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Simple vertex operator algebras are nondegenerate

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Abstract

In this paper it is shown that every irreducible vertex algebra of countable dimension is nondegenerate in the sense of Etingof and Kazhdan and that every simple vertex operator algebra is nondegenerate.

1. Introduction

In one of their series papers on quantization of Lie bialgebras [6], Etingof and Kazhdan introduced and studied certain fundamental notions of braided vertex operator algebra and quantum vertex operator algebra. While studying the axiomatic properties of braided (and quantum) vertex operator algebras, they introduced a notion of nondegeneracy of a vertex operator algebra. A vertex operator algebra V is said to be nondegenerate if for every positive integer n the linear map Z_n from $V^{\otimes n} \otimes \mathbb{C}((z_1))\cdots((z_n))$ to $V((z_1))\cdots((z_n))$ defined by

$$Z_n\big(v^{(1)} \otimes \cdots \otimes v^{(n)} \otimes f\big) = f\,Y\big(v^{(1)}, z_1\big)\cdots Y\big(v^{(n)}, z_n\big)\mathbf{1}$$

is injective. It was proved therein that in their definition of the notion of braided vertex operator algebra, if the classical limit vertex operator algebra is nondegenerate, two of

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the main axioms (the quantum Yang–Baxter equation and the unitarity condition) are automatically satisfied, and that in their notion of quantum vertex operator algebra, the main axiom (the hexagon relation) is equivalent to a certain natural associativity property. In view of this, nondegeneracy is very useful in the study of braided (and quantum) vertex operator algebras.

Let $\mathfrak{g}$ be a finite-dimensional Lie algebra equipped with a nondegenerate symmetric invariant bilinear form $\langle\cdot,\cdot\rangle$. Denote by $\hat{\mathfrak{g}}$ the associated affine Lie algebra (without the degree operator added). Associated to the affine Lie algebra $\hat{\mathfrak{g}}$ and a complex number ℓ one has a vertex (operator) algebra $V_{\hat{\mathfrak{g}}}(\ell,0)$, whose underlying vector space is a certain generalized Verma module or Weyl module for the affine Lie algebra $\hat{\mathfrak{g}}$ of level ℓ (cf. [7, 10,16,17]). It was proved in [6] that if $V_{\hat{\mathfrak{g}}}(\ell,0)$ is an irreducible $\hat{\mathfrak{g}}$-module, then $V_{\hat{\mathfrak{g}}}(\ell,0)$ is nondegenerate. Notice that the irreducibility of $V_{\hat{\mathfrak{g}}}(\ell,0)$ as a $\hat{\mathfrak{g}}$-module amounts to the irreducibility of $V_{\hat{\mathfrak{g}}}(\ell,0)$ as a module for the vertex (operator) algebra $V_{\hat{\mathfrak{g}}}(\ell,0)$. We also notice that for a general vertex operator algebra V (equipped with a conformal vector) the irreducibility of the adjoint module V amounts to the simplicity of the vertex operator algebra V. In view of this, one may conjecture that *general* simple vertex operator algebras are nondegenerate. In this paper, we shall prove that this conjecture is indeed true.

In the literature, there are certain results closely related to the injectivity of the maps Z_n. In [4], among other results it was proved that if V is a simple vertex operator algebra and W is an irreducible V-module, then $Y(v,z)w \neq 0$ for $0 \neq v \in V$, $0 \neq w \in W$. Furthermore, among other results it was proved in [5] that the linear map Y viewed as a map from $V \otimes W$ to $W((z))$ is injective. In view of this, one might guess that ideas in [4,5] and [6] would be highly valuable for proving the conjecture. Indeed, part of our proof uses some of their ideas.

Notice that in the notion of vertex operator algebra used in [6], no conformal vector and no $\mathbb{Z}$-grading are assumed. A vertex operator algebra in the sense of [6] is often called a vertex algebra. In this paper we consider a general vertex algebra V. For any V-module W and any positive integer n we define a linear map Z_n^W from $V^{\otimes n} \otimes W \otimes \mathbb{C}((z_1))\cdots((z_n))$ to $W((z_1))\cdots((z_n))$ by

$$Z_n^W\big(v^{(1)} \otimes \cdots \otimes v^{(n)} \otimes w \otimes f\big) = fY\big(v^{(1)},z_1\big)\cdots Y\big(v^{(n)},z_n\big)w.$$

