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Combinatorial Gray codes for classes of pattern avoiding permutations

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Abstract

The past decade has seen a flurry of research into pattern avoiding permutations but little of it is concerned with their exhaustive generation. Many applications call for exhaustive generation of permutations subject to various constraints or imposing a particular generating order. In this paper we present generating algorithms and combinatorial Gray codes for several families of pattern avoiding permutations. Among the families under consideration are those counted by Catalan, large Schröder, Pell, even-index Fibonacci numbers and the central binomial coefficients. We thus provide Gray codes for the set of all permutations of $\{1, \ldots, n\}$ avoiding the pattern τ for all $\tau \in \mathfrak{S}_3$ and the Gray codes we obtain have distances 4 or 5.
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1. Introduction

A number of authors have been interested in Gray codes and generating algorithms for permutations and their restrictions (unrestricted [10], with given *ups* and *downs* [14,18], involutions, and fixed-point free involutions [24], derangements [5], permutations with a fixed number of cycles [2]) or their generalizations (multiset permutations [13,23]). A recent paper [12] presented Gray codes and generating algorithms for the three classes of pattern avoiding permutations: $\mathfrak{S}_n(123, 132)$, $\mathfrak{S}_n(123, 132, p(p-1)\ldots1(p+1))$, and permutations in $\mathfrak{S}_n(123, 132)$ which have exactly $\binom{n}{2} - k$ inversions. In [6] a general technique is presented for the generation of Gray codes for a large class of combinatorial families; it is based on the ECO method and produces objects by their encoding given by

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the generating tree (in some cases the obtained encodings can be translated into the objects). Motivated by these papers, we investigate the related problem for several new classes of pattern avoiding permutations.

More specifically, we give combinatorial Gray codes for classes of pattern avoiding permutations which are counted by Catalan, Schröder, Pell, even-index Fibonacci numbers and the central binomial coefficients; the Gray codes we obtain have distances 4 or 5. Our work is different from similar work for combinatorial classes having the same counting sequence, see for instance [6,22]. Indeed, as Savage [21, Section 7] points out: '*Since bijections are known between most members of the Catalan family, a Gray code for one member of the family gives implicitly a listing scheme for every other member of the family. However, the resulting list may not look like Gray codes, since bijections need not preserve minimal changes between elements*'.

Some direct constructions for $\mathfrak{S}_n(231)$ exist but are, however, not Gray codes. For example, Bóna [8, Section 8.1.2] provides an algorithm for generating $\mathfrak{S}_n(231)$. This algorithm is such that the successor of the permutation $\pi = (n, n-1, \dots 2, 1, 2n+1, 2n, 2n-1, \dots, n+2, n+1)$ is $\pi' = (1, 2, \dots, n-1, 2n+1, n, n+1, \dots, 2n)$. The number of places in which these two permutations differ is linear in n.

In Section 2 we present a combinatorial Gray code for $\mathfrak{S}_n(231)$ with distance 4. In Section 3 we present a Gray code for the Schröder permutations, $\mathfrak{S}_n(1243, 2143)$, with distance 5. In Section 4 we present a general generating algorithm and Gray codes for some classes of pattern avoiding permutations and discuss its limits.

The techniques we will use are: in Sections 2 and 3 reversing sublists [20]; in Section 3 combinatorial bijections [12]; and in Section 4 generating trees [6].

Throughout this paper, it is convenient to use the following notation. The number $c_n = \frac{1}{n+1}\binom{2n}{n}$ is the nth Catalan number. The large Schröder numbers r_n are defined by $r_0 = 1$ and for all $n > 0$,

$$r_n = r_{n-1} + \sum_{k=1}^{n} r_{k-1} r_{n-k}. \tag{1.1}$$

Let $A(1) = 0$, $B(1) = 0$ and for all $i > 1$,

$$A(i) = c_0 + \dots + c_{i-2}, \text{ and} \tag{1.2}$$

$$B(i) = r_0 + \dots + r_{i-2}. \tag{1.3}$$

The parity of these numbers will be extremely important in proving the Gray code properties of the generating algorithms for permutations we define later on in the paper. However, the parity of $A(i)$ and $B(i)$ are not explicitly used in the algorithms. Note that for all $0 < k \le 2^n$, $A(2^n + k)$ is odd iff n is even. One can easily show that $B(i)$ is odd iff $i = 2$. For two permutations $\sigma = \sigma_1\sigma_2 \dots \sigma_n$ and $\tau = \tau_1\tau_2 \dots \tau_n$ in $\mathfrak{S}_n$, the metric $d(\sigma, \tau)$ is the number of places in which they differ; and we denote by $\sigma \circ \tau$ (or more compactly as $\sigma\tau$) their product, that is, the permutation π in $\mathfrak{S}_n$ with $\pi_i = \tau_{\sigma_i}$ for all i, $1 \le i \le n$. In particular, when σ is the transposition (u, v), then $(u, v) \circ \tau$ is the permutation π with $\pi_i = \tau_i$ for all i, except that $\pi_u = \tau_v$ and $\pi_v = \tau_u$.

2. A Gray code for $\mathfrak{S}_n(231)$

Note that if $(\pi(1), \dots, \pi(c_n))$ is an ordered list of elements of $\mathfrak{S}_n(231)$ such that $d(\pi(i), \pi(i+1)) \le 4$, then the operations of reverse, complement and their composition provide lists for $\mathfrak{S}_n(132)$, $\mathfrak{S}_n(213)$ and $\mathfrak{S}_n(312)$, respectively, which preserve the distance between two adjacent permutations.

2.1. Generating 231-avoiding permutations

First we introduce some general notation concerning the list $\mathcal{D}_n$ that our algorithm will generate and then provide the necessary proofs to show that $\mathcal{D}_n$ is the desired object.

For every $n \ge 0$, let $\mathcal{D}_n$ denote a list consisting of c_n entries, each of which is some permutation of $\{1, \dots, n\}$. The jth entry is denoted $\mathcal{D}_n(j)$. In order that we may copy such a list, either in its natural or reversed order, we define $\mathcal{D}_n^i$ to be $\mathcal{D}_n$ if i is odd, and $\mathcal{D}_n$ reversed if i is even, for every positive integer i. Thus $\mathcal{D}_n^i(j) = \mathcal{D}_n^{i+1}(c_n + 1 - j)$ for all $1 \le j \le c_n$.

By $\mathcal{D}_n(j) + l$ we shall mean $\mathcal{D}_n(j)$ with every element incremented by the value l. Concatenation of lists is defined in the usual way, concatenation of any permutation with the null permutation yields the same permutation, i.e. $[\tau, \emptyset] = [\emptyset, \tau] = \tau$.

