}, Z^n_t)$, we see $\{g^n\}_{n\in\mathbb{N}}$ is bounded and hence weakly compact in H^2_μ. Therefore, by extracting subsequences, we have the existence of weak limits $Z^n \rightharpoonup Z$ and $g^n \rightharpoonup g^\infty$. For any stopping time $\tau \leq T$, we also then have the weak convergence of the integrals $\int_{]0,\tau]} Z^n_u \cdot dM_u$ and $\int_{]0,\tau]} g^n_u d\mu_u$ in $L^2(\mathcal{F}_T)$. As

$$A^n_t = Y^n_0 - Y^n_t - \int_{]0,t]} g^n_u d\mu_u + \int_{]0,t]} Z^n_u \cdot dM_u$$

we also have the existence of a weak L^2-limit

$$A^n_t \rightharpoonup A_t = Y_0 - Y_t - \int_{]0,t]} g^\infty_u d\mu_u + \int_{]0,t]} Z_u \cdot dM_u$$

and clearly, A is a nondecreasing process with $A_T \in L^2(\mathcal{F}_T)$. By a result of Peng [18, Lemma 2.2], A is càdlàg. As Y is given, we see that Z is uniquely defined, and hence the sequence $\{Z^n\}$ (rather than a subsequence) must weakly converge.

We now write $\delta_n Y = Y - Y^n$, $\delta_n Z = Z - Z^n$, $\delta_n g = g^\infty - g^n$ and $\delta_n A = A - A^n$. Considering the dynamics of $(\delta_n Y)^2$, from Itô's formula we have

$$\begin{aligned} 0 &= E[(\delta_n Y)_T^2] \\ &= E[(\delta_n Y)_0^2] - 2E\left[\int_{]0,T]} (\delta_n Y)_{u-}((\delta_n g)_u d\mu_u + d(\delta_n A)_u)\right] \\ &\quad + E\left[\int_{]0,T]} (\delta_n Z)_u^2 \cdot d\langle M\rangle_u\right] + E\left[\sum_{u\in]0,T]} ((\delta_n g)_u \Delta\mu_u + \Delta(\delta_n A)_u)^2\right] \end{aligned}$$

from which we obtain

$$E\left[\int_{]0,T]} (\delta_n Z)_u^2 \cdot d\langle M\rangle_u\right] \le 2E\left[\int_{]0,T]} (\delta_n Y)_{u-}((\delta_n g)_u d\mu_u + d(\delta_n A)_u)\right].$$

We then see that, by the Cauchy–Schwartz inequality, for C a bound on the norms of $\delta_n g$ in H_μ^2,

$$\begin{aligned} E\left[\int_{]0,T]} (\delta_n Y)_{u-}(\delta_n g)_u d\mu\right] &\le E\left[\int_{]0,T]} (\delta_n Y)_{u-}^2\right]^{1/2} E\left[\int_{]0,T]} (\delta_n g)_u^2 d\mu_u\right]^{1/2} \\ &\le C\, E\left[\int_{]0,T]} (\delta_n Y)_{u-}^2\right]^{1/2} \\ &\to 0. \end{aligned}$$

Also

$$\begin{aligned} E\left[\int_{]0,T]} (\delta_n Y)_{u-} d(\delta_n A)_u\right] &= E\left[\int_{]0,T]} (\delta_n Y)_{u-} dA_u\right] - E\left[\int_{]0,T]} (\delta_n Y)_{u-} dA_u^n\right] \\ &\le E\left[\int_{]0,T]} (\delta_n Y)_{u-} dA_u\right] \\ &\le E[A_T \sup_u\{(\delta_0 Y)_u\}] \\ &\le E[A_T^2] + E[\sup_u\{(\delta_0 Y)_u^2\}] < \infty \end{aligned}$$

and so, by the Dominated convergence theorem,

$$E\left[\int_{]0,T]} (\delta_n Y)_{u-} d(\delta_n A)_u\right] \le E\left[\int_{]0,T]} (\delta_n Y)_{u-} dA_u\right] \to 0.$$

Hence we see that,

$$E\left[\int_{]0,T]} (\delta_n Z)_u^2 \cdot d\langle M\rangle_u\right] \to 0.$$

Given this strong convergence, it is clear that $g^n \to g^\infty$ strongly in H_μ^2, and that $g_t^\infty = g(t, Y_{t-}, Z_t)$ $\mathbb{P} \times \mu$-a.e., yielding the desired representation.