It follows from [8] that $V^{\otimes(n+1)} \otimes \mathbb{C}((z_1))\cdots((z_n))$ is a natural vertex algebra with $V^{\otimes n} \otimes W \otimes \mathbb{C}((z_1))\cdots((z_n))$ as a module. We first prove that the vector subspace $\ker Z_n^W$ of $V^{\otimes n} \otimes W \otimes \mathbb{C}((z_1))\cdots((z_n))$ is a submodule. To describe submodules of $V^{\otimes n} \otimes W \otimes \mathbb{C}((z_1))\cdots((z_n))$ we slightly generalize the result of [8] on the irreducibility of tensor product modules for tensor product vertex operator algebras in the context of vertex algebras of countable dimension. Using this, we show that if V is of countable dimension and irreducible in the sense that V is an irreducible V-module and if W is a V-module, then any submodule of $V^{\otimes n} \otimes W \otimes \mathbb{C}((z_1))\cdots((z_n))$ is of the form $V^{\otimes n} \otimes U$, where U is a $V \otimes \mathbb{C}((z_1))\cdots((z_n))$-submodule of $W \otimes \mathbb{C}((z_1))\cdots((z_n))$. Then it follows that $\ker Z_n^W = 0$. From this we show that every irreducible vertex algebra of countable dimension is nondegenerate. In particular, this implies that every simple vertex operator algebra in the sense of [9] and [8] is nondegenerate.

2. The main results

We here recall the notion of nondegeneracy of a vertex algebra from [6] and prove that every irreducible vertex algebra of countable dimension is nondegenerate and that every simple vertex operator algebra is nondegenerate. In the course of proving our main results we also extend a result of Frenkel, Huang and Lepowsky on the irreducibility of tensor product modules for tensor product vertex operator algebras in the context of vertex algebras of countable dimension.

Throughout this paper, vector spaces are considered over the field $\mathbb{C}$ of complex numbers. In this paper we use the standard formal variable notations and conventions (see [8,9]) and we use the following definition of the notion of vertex algebra ([14]; cf. [1, 4,8,9,16]):

Definition 2.1. A *vertex algebra* is a vector space V equipped with a linear map, called the *vertex operator map*,

$$\begin{aligned} Y: V &\to (\operatorname{End} V)\big[\big[z, z^{-1}\big]\big], \\ v &\mapsto Y(v,z) = \sum_{n\in\mathbb{Z}} v_n z^{-n-1} \quad (v_n \in \operatorname{End} V) \end{aligned} \tag{2.1}$$

and equipped with a distinguished vector $\mathbf{1} \in V$, called the *vacuum vector*, such that the following axioms hold: For $u, v \in V$,

$$u_n v = 0 \quad \text{for } n \text{ sufficiently large;} \tag{2.2}$$

$$Y(\mathbf{1}, z) = 1; \tag{2.3}$$

for $v \in V$,

$$Y(v,z)\mathbf{1} \in V[[z]] \quad \text{and} \quad Y(v,z)\mathbf{1}|_{z=0} \,(= v_{-1}\mathbf{1}) = v; \tag{2.4}$$

and for $u, v \in V$,

$$\begin{aligned} &z_0^{-1}\delta\left(\frac{z_1 - z_2}{z_0}\right) Y(u, z_1) Y(v, z_2) - z_0^{-1}\delta\left(\frac{z_2 - z_1}{-z_0}\right) Y(v, x_2) Y(u, x_1) \\ &\quad = z_2^{-1}\delta\left(\frac{z_1 - z_0}{z_2}\right) Y\big(Y(u, z_0)v, z_2\big) \end{aligned} \tag{2.5}$$

(the *Jacobi identity*).

Remark 2.2. In the literature, there are variant definitions of the notion of vertex algebra (cf. [1,4,13,16]). It was proved in [16] that the definition given in [1] is equivalent to the current definition (with ground field $\mathbb{C}$). The definition used in [6] and [13] (where the vacuum vector is denoted by Ω), whose defining axioms had been proved before to give rise to an equivalent definition (see [4,8,11,16]), is also equivalent to the current definition.