Table 1
The Gray code $\mathcal{D}_6$ for the set $\mathfrak{S}_6(231)$ given by relation (2.1) and produced by Algorithm 1

612345	6215**34**	165**243**	**3**1**26**45	4**213**65	215**43**6
6**21**345	6**152**34	165**432**	3126**54**	4**12**365	**12**5436
6**132**45	615**32**4	1654**23**	3**21**654	**5**123**46**	125**34**6
6**321**45	615**243**	16**254**3	3216**45**	5**21**346	12**35**46
63**12**45	615**432**	1625**34**	**132**645	5**132**46	**21**3546
6**2**1**43**5	6154**23**	162**35**4	1326**54**	5**321**46	**132**546
6**12**435	6**541**23	16**32**54	**213**654	53**12**46	**321**546
61**42**35	654**21**3	16**4**2**35**	2136**45**	5**2**1**43**6	3**12**546
614**32**5	654**132**	164**32**5	**12**3645	5**12**436	**4**12**35**6
6**431**25	654**321**	16**243**5	1236**54**	51**42**36	4**21**356
643**21**5	6543**12**	16**324**5	123**465**	514**32**6	4**132**56
64**132**5	65**143**2	16**23**45	**21**3465	5**431**26	4**321**56
64**213**5	6514**23**	1**26**345	**132**465	543**21**6	43**12**56
64**12**35	651**24**3	126**43**5	**321**465	54**132**6	**143**256
6**3**12**54**	65**21**43	126**354**	3**12**465	54**213**6	14**23**56
63**21**54	65**3**1**24**	126**543**	**2**1**43**65	54**12**36	1**24**356
6**132**54	653**21**4	1265**34**	**12**4365	**154**236	**21**4356
6**213**54	65**132**4	**21**6534	1**42**365	154**32**6	**3**1**24**56
6**12**354	65**213**4	2165**43**	14**32**65	15**243**6	3**21**456
612**53**4	65**12**34	216**354**	**431**265	15**324**6	**132**456
6125**43**	**165**234	216**43**5	43**21**65	15**23**46	**213**456
6**21**543	165**32**4	216**34**5	4**132**65	**215**346	**12**3456

Permutations are listed column-wise and changed entries are in bold.

The list $\mathcal{D}_n$ is defined recursively as follows; $\mathcal{D}_0$ consists of a single entry which contains the null permutation that we denote as $\emptyset$. For any $n \geq 1$,

$$\mathcal{D}_n = \bigoplus_{i=1}^{n} \bigoplus_{j=1}^{c_{i-1}} \bigoplus_{k=1}^{c_{n-i}} \left[\mathcal{D}_{i-1}^{n+i-1}(j), n, \mathcal{D}_{n-i}^{j+A(i)+1}(k) + (i-1) \right], \tag{2.1}$$

where $A(i)$ is defined in Eq. (1.2) and $\oplus$ denotes the concatenation operator, e.g.

$$\bigoplus_{i=1}^{2} \bigoplus_{j=1}^{2} (f(i,j)) = (f(1,1),\ f(1,2),\ f(2,1),\ f(2,2)).$$

Lemma 2.1. *The list $\mathcal{D}_n$ contains all 231-avoiding permutations exactly once.*

Proof. Every permutation $\pi \in \mathfrak{S}_n(231)$ may be decomposed as $\pi = \tau n \sigma$, where $\tau \in \mathfrak{S}_{i-1}(231)$ and σ is a 231-avoiding permutation on the set $\{i, \ldots, n-1\}$ which is order-isomorphic to a $\sigma' \in \mathfrak{S}_{n-i}$. In $\mathcal{D}_n$, n assumes the positions $i = 1, 2, \ldots, n$. For each position i of n, τ runs through $\mathcal{D}_{i-1}$ alternately forwards and backwards, forwards the last time. For each τ, σ runs through $\mathcal{D}_{i-1} + (i-1)$ alternately forwards and backwards, backwards the first time (see Table 1). The result follows by strong induction on n. $\square$

Lemma 2.2. *For all $n \geq 2$,*

$$\mathcal{D}_n(1) = n123\cdots(n-1) \text{ and } \mathcal{D}_n(c_n) = 123\cdots n.$$

Proof. The proof proceeds by induction on n. We have $\mathcal{D}_0 = \emptyset$. Assume that the result holds for each $i = 0, 1, 2, \ldots n-1$. Then by Eq. (2.1), $\mathcal{D}_n(1)$ corresponds to the expression with $i = 1$, $j = 1$ and $k = 1$;

$$\mathcal{D}_n(1) = n\, \mathcal{D}_{n-1}^{1+A(1)+1}(1) = n\, \mathcal{D}_{n-1}^{2}(1) = n\, \mathcal{D}_{n-1}(c_{n-1}) = n123\cdots(n-1).$$

The last entry $\mathcal{D}_n(c_n)$ corresponds to the expression in Eq. (2.1) with $i = n$, $j = c_{i-1}$ and $k = c_{n-i}$;

$$\mathcal{D}_n(c_n) = \mathcal{D}_{n-1}^{2n-1}(c_{n-1})\, n = 123\cdots n. \quad \square$$

Theorem 2.3. *For each $q \in \{1, 2, \ldots c_n - 1\}$, $\mathcal{D}_n(q)$ differs from its successor $\mathcal{D}_n(q+1)$ by a rotation of two, three or four elements.*

Proof. The proof proceeds by induction. The result holds trivially for $n = 1$ since $\mathcal{D}_1$ consists of a single permutation. Assume that the result holds for $\mathcal{D}_i$ for each $i = 1, 2, \ldots n-1$. From Eq. (2.1), there are 3 cases:

(i) The current permutation corresponds to $(i; j; k = t)$ and the next permutation corresponds to $(i; j; k = t+1)$, where $t \in \{1, 2, \ldots c_{n-i} - 1\}$. Therefore

$$\mathcal{D}_n(q) = \mathcal{D}_{i-1}^{n+i-1}(j)\ n\ \mathcal{D}_{n-i}^{j+A(i)+1}(t) + (i-1)$$
$$\mathcal{D}_n(q+1) = \mathcal{D}_{i-1}^{n+i-1}(j)\ n\ \mathcal{D}_{n-i}^{j+A(i)+1}(t+1) + (i-1),$$

and by the induction hypothesis,

$$d(\mathcal{D}_n(q), \mathcal{D}_n(q+1)) = d(\mathcal{D}_{n-i}(t), \mathcal{D}_{n-i}(t+1)) \leq 4.$$

(ii) The current permutation corresponds to $(i, j = t, k = c_{n-i})$ and the next permutation corresponds to $(i; j = t+1; k = 1)$, where $t \in \{1, 2, \ldots c_{i-1} - 1\}$. Therefore

$$\mathcal{D}_n(q) = \mathcal{D}_{i-1}^{n+i-1}(t)\ n\ \mathcal{D}_{n-i}^{t+A(i)+1}(c_{n-i}) + (i-1)$$
$$\mathcal{D}_n(q+1) = \mathcal{D}_{i-1}^{n+i-1}(t+1)\ n\ \mathcal{D}_{n-i}^{t+A(i)+2}(1) + (i-1).$$

Since $\mathcal{D}_{n-i}^{t+A(i)+1}(c_{n-i}) = \mathcal{D}_{n-i}^{t+A(i)+2}(1)$, the induction hypothesis gives

$$d(\mathcal{D}_n(q), \mathcal{D}_n(q+1)) = d(\mathcal{D}_{i-1}(t), \mathcal{D}_{i-1}(t+1)) \leq 4.$$

(iii) The current permutation corresponds to $(i = t; j = c_{i-1}; k = c_{n-i})$ and the next permutation corresponds to $(i = t+1; j = 1; k = 1)$, where $t \in \{1, \ldots n-1\}$. Therefore

$$\mathcal{D}_n(q) = \mathcal{D}_{t-1}^{n+t-1}(c_{t-1})\ n\ \mathcal{D}_{n-t}^{c_{t-1}+A(t)+1}(c_{n-t}) + (t-1)$$
$$\mathcal{D}_n(q+1) = \mathcal{D}_t^{n+t}(1)\ n\ \mathcal{D}_{n-t-1}^{1+A(t+1)+1}(1) + t.$$