To show uniqueness, suppose Y had two representations of this form. That is, suppose there were processes $Z^1, Z^2 \in H^2_M$ and A^1, A^2 predictable càdlàg and nondecreasing such that

$$\begin{aligned} Y_t &= Y_0 - \int_{]0,t]} g(u, Y_{u-}, Z^1_u) d\mu_u - A^1_t + \int_{]0,t]} Z^1_u \cdot dM_u \\ &= Y_0 - \int_{]0,t]} g(u, Y_{u-}, Z^2_u) d\mu_u - A^2_t + \int_{]0,t]} Z^2_u \cdot dM_u. \end{aligned}$$

As Y_t is a special semimartingale, its martingale part is uniquely defined, so we see $\int_{]0,t]} Z^1_u \cdot dM_u = \int_{]0,t]} Z^2_u \cdot dM_u$ up to evanescence, that is, $Z^1 = Z^2$ in H^2_M. Consequently, $g(u, Y_{u-}, Z^1_u) = g(u, Y_{u-}, Z^2_u)\mu \times \mathbb{P}$-a.e., and we have the equality $A^1_t = A^2_t$, so the decomposition is unique. □

To compare this with the classical Doob–Meyer decomposition, we have the following corollary.

Corollary 1. *Consider $\mathcal{E}_g$ a g-expectation, where $g(u, z)$ does not depend on y (and, hence, $\mathcal{E}_g$ is translation invariant). Then a càdlàg $\mathcal{E}_g$-supermartingale Y in S^2 has a unique decomposition $Y = Y_0 + X - A$, where A is a nondecreasing adapted càdlàg process with $A_T \in L^2(\mathcal{F}_T)$, and X is a càdlàg $\mathcal{E}_g$-martingale in S^2 with $X_0 = 0$.*

Proof. From Theorem 8, we have the representation

$$Y_t = Y_0 - \int_{]0,t]} g(u, Z_u) d\mu_u - A_t + \int_{]0,t]} Z_u \cdot dM_u,$$

and note that

$$\begin{aligned} X_t &= - \int_{]0,t]} g(u, Z_u) d\mu_u + \int_{]t,T]} Z_u \cdot dM_u \\ &= X_T + \int_{]t,T]} g(u, Z_u) d\mu_u - \int_{]t,T]} Z_u \cdot dM_u = \mathcal{E}_g(X_T|\mathcal{F}_t) \end{aligned}$$

is a g-martingale. □

We can now show that $\mathcal{E}^r$-domination implies that the drift (under $\mathbb{P}$) of any $\mathcal{E}$-martingale must be μ-absolutely continuous.

Theorem 9. *Let $\mathcal{E}$ be an $\mathcal{F}$-expectation, $\mathcal{E}^r$-dominated for some $r \in \mathfrak{D}$. Let Y be a càdlàg $\mathcal{E}$-martingale. Then there exist unique predictable processes $g \in H^2_\mu$, $Z \in H^2_M$ such that*

$$Y_T = Y_t - \int_{]t,T]} g_u d\mu_u + \int_{]t,T]} Z_u \cdot dM_u$$

up to indistinguishability. These processes satisfy $|g_u| \leq \|r_u Z_u\|_{M_u}$.

Proof. As $\mathcal{E}$ is $\mathcal{E}^r$-dominated, we know that

$$\mathcal{E}^{-r}(Y_T|\mathcal{F}_t) \leq Y_t \leq \mathcal{E}^r(Y_T|\mathcal{F}_t) = -\mathcal{E}^{-r}(-Y_T|\mathcal{F}_t),$$

and so both Y and $-Y$ are $\mathcal{E}^{-r}$-supermartingales. From the nonlinear Doob–Meyer decomposition (Theorem 8), we can find nondecreasing càdlàg processes A^r, A^{-r} and processes

$Z^r, Z^{-r} \in H^2_M$ such that

$$\begin{aligned} Y_t &= Y_0 + \int_{]0,t]} \|r_u Z_u^{-r}\|_{M_u} d\mu_u + \int_{]0,t]} Z_u^{-r} \cdot dM_u - A_t^{-r} \\ -Y_t &= -Y_0 + \int_{]0,t]} \|r_u Z_u^{r}\|_{M_u} d\mu_u + \int_{]0,t]} Z_u^{r} \cdot dM_u - A_t^{r}. \end{aligned} \tag{13}$$