For a vertex algebra V, the vertex operator map Y is a linear map from V to $\mathrm{Hom}(V, V((z)))$. The map Y can be considered as a linear map from $V \otimes V$ to $V((z))$. Following [6], we alternatively denote this map by $Y(z)$. Define a linear operator $\mathcal{D} \in \mathrm{End}\, V$ by

$$D(v) = v_{-2}\mathbf{1} \quad \left(= \left(\frac{d}{dz} Y(v, z)\mathbf{1} \right) \bigg|_{z=0} \right). \tag{2.6}$$

Then (cf. [14])

$$\big[\mathcal{D}, Y(u, z)\big] = Y(\mathcal{D}u, z) = \frac{d}{dz} Y(u, z), \tag{2.7}$$

$$Y(u, z)v = e^{z\mathcal{D}} Y(v, -z)u \quad \text{for } u, v \in V. \tag{2.8}$$

We also have

$$Y(u, z)\mathbf{1} = e^{z\mathcal{D}} u \quad \text{for } u \in V. \tag{2.9}$$

It was proved ([4,16], cf. [8,9]) that the Jacobi identity is equivalent to the following *weak commutativity and associativity*: For any $u, v \in V$, there exists a nonnegative integer k such that

$$(z_1 - z_2)^k Y(u, z_1) Y(v, z_2) = (z_1 - z_2)^k Y(v, z_2) Y(u, z_1); \tag{2.10}$$

and for any $u, v, w \in V$ there exists a nonnegative integer l such that

$$(z_0 + z_2)^l Y(u, z_0 + z_2) Y(v, z_2) w = (z_0 + z_2)^l Y\big(Y(u, z_0)v, z_2\big) w. \tag{2.11}$$

A *V-module* [14,16] is a vector space W equipped with a linear map Y from V to $(\mathrm{End}\, W)[[z, z^{-1}]]$ such that all the axioms defining the notion of vertex algebra that make sense hold. That is, the truncation condition (2.2), the vacuum property (2.3) and the Jacobi identity (2.5) hold.

The notion of ideal is defined in the obvious way; an ideal of a vertex algebra V is a subspace U such that $u_m v,\ v_m u \in U$ for all $v \in V,\ u \in U$. Every vertex algebra V has trivial ideals 0 and V.

Definition 2.3. A vertex algebra V is said to be *simple* if there is no nontrivial ideal and V is said to be *irreducible* if V is an irreducible V-module.

Clearly, an irreducible vertex algebra is simple, but simple vertex algebras are not necessarily irreducible. For example, the vertex algebra constructed in [1] from the commutative associative algebra $\mathbb{C}[x]$ with the standard derivation is simple, but not irreducible. On the other hand, simple vertex operator algebras in the sense of [9] and [8] are always irreducible because any submodule of V is an ideal (cf. [8]).

Let V be a vertex algebra, fixed throughout this section. Following [6], for a positive integer n, we define a linear map

$$\begin{aligned} Z_n : V^{\otimes n} \otimes \mathbb{C}((z_1))\cdots((z_n)) &\to V((z_1))\cdots((z_n)), \\ v^{(1)} \otimes \cdots \otimes v^{(n)} \otimes f &\mapsto fY\big(v^{(1)}, z_1\big)\cdots Y\big(v^{(n)}, z_n\big)\mathbf{1}. \end{aligned} \tag{2.12}$$

In [6], Z_n was defined as

$$Z_n = Y(z_1)\big(1 \otimes Y(z_2)\big)\cdots\big(1^{\otimes n-1} \otimes Y(z_n)\big)\big(1^{\otimes n} \otimes \mathbf{1}\big). \tag{2.13}$$

The following notion is due to Etingof and Kazhdan [6]:

Definition 2.4. A vertex algebra V is said to be *nondegenerate* if the linear maps Z_n are injective for all positive integers n.

Remark 2.5. Consider the case $n = 1$. For $v \in V,\ f \in \mathbb{C}((z))$, we have

$$Z_1(v \otimes f) = fY(v, z)\mathbf{1} = fe^{z\mathcal{D}}v = e^{z\mathcal{D}}fv.$$

Let π be the natural embedding of $V \otimes \mathbb{C}((z))$ into $V((z))$. Then $Z_1 = e^{z\mathcal{D}}\pi$. It follows immediately that Z_1 is injective.

Now, let W be a V-module. For $n \geqslant 1$, we define a linear map

$$\begin{aligned} Z_n^W : V^{\otimes n} \otimes W \otimes \mathbb{C}((z_1))\cdots((z_n)) &\to W((z_1))\cdots((z_n)), \\ v^{(1)} \otimes \cdots \otimes v^{(n)} \otimes w \otimes f &\mapsto fY\big(v^{(1)}, z_1\big)\cdots Y\big(v^{(n)}, z_n\big)w \end{aligned} \tag{2.14}$$