This divides into four cases, where in each case we use Lemma 2.2 and the fact that $A(t+1) = A(t) + c_{t-1}$:
(a) If $n + t$ is odd and $c_{t-1} + A(t) + 1 = A(t+1) + 1$ is odd, then

$$\mathcal{D}_n(q) = 1\,2\,3\,\ldots\,(t-1)\,n\,t\,(t+1)\,\ldots\,(n-1)$$
$$\mathcal{D}_n(q+1) = 1\,2\,3\,\ldots\,(t-1)\,t\,n\,(t+1)\,\ldots\,(n-1).$$

Here $\mathcal{D}_n(q+1)$ is obtained from $\mathcal{D}_n(q)$ via a single transposition of elements at positions $(t, t+1)$.
(b) If $n + t$ is odd and $c_{t-1} + A(t) + 1$ is even, then

$$\mathcal{D}_n(q) = 1\,2\,\ldots\,(t-1)\,n\,(n-1)\,t\,(t+1)\,\ldots\,(n-2)$$
$$\mathcal{D}_n(q+1) = 1\,2\,\ldots\,(t-1)\,t\,n\,(n-1)\,(t+1)\,\ldots\,(n-2),$$

for all $t \leq n-3$. Here $\mathcal{D}_n(q+1)$ is obtained from $\mathcal{D}_n(q)$ via a rotation of the 3 elements at positions $(t, t+1, t+2)$. If $t = n-2$ then

$$\mathcal{D}_n(q) = 1\,2\,\ldots\,(n-3)\,n\,(n-1)\,(n-2) \text{ and}$$
$$\mathcal{D}_n(q+1) = 1\,2\,\ldots\,(n-3)\,(n-2)\,n\,(n-1).$$

These permutations differ by a rotation of the 3 elements at positions $(n-2, n-1, n)$. If $t = n-1$ then

$$\mathcal{D}_n(q) = (n-2)\,1\,2\,\ldots\,(n-3)\,n\,(n-1) \text{ and}$$
$$\mathcal{D}_n(q+1) = (n-1)\,1\,2\,\ldots\,(n-3)\,(n-2)\,n.$$

These permutations differ by a rotation of the 3 elements at positions $(1, n-1, n)$.
(c) If $n + t$ is even and $c_{t-1} + A(t) + 1$ is odd, then

$$\mathcal{D}_n(q) = (t-1)\,1\,2\,\ldots\,(t-2)\,n\,t\,(t+1)\,\ldots\,(n-1) \text{ and}$$
$$\mathcal{D}_n(q+1) = t\,1\,2\,\ldots\,(t-2)\,(t-1)\,n\,(t+1)\,\ldots\,(n-1)$$

Algorithm 1 Pseudocode for generating $\mathfrak{S}_N(231)$ using Eq. (2.1). The list $\mathcal{D}_n$ is computed for each $1 \le n \le N$. Here $\mathcal{D}_n^R$ denotes the reversal of list $\mathcal{D}_n$.

```
set D_0 to a 1 × 0 matrix
set D_1 := [1]
for n := 2 to N do
  τstate := n (mod 2)      {1 means forwards and 0 means backwards}
  σstate := 0
  for i := 1 to n do
    for l := 1 to i − 1 do
      if τstate = 0 then
        τ =: D^R_{i−1}(l)
      else
        τ := D_{i−1}(l)
      end if
      for r := 1 to c_{n−i} do
        if σstate = 0 then
          σ := D^R_{n−i}(r) + (i − 1)
        else
          σ := D_{n−i}(r) + (i − 1)
        end if
        new_row:= [τ, n, σ]
        Append new_row to D_n
      end for
      σstate := σstate + 1 (mod 2)
    end for
    τstate := τstate + 1 (mod 2)
  end for
end for
```

for all $t \geq 3$. Here $\mathcal{D}_n(q+1)$ is obtained from $\mathcal{D}_n(q)$ via a rotation of the 3 elements at positions $(1, t, t+1)$. The degenerate cases $t = 1, 2$ are dealt with in the same manner as those at the end of part (b).
(d) If $n + t$ is even and $c_{t-1} + A(t) + 1$ is even, then

$$\begin{aligned}\mathcal{D}_n(q) &= (t-1)\,1\,2\,\ldots\,(t-2)\,n\,(n-1)\,t\,(t+1)\,\ldots\,(n-2)\\ \mathcal{D}_n(q+1) &= t\,1\,2\,\ldots\,(t-2)\,(t-1)\,n\,(n-1)\,(t+1)\,\ldots\,(n-2),\end{aligned}$$

for all $3 \le t \le n-3$. Here $\mathcal{D}_n(q+1)$ is obtained from $\mathcal{D}_n(q)$ via a rotation of the 4 elements at positions $(1, t, t+1, t+2)$. The degenerate cases $t = 1, 2, n-2, n-1$ are dealt with in the same manner as those at the end of part (b). □

In Table 1 is given the list $\mathcal{D}_6$ obtained by relation (2.1). The alert reader will note that there is no rotation of 4 elements in Table 1. Such a rotation is first observed when $n = 7$ and $t = 3$ (the permutation 2176345 becomes 3127645).

3. A Gray code for Schröder permutations

The permutations $\mathfrak{S}_n(1243, 2143)$ are called Schröder permutations and are just one of the classes of permutations enumerated by the Schröder numbers mentioned in the Introduction. Let $\mathcal{S}_n$ be the class of Schröder paths from $(0, 0)$ to $(2n, 0)$ (such paths may take steps $\mathsf{u} = (1, 1)$, $\mathsf{d} = (1, -1)$ and $\mathsf{e} = (2, 0)$ but never go below the x-axis). This class $\mathcal{S}_n$ is enumerated by r_n, see for instance [9].

In what follows, we will present a recursive procedure for generating all Schröder paths of length n. This procedure has the property that if the paths in $\mathcal{S}_n$ are listed as $(p_1, p_2, \ldots)$, then the sequence of permutations $(\varphi(p_1), \varphi(p_2), \ldots)$

is a Gray code for $\mathfrak{S}_{n+1}(1243, 2143)$ with distance 5. First we briefly describe Egge and Mansour's [9, Section 4] bijection $\varphi : \mathcal{S}_n \mapsto \mathfrak{S}_{n+1}(1243, 2143)$.

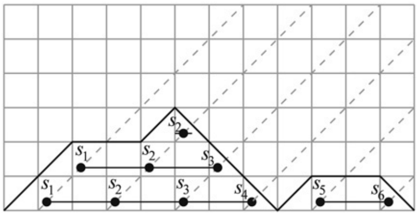
Let $p \in \mathcal{S}_n$ and let s_i be the transposition $(i, i+1)$.

Step 1. For all integers a, m with $0 \le a, m < n$, if either of the points $((8m+1)/4, (8a+5)/4)$ or $((8m+5)/4, (8a+1)/4)$ is contained in the region beneath p and above the x-axis, then place a dot at that point. For such a dot, with coordinates (x, y), associate the label s_i where $i = (1+x-y)/2$. Let $j = 1$.

Step 2. Choose the rightmost dot that has no line associated with it (with label s_k, say). Draw a line parallel to the x-axis from this dot to the leftmost dot that may be reached without crossing p (which has label s_l, say). Let $\sigma_j = s_k s_{k-1} \dots s_l$, where s_i, applied to a permutation π, exchanges π_i with π_{i+1}. If all dots have lines running through them, then go to step 3. Otherwise increase j by 1 and repeat step 2.