As Y is a special semimartingale, its canonical decomposition (into martingale and predictable finite-variation components) is unique (see [13, Def 4.22]). Hence we have $\int_{]0,t]} Z_u^{-r} \cdot dM_u = -\int_{]0,t]} Z_u^r \cdot dM_u$ up to indistinguishability, and furthermore $Z^{-r} = -Z^r$ in H^2_M. Taking the sum of the two equations in (13), we have

$$0 = 2\int_{]0,t]} \|r_u Z_u^r\|_{M_u} d\mu_u - A_t^{-r} - A_t^r.$$

Differentiating yields

$$d(A^r + A^{-r})_u = 2\|r_u Z_u^r\|_{M_u} d\mu_u$$

and, as both A^r and A^{-r} are nondecreasing, we see that they are both absolutely continuous with respect to μ. Therefore, as $A_T^{-r} \in L^2(\mathcal{F}_T)$, we can write $dA_t^{-r} = a_t^{-r} d\mu$ for some $a^{-r} \in H^2_\mu$. Defining $g_u := -\|r Z_u^{-r}\|_{M_u} + a_u^{-r}$, we have

$$Y_t = Y_0 - \int_{]0,t]} g_u d\mu_u + \int_{]0,t]} Z_u^{-r} dM_u.$$

This g is unique among predictable processes in H^2_μ, again by the uniqueness of the canonical decomposition of a special semimartingale. Furthermore, as A^{-r} and A^r are nondecreasing, we have that $0 \le a^{-r} \le 2\|r_u Z_u^r\|_{M_u}$, and so $|g_u| \le \|r_u Z_u\|_{M_u}$. □

Theorem 10. *Let $\mathcal{E}$ be as in Theorem 9, and Y and Y' be two càdlàg $\mathcal{E}$-martingales, with associated processes g, g' and Z, Z'. Then*

$$|g_t - g_t'| \le \|r_t(Z_t - Z_t')\|_{M_t}$$

up to evanescence.

Proof. As all of $Y, -Y, Y'$ and $-Y'$ are $\mathcal{E}^{-r}$-supermartingales, by Lemma 6 we know that $\delta Y := Y - Y'$ and $-\delta Y$ are both $\mathcal{E}^{-r}$-supermartingales. By precisely the same argument as in Theorem 9, we can find predictable processes $g^\delta \in H^2_\mu, Z^\delta \in H^2_M$ such that

$$\delta Y_t = \delta Y_0 - \int_{]0,t]} g_u^\delta d\mu_u + \int_{]0,t]} Z_u^\delta \cdot dM_u$$

and $|g_t^\delta| \le \|r_t Z_t^\delta\|_{M_t}$ up to evanescence. However, we also have

$$\delta Y_t = \delta Y_0 - \int_{]0,t]} (g_u - g_u') d\mu_u + \int_{]0,t]} (Z_u - Z_u') \cdot dM_u$$

and uniqueness of the canonical decomposition of δY_t yields

$$|g_t - g_t'| = |g_t^\delta| \le \|r_t Z_t^\delta\|_{M_t} = \|r_t(Z_t - Z_t')\|_{M_t}. \quad \square$$

5. $\mathcal{E}^r$-dominated Doob–Meyer decomposition

We shall need to extend our decomposition to the case where $\mathcal{E}$ is $\mathcal{E}^r$-dominated for some $r \in \mathfrak{D}$, but where we do not know *a priori* that it is a g-expectation.

We need the following generalisation of our existence result. A more general result than this is possible (where $n(Y_{t-} - y_{t-})^+$ is replaced by an appropriately Lipschitz function with sufficiently bounded upward jumps). Obtaining this extension directly is unnecessary given the representation we shall prove further on (Theorem 13), which implies these results are equivalently given by Theorem 8.