for $v^{(1)}, \ldots, v^{(n)} \in V,\ w \in W,\ f \in \mathbb{C}((z_1))\cdots((z_n))$. Notice that $\mathbb{C}((z_1))\cdots((z_n))$ is a unital commutative associative algebra (which is in fact a field) and that for any vector space U, $U((z_1))\cdots((z_n))$ is a $\mathbb{C}((z_1))\cdots((z_n))$-module. It is clear that Z_n^W is $\mathbb{C}((z_1))\cdots((z_n))$-linear where $V^{\otimes n} \otimes W \otimes \mathbb{C}((z_1))\cdots((z_n))$ is considered as a $\mathbb{C}((z_1))\cdots((z_n))$-module in the obvious way. Let E_n be the embedding of $V^{\otimes n}$ into $V^{\otimes(n+1)}$ defined by

$$E_n\big(v^{(1)} \otimes \cdots \otimes v^{(n)}\big) = v^{(1)} \otimes \cdots \otimes v^{(n)} \otimes \mathbf{1}. \tag{2.15}$$

Then

$$Z_n = Z_n^V E_n. \tag{2.16}$$

In the literature, there are certain results which are closely related to the injectivity of linear maps Z_1^W. The following result is due to Dong and Mason [5] while the particular case is due to Dong and Lepowsky [4]:

Proposition 2.6. *Let V be a simple vertex operator algebra and let W be an irreducible V-module. Let $v^{(1)}, \ldots, v^{(r)} \in V$ be nonzero vectors and let $w^{(1)}, \ldots, w^{(r)} \in W$ be linearly independent vectors. Then*

$$\sum_{i=1}^{r} Y\big(v^{(i)}, z\big)w^{(i)} \neq 0 \quad \textit{in } W((z)). \tag{2.17}$$

In particular,

$$Y(v, z)w \neq 0 \quad \textit{for } 0 \neq v \in V,\ 0 \neq w \in W. \tag{2.18}$$

Remark 2.7. Proposition 2.6 exactly asserts that Y viewed as a linear map from $V \otimes W$ to $W((z))$ is injective.

From [8], for any positive integer n, $V^{\otimes(n+1)}$ has a natural vertex algebra structure and for V-modules $W_1, \ldots, W_{n+1}$, $W_1 \otimes \cdots \otimes W_{n+1}$ has a natural $V^{\otimes(n+1)}$-module structure. For $1 \leqslant i \leqslant n+1$, let π_i be the embedding of V into $V^{\otimes(n+1)}$:

$$\begin{aligned} \pi_i:\ V &\rightarrow V^{\otimes(n+1)}, \\ v &\mapsto \mathbf{1}^{\otimes(i-1)} \otimes v \otimes \mathbf{1}^{\otimes(n+1-i)}. \end{aligned} \tag{2.19}$$

Let U be any subspace of $W_1 \otimes \cdots \otimes W_{n+1}$. It is clear (cf. [8]) that U is a submodule if and only if for $i = 1, \ldots, n+1$,

$$Y\big(\pi_i(v), z\big)U \subset U((z)) \quad \text{for all } v \in V. \tag{2.20}$$

Note that any commutative associative algebra with identity is naturally a vertex algebra. Then from [8], $V^{\otimes(n+1)} \otimes \mathbb{C}((z_1)) \cdots ((z_n))$ is a vertex algebra. Now we state our first and key result:

Proposition 2.8. *Let W be any V-module. For any positive integer n, subspace* $\ker Z_n^W$ *is a $V^{\otimes(n+1)} \otimes \mathbb{C}((z_1)) \cdots ((z_n))$-submodule of $V^{\otimes n} \otimes W \otimes \mathbb{C}((z_1)) \cdots ((z_n))$.*

For convenience we first prove the following simple fact:

Lemma 2.9. *Let U be a vector space and let*

$$f(z_1, \ldots, z_n) \in U((z_1)) \cdots ((z_n)). \tag{2.21}$$

Assume that there exist nonnegative integers k_{ij} for $1 \leqslant i < j \leqslant n$ such that

$$\left(\prod_{1 \leqslant i < j \leqslant n} (z_i - z_j)^{k_{ij}}\right) f(z_1, \ldots, z_n) = 0. \tag{2.22}$$

Then $f(z_1, \ldots, z_n) = 0$.