Step 3. Let $\varphi(p) = \sigma_j \dots \sigma_2\sigma_1(n+1, n, \dots, 1)$.

Example 3.1. Consider the path $p \in \mathcal{S}_6$ in the diagram.



The dots indicate the points realized in Step 1 and the lines joining them indicate how each of the σ's are formed. We have $\sigma_1 = s_6s_5$, $\sigma_2 = s_4s_3s_2s_1$, $\sigma_3 = s_3s_2s_1$ and $\sigma_4 = s_2$. So

$$\begin{aligned}\varphi(p) &= \sigma_4\sigma_3\sigma_2\sigma_1(7, 6, 5, 4, 3, 2, 1)\\ &= s_2\ s_3s_2s_1\ s_4s_3s_2s_1\ s_6s_5(7, 6, 5, 4, 3, 2, 1)\\ &= (5, 2, 4, 6, 7, 1, 3).\end{aligned}$$

3.1. Generating all Schröder paths

There are many ways to recursively generate all Schröder paths of length n. In what follows, we give one such procedure for generating the list $\mathcal{S}_n$. This list has the property that the corresponding permutations, under the bijection φ, are a Gray code for Schröder permutations of distance 5.

As in Section 2, we will use the convention that for any integer i, $\mathcal{S}_n^i = \mathcal{S}_n$ if i is odd and $\mathcal{S}_n^i$ is $\mathcal{S}_n$ reversed, if i is even. Entry j of $\mathcal{S}_n$ is denoted $\mathcal{S}_n(j)$. In this notation we will have

$$\mathcal{S}_n^i(j) = \begin{cases}\mathcal{S}_n(j) & \text{if } i \text{ is odd,}\\ \mathcal{S}_n(r_n+1-j) & \text{if } i \text{ is even.}\end{cases}$$

Define $\mathcal{S}_0$ to be the list consisting of the single null Schröder path, denoted $\emptyset$. For all $n \ge 1$, the paths are generated recursively via

$$\mathcal{S}_n = \bigoplus_{i=1}^{r_{n-1}} (\mathsf{e}\,\mathcal{S}_{n-1}(i)) \oplus \bigoplus_{i=1}^{n}\bigoplus_{j=1}^{r_{i-1}}\bigoplus_{k=1}^{r_{n-i}} \left(\mathsf{u}\,\mathcal{S}_{i-1}^{n+i}(j)\,\mathsf{d}\,\mathcal{S}_{n-i}^{j+B(i)+1}(k)\right). \tag{3.1}$$

$\mathcal{S}_n$ starts with $\mathcal{S}_{n-1}$ with each path preceded by e. There follow all the Schröder paths beginning with u. Let d be the partner of this u (the d that returns the path to the x-axis). Then d assumes positions $i = 2, 4, 6, \dots, 2n$ in the path. For each i, we have the paths in $\mathsf{u}\,\alpha\,\mathsf{d}\,\beta$, where α runs through $\mathcal{S}_{i-1}$ alternately forwards and backwards, backwards the last time, and for each α, β runs through $\mathcal{S}_{n-i}$ alternately forwards and backwards, backwards the first time.

Furthermore, we define $\Phi_n(j) := \varphi(\mathcal{S}_n(j))$ and

$$\Phi_n := \bigoplus_{j=1}^{r_n} \Phi_n(j). \tag{3.2}$$

Table 2
The lists $\mathcal{S}_3$ and Φ_3

n	$\mathcal{S}_3(n)$	$\Phi_3(n)$	n	$\mathcal{S}_3(n)$	$\Phi_3(n)$	n	$\mathcal{S}_3(n)$	$\Phi_3(n)$
1	eee	4321	9	udude	2431	17	uuedd	2134
2	eeud	4312	10	ududud	1432	18	uuuddd	1234
3	eudud	4132	11	udeud	3412	19	uuded	2314
4	eude	4231	12	udee	3421	20	uududd	1324
5	euudd	4123	13	udee	3241	21	ueudd	3124
6	eued	4213	14	uedud	3142	22	ueed	3214
7	udued	2413	15	uuddud	1342			
8	uduudd	1423	16	uudde	2341			

Table 3
The lists $\mathcal{S}_4$ and Φ_4

n	$\mathcal{S}_4(n)$	$\Phi_4(n)$	n	$\mathcal{S}_4(n)$	$\Phi_4(n)$	n	$\mathcal{S}_4(n)$	$\Phi_4(n)$
1	eeee	54321	31	uduedud	35142	61	uududde	24351
2	eeeud	54312	32	ududee	35241	62	uududdud	14352
3	eeudud	54132	33	ududee	35421	63	uudedud	34152
4	eeude	54231	34	ududeud	35412	64	uudede	34251
5	eeuudd	54123	35	udududud	15432	65	uuuddde	23451
6	eeued	54213	36	ududude	25431	66	uuudddud	13452
7	eudued	52413	37	ududuudd	15423	67	uueddud	31452
8	euduudd	51423	38	ududued	25413	68	uuedde	32451
9	eudude	52431	39	udeued	45213	69	uueedd	32145
10	eududud	51432	40	udeuudd	45123	70	uueuddd	31245
11	eudeud	53412	41	udeude	45231	71	uuududdd	13245
12	eudee	53421	42	udeudud	45132	72	uudeddd	23145
13	eudee	53241	43	udeeud	45312	73	uuuudddd	12345
14	euedud	53142	44	udeee	45321	74	uuueddd	21345
15	euuddud	51342	45	uuddee	34521	75	uuudded	23415
16	euudde	52341	46	uuddeud	34512	76	uuudddd	13425
17	euuedd	52134	47	uuddudud	14532	77	uuedudd	31425
18	euuuddd	51234	48	uuddude	24531	78	uudeed	32415
19	euuded	52314	49	uudduudd	14523	79	uudeed	34215
20	euududd	51324	50	uuddued	24513	80	uudeudd	34125
21	eueudd	53124	51	uedued	42513	81	uududdd	14325
22	eueed	53214	52	ueduudd	41523	82	uududed	24315
23	udueed	35214	53	uedude	42531	83	uuduuddd	14235
24	udueudd	35124	54	uedudud	41532	84	uuduedd	24135
25	uduududd	15324	55	uedeud	43512	85	ueuedd	42135
26	uduuded	25314	56	uedee	43521	86	ueuuddd	41235
27	uduuuddd	15234	57	ueede	43251	87	ueuded	42315
28	uduuedd	25134	58	ueedud	43152	88	ueududd	41325
29	uduudde	25341	59	ueuddud	41352	89	ueeudd	43125
30	uduuddud	15342	60	ueudde	42351	90	ueeed	43215

For example, we have $\mathcal{S}_1 = (\mathsf{e}, \mathsf{ud})$ and $\mathcal{S}_2 = (\mathsf{ee}, \mathsf{eud}, \mathsf{udud}, \mathsf{ude}, \mathsf{uudd}, \mathsf{ued})$. Thus $\Phi_1 = (21, 12)$ and $\Phi_2 = (321, 312, 132, 231, 123, 213)$. The paths and permutations $\mathcal{S}_3$, Φ_3, $\mathcal{S}_4$ and Φ_4 are listed in Tables 2 and 3. For two paths $p_1, p_2 \in \mathcal{S}_n$, we write $d(p_1, p_2)$ for the number of places in which the two paths differ when each e is replaced by rr where r represents $(1, 0)$; e.g. $d(\mathsf{e}, \mathsf{ud}) = 2$ and $d(\mathsf{ued}, \mathsf{eud}) = 2$.