Theorem 11. *Consider $\mathcal{E}$ any translation invariant $\mathcal{F}$-expectation, $\mathcal{E}^r$-dominated for some $r \in \mathfrak{D}$. For any $Q \in L^2(\mathcal{F}_T)$, any càdlàg $\mathcal{E}$-supermartingale Y in H^2_μ with $Y_T = Q$, the equation*

$$Y_t^n = \mathcal{E}\left(Q + n\int_{]t,T]} (Y_{u-} - Y_{u-}^n)^+ d\mu_u \middle| \mathcal{F}_t\right)$$

has a unique càdlàg solution in H^2_μ.

Proof. Our approach is similar to that in Theorem 4. For any $s < t$, any $Q' \in L^2(\mathcal{F}_{t-})$, define a mapping

$$\Phi^{Q'}_{]s,t[} : H^2_\mu \to H^2_\mu, \qquad y \mapsto \mathcal{E}\left(Q' + n\int_{]s,t[} (Y_{u-} - y_{u-})^+ d\mu_u \middle| \mathcal{F}_s\right).$$

For any two approximations $y, y' \in H^2_\mu$, define $\delta y = y - y'$ and $\delta\Phi(y) = \Phi^{Q'}_{]s,t[}(y) - \Phi^{Q'}_{]s,t[}(y')$. Then as $\mathcal{E}$ is $\mathcal{E}^r$-dominated, and r is assumed to be bounded (as it is uniformly balanced), it is easy to show (see [7, Lemma 6.1] and use Lemma 11)

$$\begin{aligned} E[(\delta\Phi(y))^2] &\le E\left[\left(n\mathcal{E}^r\left(\int_{]s,t[} |\delta y|_u d\mu_u \middle| \mathcal{F}_s\right)\right)^2\right] \\ &\le n^2 e^{(\sup_u \|r_u\|^2_{D_u})(\mu_{t-} - \mu_s)} E\left[\left(\int_{]s,t[} |\delta y|_u d\mu_u\right)^2\right] \\ &\le n^2 e^{\sup_u \|r_u\|^2_{D_u}\mu_T} (\mu_{t-} - \mu_s) E\left[\int_{]s,t[} |\delta y|^2_u d\mu_u\right]. \end{aligned}$$

As μ is summable, using the result of [4, Lemma 6.1], we can find a finite set $\{0 = t_1, t_2, \ldots, t_m = T\}$ where $n^2 e^{\sup_u \|r_u\|^2_{D_u}\mu_T}(\mu_{t_{i+1}-} - \mu_{t_i}) < 1$ for all i. Hence we have a contraction on each of the subintervals $]t_i, t_{i+1}[$. Therefore, for any $Y^n_{t_{i+1}-} = Q' \in L^2(\mathcal{F}_{t_{i+1}-})$, we can solve our equation uniquely back to time t_i.

At each t_i, we shall solve the equation directly. Suppose we have a solution Y_u^n for all $u \ge t_i$. In particular, we have the value $Y^n_{t_i} \in L^2(\mathcal{F}_{t_i})$. Then we have the equation

$$Y^n_{t_i-} = \mathcal{E}\left(Y_{t_i} + n(Y_{t_i-} - Y^n_{t_i-})^+ \Delta\mu_{t_i} | \mathcal{F}_{t_i-}\right)$$

which, by translation invariance of $\mathcal{E}$, gives

$$Y^n_{t_i-} = \left(\frac{1}{1 + n\Delta\mu_{t_i}} \mathcal{E}\left(Y^n_{t_i} + n\Delta\mu_{t_i} Y_{t_i-} | \mathcal{F}_{t_i-}\right)\right) \wedge \mathcal{E}(Y^n_{t_i}|\mathcal{F}_{t_i-}).$$

Note as $n\Delta\mu_{t_i} > 0$, $Y^n_{t_i-}$ is clearly in $L^2(\mathcal{F}_{t-})$. Therefore, at each time t_i, we can take our solution $Y^n_{t_i} \in L^2(\mathcal{F}_{t_i})$, and obtain a unique value $Y^n_{t_i-} \in L^2(\mathcal{F}_{t_i-})$.

Using backward induction and alternating between the contraction mapping approach and the direct approach yields a unique solution. It is then straightforward to verify (as in Theorem 4) that this solution is càdlàg and in H^2_μ. □

Lemma 19. *For Y, Y^n as in Theorem 11,*

$$\mathcal{E}(Q|\mathcal{F}_t) = Y^0 \leq Y^n \leq Y^{n+1} \leq Y.$$

Proof. That $Y^n \geq Y^0 = \mathcal{E}(Q|\mathcal{F}_t)$ is easy from the monotonicity of $\mathcal{E}$.