Proof. The key issue here is the cancellation law. From [9], for any three formal series A, B and C, if ABC, AB and BC all exist (algebraically), then

$$A(BC) = (AB)C = ABC. \tag{2.23}$$

Set

$$A = \prod_{1\leqslant i<j\leqslant n} (z_i - z_j)^{-k_{ij}}, \qquad B = \prod_{1\leqslant i<j\leqslant n} (z_i - z_j)^{k_{ij}}, \qquad C = f(z_1, \dots, z_n),$$

where we use the usual binomial expansion convention (cf. [9]). Then it follows immediately from (2.22) that $f(z_1, \dots, z_n) = 0$. □

Proof of Proposition 2.8.. Since Z_n^W is already $\mathbb{C}((z_1))\cdots((z_n))$-linear, what we must prove is that for $i = 1, \dots, n+1$,

$$Y\big(\pi_i(v), z\big) \ker Z_n^W \subset \big(\ker Z_n^W\big)((z)) \quad \text{for } v \in V. \tag{2.24}$$

That is, we must prove that if $X \in V^{\otimes n} \otimes W \otimes \mathbb{C}((z_1))\cdots((z_n))$ with $Z_n^W(X) = 0$, then for $i = 1, \dots, n+1$,

$$Z_n^W\big(Y\big(\pi_i(v), z\big)X\big) = 0 \quad \text{for all } v \in V. \tag{2.25}$$

We shall prove this in four steps.

Claim 1: For $i = n+1$, (2.25) holds.

Let

$$X = \sum_{\alpha_i, \beta, \gamma} u^{1\alpha_1} \otimes \cdots \otimes u^{n\alpha_n} \otimes w^\beta \otimes f_\gamma \tag{2.26}$$

(a *finite* sum). We shall use this general X for the whole proof. Then

$$\sum_{\alpha_i, \beta, \gamma} f_\gamma Y\big(u^{1\alpha_1}, z_1\big) \cdots Y\big(u^{n\alpha_n}, z_n\big) w^\beta = Z_n^W(X) = 0. \tag{2.27}$$

Let $v \in V$. By the weak commutativity there exists a nonnegative integer k such that

$$(z - z_i)^k Y(v, z) Y\big(u^{i\alpha_i}, z_i\big) = (z - z_i)^k Y\big(u^{i\alpha_i}, z_i\big) Y(v, z) \tag{2.28}$$

for all of the indices i and α_i. Multiplying (2.27) by $(z - z_1)^k \cdots (z - z_n)^k Y(v, z)$ (from left) and then using (2.28) we get

$$(z - z_1)^k \cdots (z - z_n)^k \sum_{\alpha_i, \beta, \gamma} f_\gamma Y\big(u^{1\alpha_1}, z_1\big) \cdots Y\big(u^{n\alpha_n}, z_n\big) Y(v, z) w^\beta = 0. \tag{2.29}$$

In view of Lemma 2.9 we have

$$\sum_{\alpha_i,\beta,\gamma} f_\gamma Y\big(u^{1\alpha_1}, z_1\big)\cdots Y\big(u^{n\alpha_n}, z_n\big) Y(v,z) w^\beta = 0. \tag{2.30}$$

Thus $Z_n^W(Y(\pi_{n+1}(v), z)X) = 0$.

Claim 2. For $i = n$, (2.25) holds. From (2.27), using (2.28) we get

$$\left(\prod_{i=1}^{n-1}(z-z_i)^k\right) \sum_{\alpha_i,\beta,\gamma} f_\gamma Y\big(u^{1\alpha_1}, z_1\big)\cdots Y\big(u^{(n-1)\alpha_{n-1}}, z_{n-1}\big) Y(v,z) Y\big(u^{n\alpha_n}, z_n\big) w^\beta = 0. \tag{2.31}$$

In view of Lemma 2.9 we have

$$\sum_{\alpha_i,\beta,\gamma} f_\gamma Y\big(u^{1\alpha_1}, z_1\big)\cdots Y\big(u^{(n-1)\alpha_{n-1}}, z_{n-1}\big) Y(v,z) Y\big(u^{n\alpha_n}, z_n\big) w^\beta = 0. \tag{2.32}$$

By the weak associativity, there exists a nonnegative integer l such that

$$(z_0+z_n)^l Y(v, z_0+z_n) Y\big(u^{n\alpha_n}, z_n\big) w^\beta = (z_0+z_n)^l Y\big(Y(v,z_0)u^{n\alpha_n}, z_n\big) w^\beta \tag{2.33}$$

for all of the indices α_n and β, (which are finitely many). Then

$$\begin{aligned} &(z_0+z_n)^l \sum_{\alpha_i,\beta,\gamma} f_\gamma Y\big(u^{1\alpha_1}, z_1\big)\cdots Y\big(u^{(n-1)\alpha_{n-1}}, z_{n-1}\big) Y\big(Y(v,z_0)u^{n\alpha_n}, z_n\big) w^\beta \\ &\quad = (z_0+z_n)^l \sum_{\alpha_i,\beta,\gamma} f_\gamma Y\big(u^{1\alpha_1}, z_1\big)\cdots Y\big(u^{(n-1)\alpha_{n-1}}, z_{n-1}\big) Y(v, z_0+z_n) Y\big(u^{n\alpha_n}, z_n\big) w^\beta \\ &\quad = 0. \end{aligned} \tag{2.34}$$