Lemma 3.2. *Eq. (3.1) generates all Schröder paths of length n.*

Proof. This is routine by induction. The first concatenation operator forms all paths that begin with step e. If a path does not begin with e, then it does not touch the x-axis for the first time until $(2i, 0)$. A path of this form is uniquely expressed as $\mathsf{u}\alpha\mathsf{d}\beta$ where $\alpha \in \mathcal{S}_{i-1}$ and $\beta \in \mathcal{S}_{n-i}$. □

Lemma 3.3. *For all $n \geq 1$, $\mathcal{S}_n(1) = \mathsf{e}^n$ and $\mathcal{S}_n(r_n) = \mathsf{ue}^{n-1}\mathsf{d}$.*

Proof. By Eq. (3.1) we have that $\mathcal{S}_1(1) = \mathsf{e}$ and $\mathcal{S}_1(2) = \mathsf{ud}$; so the result is true for $n = 1$. Assume it to be true for all $m \le n-1$. Then $S_n(1) = \mathsf{e}\,\mathcal{S}_{n-1}(1) = \mathsf{e}\,\mathsf{e}^{n-1} = \mathsf{e}^n$.

Similarly, $\mathcal{S}_n(r_n)$ corresponds to Eq. (3.1) with $i = n$, $j = r_{n-1}$, $k = r_0$, thus

$$\mathcal{S}_n(r_n) = \mathsf{u}\,\mathcal{S}^{2n}_{n-1}(r_{n-1})\,\mathsf{d} = \mathsf{u}\,\mathsf{e}^{n-1}\,\mathsf{d} = \mathsf{u}\,\mathsf{e}^{n-1}\,\mathsf{d}.$$

Hence by induction the result is true for all $n \ge 1$. $\square$

Under the bijection φ, we thus have

Corollary 3.4. *For all $n > 0$,*

$$\begin{aligned} \Phi_n(1) &= (n+1)\,n\,\ldots\,1,\\ \Phi_n(r_n) &= n\,\ldots\,1\,(n+1). \end{aligned}$$

Theorem 3.5. *For each $1 \le q < r_n$, $\mathcal{S}_n(q)$ differs from $\mathcal{S}_n(q+1)$ in at most 5 places and $d(\Phi_n(q), \Phi_n(q+1)) \le 5$.*

Proof. This proof follows by strong induction and analyzing the different successors that occur in Eq. (3.1). The statement in the Theorem holds for $n = 0$ because there is only one permutation. We assume that the statement in the Theorem holds true for all $0 \le i \le n-1$. From Eq. (3.1) there are five cases to consider:

(i) If $1 \le q < r_{n-1} - 1$, then $\mathcal{S}_n(q) = \mathsf{e}\,\mathcal{S}_{n-1}(q)$ and $\mathcal{S}_n(q+1) = \mathsf{e}\,\mathcal{S}_{n-1}(q+1)$. This gives

$$d(\mathcal{S}_n(q), \mathcal{S}_n(q+1)) = d(\mathcal{S}_{n-1}(q), \mathcal{S}_{n-1}(q+1)),$$

which is ≤ 5 by our hypothesis. Thus

$$\begin{aligned} \Phi_n(q) &= (n+1)\;\Phi_{n-1}(q) \text{ and}\\ \Phi_n(q+1) &= (n+1)\;\Phi_{n-1}(q+1), \end{aligned}$$

and so $d(\Phi_n(q), \Phi_n(q+1)) \le 5$.

(ii) If $q = r_{n-1}$ then by Eq. (3.1) with $(i = 1;\ j = 1;\ k = 1)$ and Lemma 3.3 we have

$$\begin{aligned} \mathcal{S}_n(r_{n-1}) &= \mathsf{e}\,\mathcal{S}_{n-1}(r_{n-1}) = \mathsf{e}\,\mathsf{u}\,\mathsf{e}^{n-2}\,\mathsf{d} \text{ and}\\ \mathcal{S}_n(r_{n-1}+1) &= \mathsf{u}\,\mathsf{d}\,\mathcal{S}^2_{n-1}(1) = \mathsf{u}\,\mathsf{d}\,\mathsf{u}\,\mathsf{e}^{n-2}\,\mathsf{d}. \end{aligned}$$

Thus $d(\mathcal{S}_n(r_{n-1}), \mathcal{S}_n(r_{n-1}+1)) = d(\mathsf{eue}^{n-2}\mathsf{d}, \mathsf{udue}^{n-2}\mathsf{d}) = 2$. The corresponding permutations are

$$\begin{aligned} \Phi_n(r_{n-1}) &= (n+1)\,(n-1)\,(n-2)\,\ldots\,2\,1\,n \text{ and}\\ \Phi_n(r_{n-1}+1) &= (n-1)\,(n+1)\,(n-2)\,\ldots\,2\,1\,n, \end{aligned}$$

so that $d(\Phi_n(r_{n-1}), \Phi_n(r_{n-1}+1)) = 2 \le 5$.

(iii) If $\mathcal{S}_n(q)$ corresponds to $(i;\ j = r_{i-1};\ k = t)$ for some $1 \le t < r_{n-i}$ in Eq. (3.1) then

$$\begin{aligned} \mathcal{S}_n(q) &= \mathsf{u}\,\mathcal{S}^{n+i}_{i-1}(r_{i-1})\,\mathsf{d}\,\mathcal{S}^{j+B(i)+1}_{n-i}(t) \text{ and}\\ \mathcal{S}_n(q+1) &= \mathsf{u}\,\mathcal{S}^{n+i}_{i-1}(r_{i-1})\,\mathsf{d}\,\mathcal{S}^{j+B(i)+1}_{n-i}(t+1), \end{aligned}$$

and the distance of the two paths is no greater than 5, by the induction hypothesis. Therefore

$$\begin{aligned} \Phi_n(q) &= a \circ (n+1, \ldots, n+2-i, \varphi(\mathcal{S}^{j+B(i)+1}_{n-i}(t))) \text{ and}\\ \Phi_n(q+1) &= a \circ (n+1, \ldots, n+2-i, \varphi(\mathcal{S}^{j+B(i)+1}_{n-i}(t+1))), \end{aligned}$$

where

$$a = \begin{cases} s_i s_{i-1} \ldots s_1, & \text{if } n+i \text{ even},\\ s_{i-1} \ldots s_1 s_i s_{i-1} \ldots s_1, & \text{if } n+i \text{ odd}. \end{cases}$$

Using the fact that if $d(b, b') \le x$, then $d(a \circ b, a \circ b') \le x$, we have by the induction hypothesis $d(\Phi_n(q), \Phi_n(q+1)) \le 5$.