Suppose $Y^n_t > Y^{n+1}_t$ with positive probability. By right continuity, there exists an optional interval $A =]\sigma, \tau]$, nonempty with positive probability, such that $Y^n_t \geq Y^{n+1}_t$ on $]\sigma, \tau[$ and $Y^n_\tau \geq Y^{n+1}_\tau$. On A, note that $(Y - Y^n)^+ \leq (Y - Y^{n+1})^+$, and hence for any $t \in A$,

$$\begin{aligned} I_A Y^n_t &= \mathcal{E}\left(I_A Y^n_\tau + \int_{]t,\tau]} n I_A (Y_{u-} - Y^n_{u-})^+ d\mu_u \,\middle|\, \mathcal{F}_t \right) \\ &\leq \mathcal{E}\left(I_A Y^{n+1}_\tau + \int_{]t,\tau]} (n+1) I_A (Y_{u-} - Y^{n+1}_{u-})^+ d\mu_u \,\middle|\, \mathcal{F}_t \right) \\ &= I_A Y^{n+1}_t \end{aligned}$$

which gives a contradiction. Hence $Y^n \leq Y^{n+1}$. A similar argument applies with Y^{n+1} replaced by Y. □

Lemma 20. *For Y^n as in Theorem 11, Y^n has a representation*

$$Y^n_t = Y^n_0 - \int_{]0,t]} g^n_u d\mu_u - A^n_t + \int_{]0,t]} Z^n_u \cdot dM_u$$

for some $g^n \in H^2_\mu$, $Z^n \in H^2_M$ and A^n_t nondecreasing, predictable and càdlàg with $A_0 = 0$ and $A_T \in L^2(\mathcal{F}_T)$. Furthermore, $|g^n_u| \leq \|r_u Z^n_u\|_{M_u}$, and there exists a constant C independent of n such that $E[(A^n_T)^2] < C$ and $E[\int_{]0,T]} (Z^n)^2_u \cdot d\langle M\rangle_u] < C$.

Proof. Define $A^n_t = \int_{]0,t]} n(Y_u - Y^n_u)^+ d\mu_u$. As $Y^n + \int_{]0,t]} n(Y_u - Y^n_u)^+ d\mu_u$ is a $\mathcal{E}$-martingale, we have from Theorem 9 the existence of g^n and Z^n with the required inequality between them.

For the required bound on $E[(A^n_T)^2]$ and $E[\int_{]0,T]} (Z^n)^2_u \cdot d\langle M\rangle_u]$, as $|g^n_t| < \|r_t Z^n_t\|_{M_t}$, where $r \in \mathfrak{D}$, we can precisely repeat the argument of Lemma 18. □

Theorem 12. *Let $\mathcal{E}$ be a translation invariant $\mathcal{F}$-expectation, which is $\mathcal{E}^r$-dominated for some r. A càdlàg $\mathcal{E}$-supermartingale Y has a representation of the form*

$$Y_t + A_t = \mathcal{E}(Y_T + A_T|\mathcal{F}_t)$$

where A is a nondecreasing, predictable and càdlàg process with $A_T \in L^2(\mathcal{F}_T)$.

Proof. As in the proof of Theorem 8, we see that the A^n and Z^n terms constructed in Lemma 20 are uniformly bounded, and so must weakly converge. As $|g^n_u| \leq \|r_u Z^n_u\|_{M_u}$, we can again see that the argument of Theorem 8 will hold, and so Z^n converges strongly in H^2_M. Therefore g^n converges strongly in H^2_μ, and hence A^n_t converges strongly in $L^2(\mathcal{F}_t)$. By Lemma 13, we can pass to the L^2-limit in the equation $Y^n_t + A^n_t = \mathcal{E}(Y^n_T + A^n_T|\mathcal{F}_t)$, and the theorem is proven. □

Remark 7. Unlike in Theorem 8, we have not here shown that the representation is unique. This is not a cause for concern, as uniqueness will follow from Theorem 13, from which we see that the processes A constructed in Theorem 12 and Corollary 1 are identical.

6. Representation as a g-expectation

We can now prove our main result, that any translation invariant $\mathcal{F}$-expectation which is $\mathcal{E}^r$-dominated for some $r \in \mathfrak{D}$, must be a g-expectation.