By Lemma 2.9 we get

$$\sum_{\alpha_i,\beta,\gamma} f_\gamma Y\big(u^{1\alpha_1}, z_1\big)\cdots Y\big(u^{(n-1)\alpha_{n-1}}, z_{n-1}\big) Y\big(Y(v,z_0)u^{n\alpha_n}, z_n\big) w^\beta = 0. \tag{2.35}$$

That is, $Z_n^W(Y(\pi_n(v), z)X) = 0$.

Claim 3. $\ker Z_n^W$ is stable under the natural action of S_n. (Note that the symmetric group S_n acts naturally on $V^{\otimes n} \otimes W \otimes \mathbb{C}((z_1))\cdots((z_n))$, the domain of Z_n^W.)

It suffices to prove that $\ker Z_n^W$ is stable under the actions of transpositions $\sigma_{i(i+1)}$ for $i = 1, \ldots, n-1$. Let $X \in \ker Z_n^W$ and write X as in (2.26). By the weak commutativity there exists a nonnegative integer k such that

$$(z_i - z_{i+1})^k Y\big(u^{i\alpha_i}, z_i\big) Y(u, z_{i+1}) = (z_i - z_{i+1})^k Y(u, z_{i+1}) Y\big(u^{i\alpha_i}, z_i\big) \tag{2.36}$$

for all of the indices i and α_i. Then

$$(z_i - z_{i+1})^k Z_n^W(\sigma_{i(i+1)}X) = (z_i - z_{i+1})^k Z_n^W(X) = 0. \tag{2.37}$$

In view of Lemma 2.9, we have $Z_n^W(\sigma_{i(i+1)}X) = 0$. That is, $\ker Z_n^W$ is stable under the action of $\sigma_{i(i+1)}$ for $i = 1, \ldots, n$. Therefore $\ker Z_n^W$ is S_n-stable.

Claim 4. For $1 \leqslant i \leqslant n$, (2.25) holds.

Let σ be the transposition that exchanges i with n. Then we have

$$Y\big(\pi_i(v), z\big)X = \sigma\big(Y\big(\pi_n(v), z\big)\sigma(X)\big) \quad \text{for } v \in V. \tag{2.38}$$

By Claim 3, $\sigma(X) \in \ker Z_n^W$. Furthermore, by Claim 2 we have

$$Y\big(\pi_n(v), z\big)\sigma(X) \in \big(\ker Z_n^W\big)((z)).$$

By Claim 3 again we get

$$\sigma\big(Y\big(\pi_n(v), z\big)\sigma(X)\big) \in \big(\ker Z_n^W\big)((z)).$$

Therefore

$$Z_n^W\big(Y\big(\pi_i(v), z\big)X\big) = Z_n^W\big(\sigma\big(Y\big(\pi_n(v), z\big)\sigma(X)\big)\big) = 0.$$

This proves Claim 4, completing the proof. □

Let W be a nonzero V-module. Noticing that for $0 \neq w \in W$, $0 \neq f \in \mathbb{C}((z_1))\cdots((z_n))$,

$$Z_n^W\big(\mathbf{1}^{\otimes n} \otimes w \otimes f\big) = fY(\mathbf{1}, z_1)\cdots Y(\mathbf{1}, z_n)w = fw \neq 0, \tag{2.39}$$

we have

$$\ker Z_n^W \neq V^{\otimes n} \otimes W \otimes \mathbb{C}((z_1))\cdots((z_n)). \tag{2.40}$$

If we can prove that $V^{\otimes n} \otimes W \otimes \mathbb{C}((z_1))\cdots((z_n))$ is an irreducible $V^{\otimes(n+1)} \otimes \mathbb{C}((z_1))\cdots((z_n))$-module, in view of Proposition 2.8 we will immediately have $\ker Z_n^W = 0$. When V is a simple vertex operator algebra and W is an irreducible V-module, it follows from [8] that $V^{\otimes n} \otimes W$ is an irreducible $V^{\otimes(n+1)}$-module. But it is not clear if this is still true if V is just an irreducible vertex algebra. Motivated by this we next determine all the submodules of $V^{\otimes n} \otimes W \otimes \mathbb{C}((z_1))\cdots((z_n))$ for the vertex algebra $V^{\otimes(n+1)} \otimes \mathbb{C}((z_1))\cdots((z_n))$ with V an irreducible vertex algebra and W a general (not necessarily irreducible) V-module. In this direction we first prove the following simple result (for which we have no reference):