(iv) If $\mathcal{S}_n(q)$ corresponds to Eq. (3.1) with triple $(i;\ j = t; k = r_{n-i})$, where $1 \le t < r_{i-1}$, then the successor $\mathcal{S}_n(q+1)$ corresponds to Eq. (3.1) with triple $(i;\ j = t+1; k = 1)$. Consequently,

$$\begin{aligned}\mathcal{S}_n(q) &= \mathsf{u}\,\mathcal{S}_{i-1}^{n+i}(t)\,\mathsf{d}\,\mathcal{S}_{n-i}^{t+B(i)+1}(r_{n-i}) \text{ and}\\ \mathcal{S}_n(q+1) &= \mathsf{u}\,\mathcal{S}_{i-1}^{n+i}(t+1)\,\mathsf{d}\,\mathcal{S}_{n-i}^{t+B(i)+2}(1).\end{aligned}$$

Since $\mathcal{S}_{n-i}^{t+B(i)+1}(r_{n-i}) = \mathcal{S}_{n-i}^{t+B(i)+2}(1)$, the result for $\mathcal{S}_n$ follows by the induction hypothesis applied to $\mathcal{S}_{i-1}^{n+i}$. Now if $t + B(i) + 2$ is odd, then

$$\begin{aligned}\Phi_n(q) &= \hat{\varphi}(\mathsf{u}\,\mathcal{S}_{i-1}^{n+i}(t)\,\mathsf{d})\,i\,(i-1)\,\ldots\,1 \text{ and}\\ \Phi_n(q+1) &= \hat{\varphi}(\mathsf{u}\,\mathcal{S}_{i-1}^{n+i}(t+1)\,\mathsf{d})\,i\,(i-1)\,\ldots\,1,\end{aligned}$$

where $\hat{\varphi}(\mathsf{u}\,\mathcal{S}_{i-1}^{n+i}(t)\,\mathsf{d})$ is $\varphi(\mathsf{u}\,\mathcal{S}_{i-1}^{n+i}(t)\,\mathsf{d})$ with every element incremented by i. Since $d(\mathcal{S}_{i-1}^{n+i}(t), \mathcal{S}_{i-1}^{n+i}(t+1)) \le 5$, we have that $d(\Phi_n(q), \Phi_n(q+1)) \le 5$. The case where $t + B(i) + 2$ is even is handled in a similar manner with the suffix $i(i-1)\ldots 1$ replaced by $(i-1)\ldots 1(i+1)$.

(v) If $\mathcal{S}_n(q)$ corresponds to Eq. (3.1) with triple $(i = t; j = r_{i-1}; k = r_{n-i})$, where $1 \le t < n$, then $\mathcal{S}_n(q+1)$ corresponds to Eq. (3.1) with triple $(i = t+1;\ j = 1; k = 1)$. Consequently

$$\begin{aligned}\mathcal{S}_n(q) &= \mathsf{u}\,\mathcal{S}_{t-1}^{n+t}(r_{t-1})\,\mathsf{d}\,\mathcal{S}_{n-t}^{r_{t-1}+B(t)+1}(r_{n-t}) \text{ and}\\ \mathcal{S}_n(q+1) &= \mathsf{u}\,\mathcal{S}_{t}^{n+t+1}(1)\,\mathsf{d}\,\mathcal{S}_{n-t-1}^{1+B(t+1)+1}(1).\end{aligned}$$

This divides into 4 subcases depending on the parity of the numbers $n + t$ and $r_{t-1} + B(t) + 1 = B(t+1) + 1$. Each case is easily resolved by applying Lemma 3.3.

(a) If $n + t$ is even and $B(t+1) + 1$ is even, then

$$\begin{aligned}\mathcal{S}_n(q) &= \mathsf{u}\,\mathcal{S}_{t-1}^{2}(r_{t-1})\,\mathsf{d}\,\mathcal{S}_{n-t}^{2}(r_{n-t}) \;=\; \mathsf{u}\,\mathsf{e}^{t-1}\,\mathsf{d}\,\mathsf{e}^{n-t} \text{ and}\\ \mathcal{S}_n(q+1) &= \mathsf{u}\,\mathcal{S}_{t}(1)\,\mathsf{d}\,\mathcal{S}_{n-t-1}(1) \;=\; \mathsf{u}\,\mathsf{e}^{t}\,\mathsf{d}\,\mathsf{e}^{n-t-1},\end{aligned}$$

which differ in two positions. This gives

$$\begin{aligned}\Phi_n(q) &= n\,(n-1)\,\ldots\,(n-t+1)\,(n+1)\,(n-t)\,(n-t-1)\,\ldots\,1 \text{ and}\\ \Phi_n(q+1) &= n\,(n-1)\,\ldots\,(n-t)\,(n+1)\,(n-t-1)\,\ldots\,1,\end{aligned}$$

for all $1 \le t \le n-1$. The two permutations differ by transposing the elements at positions $(t+1, t+2)$.

(b) If $n + t$ is odd and $B(t+1) + 1$ is odd, then

$$\begin{aligned}\mathcal{S}_n(q) &= \mathsf{u}\,\mathcal{S}_{t-1}(r_{t-1})\,\mathsf{d}\,\mathcal{S}_{n-t}(r_{n-t}) \;=\; \mathsf{u}\,\mathsf{u}\mathsf{e}^{t-2}\mathsf{d}\,\mathsf{d}\,\mathsf{u}\mathsf{e}^{n-t-1}\mathsf{d} \text{ and}\\ \mathcal{S}_n(q+1) &= \mathsf{u}\,\mathcal{S}_{t}^{2}(1)\,\mathsf{d}\,\mathcal{S}_{n-t-1}^{2}(1) \;=\; \mathsf{u}\,\mathsf{u}\mathsf{e}^{t-1}\mathsf{d}\,\mathsf{d}\,\mathsf{u}\mathsf{e}^{n-t-2}\mathsf{d},\end{aligned}$$

which differ in five positions. This gives

$$\begin{aligned}\Phi_n(q) &= (n-1)\cdots(n-t+2)(n-t)n(n+1)(n-t-1)\cdots 1(n-t+1) \text{ and}\\ \Phi_n(q+1) &= (n-1)\cdots(n-t+1)(n-t-1)n(n+1)(n-t-2)\cdots 1(n-t),\end{aligned}$$

for all $2 \le t \le n-2$. These two permutations differ in five places (a transposition of the positions $(t-1, n)$ and a cycle of three elements at positions $(t, t+1, t+2)$). For $t = 1$ we have

$$\begin{aligned}\Phi_n(q) &= n\,(n+1)\,(n-1)\,(n-2)\,\ldots\,1 \text{ and}\\ \Phi_n(q+1) &= (n-1)\,n\,(n+1)\,(n-2)\,\ldots\,1,\end{aligned}$$

which differ by a cycle of three elements at positions (1,2,3). Similarly, for $t = n-1$ we have

$$\begin{aligned}\Phi_n(q) &= (n-1)\,\ldots\,1\,(n+1)\,n \text{ and}\\ \Phi_n(q+1) &= (n-1)\,\ldots\,1\,n\,(n+1),\end{aligned}$$

which differ by transposing the entries in positions $(n, n+1)$.

(c) If $n+t$ is odd and $B(t+1)+1$ is even, then

$$\mathcal{S}_n(q) = \mathsf{u}\,\mathcal{S}_{t-1}(r_{t-1})\,\mathsf{d}\,\mathcal{S}^2_{n-t}(r_{n-t}) = \mathsf{u}\,\mathsf{u}\mathsf{e}^{t-2}\mathsf{d}\,\mathsf{d}\,\mathsf{e}^{n-t} \text{ and}$$
$$\mathcal{S}_n(q+1) = \mathsf{u}\,\mathcal{S}^2_{t}(1)\,\mathsf{d}\,\mathcal{S}_{n-t-1}(1) = \mathsf{u}\,\mathsf{u}\mathsf{e}^{t-1}\mathsf{d}\,\mathsf{d}\,\mathsf{e}^{n-t-1}.$$

Thus $\mathcal{S}_n(q+1)$ differs from $\mathcal{S}_n(q)$ in four positions. This gives

$$\Phi_n(q) = (n-1)\,\ldots\,(n-t+1)\,n\,(n+1)\,(n-t)\,\ldots\,1 \text{ and}$$
$$\Phi_n(q+1) = (n-1)\,\ldots\,(n-t)\,n\,(n+1)\,(n-t-1)\,\ldots\,1,$$

for all $t \geq 2$. The two permutations differ in three places (a rotation of three elements at positions $(t, t+1, t+2)$). The degenerate case $t=1$ is handled in the same manner as in part (a).