Theorem 13. *Consider a translation invariant $\mathcal{F}$-expectation $\mathcal{E}$, which is $\mathcal{E}^r$-dominated for some $r \in \mathfrak{D}$. Then there exists a unique function $g : \Omega \times [0, T] \times \mathbb{R}^\infty \to \mathbb{R}$ satisfying $E[\int_{]0,T]}(g(t, \mathbf{0}))^2 d\mu_t] < \infty$ such that*

$$\mathcal{E}(Q) = \mathcal{E}_g(Q)$$

for all $Q \in L^2(\mathcal{F}_T)$. Furthermore, $g(t, \mathbf{0}) = 0$ for μ-almost all t and g is uniformly balanced (and hence Lipschitz).

Proof. For each $z \in \mathbb{R}^\infty$, we consider the forward equation

$$dY_t^z = -\|r_t z\|_{M_t} d\mu_t + z \cdot dM_t; \qquad Y_0^z = 0.$$

We then see that Y^z is an $\mathcal{E}^r$-martingale, and hence an $\mathcal{E}$-supermartingale.

From Theorem 12, there exists a nondecreasing, predictable and càdlàg process A^z with $A_0^z = 0$ and $A_T^z \in L^2(\mathcal{F}_T)$ such that

$$Y_t^z + A_t^z = \mathcal{E}(Y_T^z + A_T^z|\mathcal{F}_t). \tag{14}$$

By Theorem 9, given A^z, there is a unique $g(z; \cdot) : \Omega \times [0, T] \to \mathbb{R}$ predictable such that

$$Y_t^z + A_t^z = Y_T^z + A_T^z + \int_{]t,T]} g(z; u) d\mu_u - \int_{]t,T]} Z_u^z \cdot dM_u$$

and $|g(z; t)| \leq \|r_t Z^z\|_{M_t}$.

As we also know

$$Y_t^z = Y_T^z + \int_{]t,T]} \|r_u z\|_{M_u} d\mu_u - \int_{]t,T]} z \cdot dM_u$$

we see that

$$A_t^z \equiv \|r_t z\|_{M_t} - \int_{]0,t]} g(z; u) d\mu_u, \quad Z^z \equiv z.$$

In particular, this implies $|g(z; t)| \leq \|r_t z\|_{M_t}$. From Theorem 10, we also see that for any $z, z' \in \mathbb{R}^\infty$, $|g(z; t) - g(z'; t)| \leq \|r_t(z - z')\|_{M_t}$. Hence, for each t, $g(\cdot; t)$ is uniformly Lipschitz continuous and uniformly balanced, as a function of z.

We can see that, for any $0 \leq r \leq t \leq T$,

$$Y_t^z + A_t^z = Y_r^z + A_r^z - \int_{]r,t]} g(z; u) d\mu_u + \int_{]r,t]} z \cdot dM_u.$$

Because of translation invariance, we have

$$\mathcal{E}\left(-\int_{]r,t]} g(z; u) d\mu_u + \int_{]r,t]} z \cdot dM_u \middle| \mathcal{F}_r\right) = 0.$$

Let $\{A_i\}_{i=1}^N \subset \mathcal{F}_r$ be a partition of Ω, and let $z_i \in \mathbb{R}^\infty$. From Lemma 5, and the fact $g(0,t) \equiv 0$, it follows that

$$\mathcal{E}\left(-\int_{]r,t]} g\left(\sum_i I_{A_i} z_i; u\right) d\mu_u + \int_{]r,t]} \left(\sum_i I_{A_i} z_i\right) \cdot dM_u \middle| \mathcal{F}_r\right) = 0.$$

Hence, by the continuity of $\mathcal{E}$ given in Lemma 13 and the fact that g is Lipschitz in z, we have, for any $Z \in H_M^2$,

$$\mathcal{E}\left(-\int_{]r,t]} g\left(Z_u; u\right) d\mu_u + \int_{]r,t]} Z_u \cdot dM_u \middle| \mathcal{F}_r\right) = 0.$$