Lemma 2.10. *Let A_1 and A_2 be associative algebras with identity element and let U_1 and U_2 be modules for A_1 and A_2, respectively. Assume that U_1 is irreducible and* $\mathrm{End}_{A_1} U_1 = \mathbb{C}$. *Then any $A_1 \otimes A_2$-submodule of $U_1 \otimes U_2$ is of the form $U_1 \otimes U_2'$, where U_2' is an A_2-submodule of U_2. Furthermore, if U_2 is irreducible, then $U_1 \otimes U_2$ is irreducible.*

Proof. Fix a basis $\{u_{1\alpha} \mid \alpha \in I\}$ for U_1. Let U be a nonzero $A_1 \otimes A_2$-submodule of $U_1 \otimes U_2$. Let u be any nonzero element of U. Then

$$u = u_{1\alpha_1} \otimes u_{21} + \cdots + u_{1\alpha_r} \otimes u_{2r},$$

where $u_{21}, \ldots, u_{2r}$ are (finitely many) nonzero vectors in U_2. Since U_1 is irreducible and $\mathrm{End}_{A_1} U_1 = \mathbb{C}$, in view of the density theorem (cf. [12]), there exists $a \in A_1$ such that

$$au_{1\alpha_1} \neq 0 \quad \text{and} \quad au_{1\alpha_i} = 0 \quad \text{for } i = 2, \ldots, r.$$

Then

$$0 \neq au_{1\alpha_1} \otimes u_{21} = (a \otimes 1)u \in U.$$

Since U_1 is irreducible, we have $A_1 au_{1\alpha_1} = U_1$. Thus $U_1 \otimes u_{21} = (A_1 a \otimes 1)u \subset U$. Similarly, we have $U_1 \otimes u_{2i} \subset U$ for $i = 2, \ldots, r$. Set

$$U_2'(u) = A_2 u_{21} + \cdots + A_2 u_{2r} \subset U_2.$$

(Notice that u_{2i}'s are uniquely determined by u.) Then

$$u \in U_1 \otimes U_2'(u) \subset U.$$

Setting

$$U_2' = \sum_{0 \neq u \in U} U_2'(u) \subset U_2, \tag{2.41}$$

we get $U = U_1 \otimes U_2'$, completing the proof. □

Closely related to Lemma 2.10 is the following result which can be found in [3], or [2]:

Lemma 2.11. *Let A be an associative algebra with identity and let U be an irreducible A-module of countable dimension. Then* $\mathrm{End}_A U = \mathbb{C}$.

We now apply Lemma 2.11 to vertex algebras.

Lemma 2.12. *Let V be a vertex algebra of countable dimension and let W be an irreducible V-module. Then W is of countable dimension and* $\mathrm{End}_V W = \mathbb{C}$.

Proof. Let w be any nonzero element of W. It was proved in [5] and [15] that the linear span of vectors $v_m w$ for $v \in V,\ m \in \mathbb{Z}$ is a submodule of W. Consequently,

$$W = \operatorname{span}\{v_m w \mid v \in V,\ m \in \mathbb{Z}\}.$$

Then we see at once that W has countable dimension. Furthermore, let A be the subalgebra of $\operatorname{End} W$, generated by all the operators v_m for $v \in V,\ m \in \mathbb{Z}$. Then A acts irreducibly on W and $\operatorname{End}_A W = \operatorname{End}_V W$. In view of Lemma 2.11 we have $\operatorname{End}_A W = \mathbb{C}$. Thus $\operatorname{End}_V W = \operatorname{End}_A W = \mathbb{C}$. □

The following result slightly generalizes the corresponding result of [8]:

Proposition 2.13. *Let $V_1, \ldots, V_r$ be vertex algebras of countable dimension and let $W_1, \ldots, W_r$ be irreducible modules for $V_1, \ldots, V_r$, respectively. Then $W_1 \otimes \cdots \otimes W_r$ is an irreducible $V_1 \otimes \cdots \otimes V_r$-module with*

$$\operatorname{End}_{V_1 \otimes \cdots \otimes V_r}(W_1 \otimes \cdots \otimes W_r) = \mathbb{C}.$$