(d) If $n+t$ is even and $B(t+1)+1$ is odd, then

$$\mathcal{S}_n(q) = \mathsf{u}\,\mathcal{S}^2_{t-1}(r_{t-1})\,\mathsf{d}\,\mathcal{S}_{n-t}(r_{n-t}) = \mathsf{u}\,\mathsf{e}^{t-1}\,\mathsf{d}\,\mathsf{u}\mathsf{e}^{n-t-1}\mathsf{d} \text{ and}$$
$$\mathcal{S}_n(q+1) = \mathsf{u}\,\mathcal{S}_{t}(1)\,\mathsf{d}\,\mathcal{S}^2_{n-t-1}(1) = \mathsf{u}\,\mathsf{e}^{t}\,\mathsf{d}\,\mathsf{u}\,\mathsf{e}^{n-t-2}\,\mathsf{d}.$$

Thus $\mathcal{S}_n(q+1)$ differs from $\mathcal{S}_n(q)$ in five positions. This gives

$$\Phi_n(q) = n(n-1)\cdots(n-t+2)(n-t)(n+1)(n-t-1)\cdots 1(n-t+1)$$

and

$$\Phi_n(q+1) = n(n-1)\cdots(n-t+1)(n-t-1)(n+1)(n-t-2)\cdots 1(n-t),$$

for all $t \leq n-2$. The two permutations differ in four places (the two disjoint transpositions of elements at positions $(t, n+1)$ and $(t+1, t+2)$). The degenerate case $t = n-1$ is handled in the same manner as in part (a). □

The lists $\mathcal{S}_3$, Φ_3, $\mathcal{S}_4$ and Φ_4 are given in Tables 2 and 3. Note that, unlike Φ_n, the list $\mathcal{S}_n$ is a circular Gray code; its first and last element have distance at most five. The choice of a Gray code for Schröder paths is critical in our construction of a Gray code for $\mathfrak{S}_n(1243, 2143)$ since Egge and Mansour's bijection φ, generally, does not preserves distances. For instance $d(\mathsf{e}^n, \mathsf{u}\mathsf{e}^{n-1}\mathsf{d}) = 2$ but $\varphi(\mathsf{e}^n) = (n+1)n\ldots 1$ differs from $\varphi(\mathsf{u}\mathsf{e}^{n-1}\mathsf{d}) = n\ldots 1(n+1)$ in all positions. Also, there already exists a distance-5 Gray code for Schröder paths [22] but it is not transformed into a Gray code for $\mathfrak{S}_n(1243, 2143)$ by a known bijection. Finally, as in the previous section, both Gray codes presented above can be implemented in exhaustive generating algorithms.

4. Regular patterns and Gray codes

Here we present a general generating algorithm and Gray codes for permutations avoiding a set of patterns T, provided T satisfies certain constraints. The operations of reverse, complement and their composition extend these to codes for T^c, T^r and T^{rc}. Our approach is based on generating trees; see [1,6,7,25] and the references therein. In [6] a general Gray code for a very large family of combinatorial objects is given; objects are encoded by their corresponding path in the generating tree and often it is possible to translate the obtained codes into codes for objects. The method we present here is, in a way, complementary to that of [6]: it works for a large family of patterns and objects are produced in 'natural' representation. It is also easily implemented by efficient generating algorithms. Its disadvantage is, for example, that it gives a distance-5 Gray code for $\mathfrak{S}(231)$, and so is less optimal than the one given in Section 2; and it does not work for $T = \{1243, 2143\}$ (the set of patterns considered in Section 3) since T does not satisfy the required criteria.

We begin by explaining the generating tree technique in the context of pattern avoidance. The *sites* of $\pi \in \mathfrak{S}_n$ are the positions between two consecutive entries, as well as before the first and after the last entry; and they are numbered, from right to left, from 1 to $n+1$. For a permutation $\pi \in \mathfrak{S}_n(T)$, with T a set of forbidden patterns, i is an *active site* if the permutation obtained from π by inserting $n+1$ into its ith site is a permutation in $\mathfrak{S}_{n+1}(T)$; we call such a permutation in $\mathfrak{S}_{n+1}(T)$ a *son* of π. Clearly, if $\pi \in \mathfrak{S}_{n+1}(T)$, by erasing $n+1$ in π one obtains a permutation in $\mathfrak{S}_n(T)$; or equivalently, any permutation in $\mathfrak{S}_{n+1}(T)$ is obtained from a permutation in $\mathfrak{S}_n(T)$ by inserting $n+1$ into one of its active sites. The active sites of a permutation $\pi \in \mathfrak{S}_n(T)$ are *right justified* if the sites to the right of

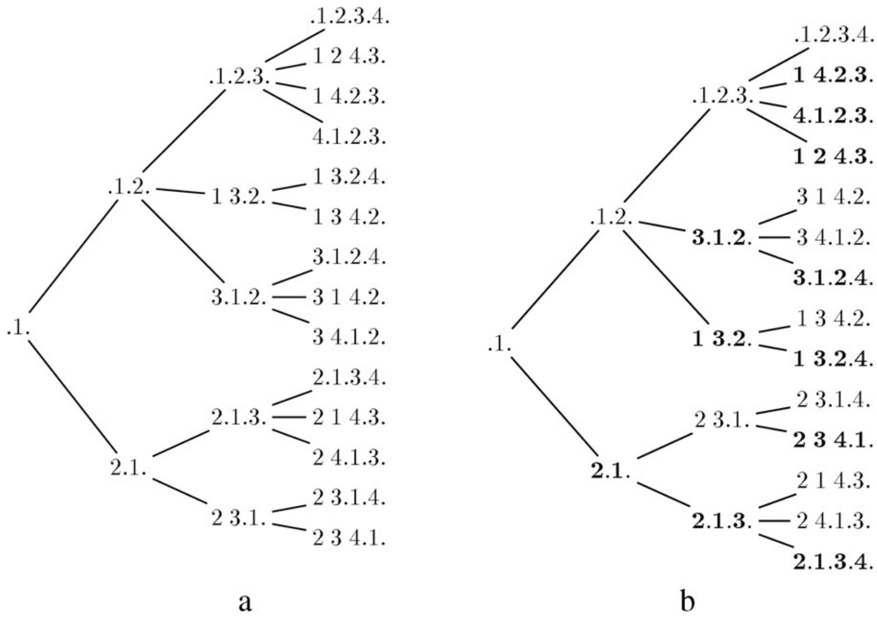


Fig. 1. (a) The generating tree induced by the call of Gen_Avoid(1,2) for $n = 4$ and with χ defined by: $\chi(1,k) = k+1$ and $\chi(i,k) = i$ if $i \neq 1$. It corresponds to the forbidden pattern $T = \{321\}$. The active sites are represented by a dot. (b) The first four levels of the generating tree induced by the definition (4.2) with the same function χ; they yield the lists $\mathcal{C}_i(321)$ for the sets $\mathfrak{S}_i(321)$, $1 \leq i \leq 4$. This tree is the Gray-code ordered version of the one in (a). Permutations in bold have direction *down* and the others direction *up*.

any active site are also active. We denote by $\chi_T(i,\pi)$ the number of active sites of the permutation obtained from π by inserting $n+1$ into its ith active site.