For any $Q \in L^2(\mathcal{F}_T)$, now solve the BSDE with driver g. As g is Lipschitz, this has a unique solution (Y, Z), and by the definition of g-expectation, $\mathcal{E}_g(Q) = Y_0$. On the other hand, we also have

$$\begin{aligned}\mathcal{E}(Q) &= \mathcal{E}\left(Y_0 - \int_{]0,T]} g\left(Z_u; u\right) d\mu_u + \int_{]0,T]} Z_u \cdot dM_u\right)\\ &= Y_0 + \mathcal{E}\left(-\int_{]0,T]} g\left(Z_u; u\right) d\mu_u + \int_{]0,T]} Z_u \cdot dM_u\right) = Y_0\end{aligned}$$

and so $\mathcal{E}_g(Q) = Y_0 = \mathcal{E}(Q)$ for all $Q \in L^2(\mathcal{F}_T)$.

To show uniqueness of the representation, we note that as we now know that $\mathcal{E}$ has *some* representation as a g-expectation, we know from Corollary 1 that the process A^z which satisfies (14) is unique. Hence Theorem 9 guarantees the uniqueness of the process g^z, and hence of the driver of the BSDE. □

7. Conclusion

We have extended the results of [7,18] to a general setting. This directly answers the question raised by Remark 7.1 of [7]; we have given a nonlinear Doob–Meyer decomposition theorem for g-expectations, and have shown that every $\mathcal{F}$-expectation satisfying a dominance relation can be expressed as a g-expectation. Our only assumption on the probability space is that $L^2(\mathcal{F}_T)$ is separable.

The exact nature of this dominance relation is quite interesting in this context. One can think of the dominance relation in [7] as being needed to guarantee that the induced driver of the BSDE exists, and is Lipschitz continuous. Our assumption guarantees both these properties, and furthermore that the driver can be integrated with respect to the (arbitrary) Stieltjes measure μ, and that it satisfies the conditions to be uniformly balanced, and so a comparison theorem will hold. Neither of these properties appears in [7], as in they assume that μ is always Lebesgue measure (a reasonable assumption, as all martingales have absolutely continuous quadratic variation), and all martingales are continuous (so the comparison theorem holds automatically). However, if our filtration is generated by finitely many Brownian motions, as in [7], then our result corresponds precisely to theirs. Furthermore, our result will also encompass the case of a filtration generated by countably many independent Brownian motions.

As $\mathfrak{D}$ contains a wide range of processes, our assumption that $\mathcal{E}$ is $\mathcal{E}^r$-dominated for some $r \in \mathfrak{D}$ has particular implications for those cases where the BSDE can be written in the form (c.f. [10])

$$dY_t = -g(t, Z_t)d\mu_t + Z_t dM_t' + dN_t$$

for some finite-dimensional martingale M', where N is a martingale orthogonal to M'. From the perspective of the Davis–Varaiya martingale representation theorem, this means that the BSDE driver looks only at a finite dimensional subspace of the space of S^2-martingales. Looking from the perspective of the $\mathcal{F}$-expectation, this is equivalent to stating that $\mathcal{E}(Q+N_T) = \mathcal{E}(Q)$ for any Q and any $\mathbb{P}$-martingale N orthogonal to M' with $N_0 = 0$. In this context, if $\mathcal{E}$ is $\mathcal{E}^r$-dominated for some $r \in \mathfrak{D}$, we can find a *degenerate* matrix $r' \in \mathfrak{D}$ such that $\mathcal{E}$ is $\mathcal{E}^{r'}$-dominated, and the representation will follow.

If we compare our results with the Lévy case considered by Royer [21], we see that our condition 'g is uniformly balanced' is equivalent to her 'assumption A_γ'. Royer shows that assumption A_γ is satisfied by the BSDEs generated by nonlinear expectations, and we similarly show that the induced g is uniformly balanced.

If we compare with earlier results in discrete time [3,5], we see that we have again shown an equivalence between BSDE solutions and translation invariant nonlinear expectations. Unlike in discrete time, we require the further assumption of $\mathcal{E}^r$-domination to ensure that the continuous-time generator is adequately Lipschitz continuous, and so our results lack the complete generality of those in discrete time.

Further work on this area may allow us to extend away from the assumption of translation invariance (see [5] in discrete time), and towards quadratic BSDEs (see [12] in the Brownian case). A further extension would also be to allow μ to be a stochastic finite-variation process. These results will require further extension of the existence results of BSDEs in general filtrations.

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