Proof. In view of Lemma 2.12, we have $\operatorname{End}_{V_i} W_i = \mathbb{C}$ for $i = 1, \ldots, r$. In particular, Proposition holds for $r = 1$. For $r = 2$, first it follows from Lemma 2.10 that $W_1 \otimes W_2$ is an irreducible $V_1 \otimes V_2$-module. Then by Lemma 2.12, $\operatorname{End}_{V_1 \otimes V_2}(W_1 \otimes W_2) = \mathbb{C}$. Now, Proposition follows immediately from induction on r and the assertion for $r = 2$. □

Now we are ready to prove our main result:

Theorem 2.14. *Let V be an irreducible vertex algebra of countable dimension and let W be any nonzero V-module. Then for every positive integer n, the linear map Z_n^W is injective. Furthermore, every irreducible vertex algebra V of countable dimension is nondegenerate.*

Proof. In view of Proposition 2.8, for any positive integer n, $\ker Z_n^W$ is a $V^{\otimes(n+1)} \otimes \mathbb{C}((z_1)) \cdots ((z_n))$-submodule of $V^{\otimes n} \otimes W \otimes \mathbb{C}((z_1)) \cdots ((z_n))$. It follows from Proposition 2.13 that $V^{\otimes n}$ is an irreducible $V^{\otimes n}$-module and that $\operatorname{End}_{V^{\otimes n}} V^{\otimes n} = \mathbb{C}$. Let A_1 be the subalgebra of $\operatorname{End} V^{\otimes n}$, generated by all the operators u_m for $u \in V^{\otimes n},\ m \in \mathbb{Z}$. Then A_1 acts irreducibly on $V^{\otimes n}$ and $\operatorname{End}_{A_1} V^{\otimes n} = \mathbb{C}$. In view of Lemma 2.10, we have

$$\ker Z_n^W = V^{\otimes n} \otimes U,$$

where U is a $V \otimes \mathbb{C}((z_1)) \cdots ((z_n))$-submodule of $W \otimes \mathbb{C}((z_1)) \cdots ((z_n))$. Let

$$F = w_1 \otimes f_1 + \cdots + w_r \otimes f_r$$

be a generic element of U, where $w_1, \ldots, w_r$ are linearly independent vectors in W and $f_i \in \mathbb{C}((z_1)) \cdots ((z_n))$. Then we have $\mathbf{1}^{\otimes n} \otimes F \in \ker Z_n^W$, so that

$$f_1 w_1 + \cdots + f_r w_r = \sum_{i=1}^{r} f_i Y(\mathbf{1}, z_1) \cdots Y(\mathbf{1}, z_n) w_i = Z_n^W\bigl(\mathbf{1}^{\otimes n} \otimes F\bigr) = 0. \tag{2.42}$$

Consequently, $f_i = 0$ for $i = 1, \dots, r$. This proves that $U = 0$, so $\ker Z_n^W = 0$. Furthermore, since $Z_n = Z_n^V E_n$, where E_n is the embedding of $V^{\otimes n}$ into $V^{\otimes(n+1)}$, Z_n must be injective. This proves that V is nondegenerate. □

It follows from the definition of the notion of vertex operator algebra [8,9] that any vertex operator algebra in the sense of [9] and [8] has countable dimension. We also know that a simple vertex operator algebra is irreducible. Then we immediately have

Corollary 2.15. *Let V be a simple vertex operator algebra in the sense of* [9] *and* [8] *and let W be a nonzero V-module. Then for every positive integer n, the linear map Z_n^W is injective. Furthermore, every simple vertex operator algebra V is nondegenerate.*

We also immediately have the following generalization of Proposition 2.6:

Corollary 2.16. *Let V be an irreducible vertex algebra of countable dimension and let W be any* (*not necessarily irreducible*) *nonzero V-module. Then for each positive integer n, the linear map*

$$F_n^W : V^{\otimes n} \otimes W \to W((z_1)) \cdots ((z_n))$$

defined by

$$F_n^W\big(v^{(1)} \otimes \cdots \otimes v^{(n)} \otimes w\big) = Y\big(v^{(1)}, z_1\big) \cdots Y\big(v^{(n)}, z_n\big)w \tag{2.43}$$

is injective.

Remark 2.17. Results of this paper can be appropriately generalized in terms of intertwining operators [8]. In fact, the corresponding result of [4] was formulated in terms of intertwining operators in the more general context of generalized vertex algebras.

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