A set of patterns T is called *regular* if for any $n \geq 1$ and $\pi \in \mathfrak{S}_n(T)$

- π has at least two active sites and they are right justified;
- $\chi_T(i,\pi)$ does not depend on π but only on the number k of active sites of π; in this case we denote $\chi_T(i,\pi)$ by $\chi_T(i,k)$.

In what follows we shall assume that T is a regular set of patterns. Several examples of regular patterns T, together with their respective χ functions, are given at the end of this section.

Now we will describe an efficient (constant amortized time) generating algorithm for permutations avoiding a regular set of patterns; then we show how we can modify it to obtain Gray codes. If $n = 1$, then $\mathfrak{S}_n(T) = \{(1)\}$; otherwise $\mathfrak{S}_n(T) = \cup_{\pi \in \mathfrak{S}_{n-1}(T)}\{\sigma \in \mathfrak{S}_n \mid \sigma \text{ is a son of } \pi\}$. An efficient implementation is based on the following considerations and its pseudocode is given in Algorithm 2. The permutation obtained from $\pi \in \mathfrak{S}_{n-1}(T)$ by inserting n into its first (rightmost) active site is πn. Let σ (resp. τ) be the permutation obtained from π by inserting n into the ith (resp. $(i+1)$th) active site of π. In this case τ is obtained by transposing the entries in positions $n-i+1$ and $n-i$ of σ. In addition, if $\chi_T(i,k)$ is calculable, from i and k, in constant time, then the obtained algorithm, Gen_Avoid (Algorithm 2), runs in constant amortized time. Indeed, this algorithm satisfies the following properties:

- the total amount of computation in each call is proportional with the number of direct calls produced by this call,
- each non-terminal call produces at least two recursive calls (i.e., there is no call of degree one), and
- each terminal call (degree-zero call) produces a new permutation,

see for instance [19] and Fig. 1(a) for an example.

Now we show how one can modify the generating procedure Gen_Avoid sketched above in order to produce a Gray-code listing. We associate to each permutation $\pi \in \mathfrak{S}_n(T)$

- a *direction*, *up* or *down*, and we denote by π^1 the permutation π with direction *up* and by π^0 the permutation π with direction *down*. A permutation together with its direction is called *directed permutation*.

- a list of successors, each of them a permutation in $\mathfrak{S}_{n+1}(T)$. The first permutation in the list of successors of π^1 has direction *up* and all others have direction *down*. The list of successors of π^0 is obtained by reversing the list of successors of π^1 and then reversing the direction of each element of the list.

Let $\pi \in \mathfrak{S}_n(T)$ with k successors (or, equivalently, k active sites), and L_k be the unimodal sequence of integers

$$L_k = \begin{cases} 1, 3, 5, \ldots, k, (k-1), (k-3), \ldots, 4, 2 & \text{if } k \text{ is odd} \\ 1, 3, 5, \ldots, (k-1), k, (k-2), \ldots, 4, 2 & \text{if } k \text{ is even.} \end{cases} \tag{4.1}$$

This list is very important in our construction of a Gray code; it has the following critical properties, independent of k: it begins and ends with the same element, and the difference between two consecutive elements is less than or equal to 2.

For a permutation π with k active sites, the list of successors of π^1, denoted by $\phi(\pi^1)$, is a list of k directed permutations in $\mathfrak{S}_{n+1}(T)$: its jth element is obtained from π by inserting $n+1$ in the $L_k(j)$th active site of π; and as stated above, the first permutation in $\phi(\pi^1)$ has direction *up* and all others have direction *down*. Also we extend ϕ in a natural way to lists of directed permutations: $\phi(\pi(1), \pi(2), \ldots)$ is simply the list $\phi(\pi(1)), \phi(\pi(2)), \ldots$. This kind of distribution of directions among the successors of an object is similar to that of [26].

Let $d_n = \mathrm{card}(\mathfrak{S}_n(T))$ and define the list

$$\mathcal{C}_n(T) = \mathcal{C}_n = \bigoplus_{q=1}^{d_{n-1}} \phi(\mathcal{C}_{n-1}(q)) \tag{4.2}$$

where $\mathcal{C}_n(q)$ is the qth directed permutation of $\mathcal{C}_n$, anchored by $\mathcal{C}_1 = (1)^1$. We will show that the list of permutations in $\mathcal{C}_n$ (regardless of their directions) is a Gray code with distance 5 for the set $\mathfrak{S}_n(T)$. With these considerations in mind we have

Lemma 4.1.

- *The list $\mathcal{C}_n$ contains all T-avoiding permutations exactly once;*
- *The first permutation in $\mathcal{C}_n$ is $(1, \ldots, n)$ and the last one is $(2, 1, 3, \ldots, n)$.*

Lemma 4.2. *If π^i is a directed permutation in $\mathcal{C}_n$ (that is, π is a length n permutation and $i \in \{0, 1\}$ is a direction), then two successive permutations in $\phi(\pi^i)$, say σ and τ, differ in at most three positions.*

Proof. Since $\phi(\pi^0)$ is the reverse of $\phi(\pi^1)$ it is enough to prove the statement for $i = 1$; so suppose that $i = 1$. Let σ and τ be the permutations obtained by inserting $n+1$ in the $L_k(j)$th and $L_k(j+1)$th active site of π, respectively, for some j. Since $|L_k(j) - L_k(j+1)| \le 2$, $d(\sigma, \tau) \le 3$. □

Let $\pi^i \in \mathcal{C}_n$ and $\ell(\pi^i)$ denote the first (leftmost) element of the list $\phi(\pi^i)$, $\ell^2(\pi^i) = \ell(\ell(\pi^i))$, and $\ell^s(\pi^i) = \ell(\ell^{s-1}(\pi^i))$. Similarly, $r(\pi^i)$ denotes the last (rightmost) element of the list $\phi(\pi^i)$, and $r^s(\pi^i)$ is defined analogously. For $\pi^i \in \mathcal{C}_n$ let $\mathrm{dir}(\pi^i) = i \in \{0, 1\}$. By the recursive application of the definition of the list $\phi(\pi^i)$ we have the following lemma.

Lemma 4.3. *If $\pi^i \in \mathcal{C}_n$, then $\mathrm{dir}(\ell^s(\pi^i)) = 1$ and $\mathrm{dir}(r^s(\pi^i)) = 0$ for any $s \ge 1$.*

Proof. $\ell(\pi^i)$, the first successor of π^i has direction *up* for any $i \in \{0, 1\}$, and generally $\mathrm{dir}(\ell^s(\pi^i)) = 1$ for $s \ge 1$. Similarly, $r(\pi^i)$, the last successor of π^i has direction *down* for any $i \in \{0, 1\}$, and $\mathrm{dir}(r^s(\pi^i)) = 0$ for $s \ge 1$. □

Lemma 4.4. *If $\sigma, \tau \in \mathfrak{S}_n(T)$ and $d(\sigma, \tau) \le p$, then, for $s \ge 1$,*

$$d(r^s(\sigma^0), \ell^s(\tau^1)) \le p.$$

Proof. $r(\sigma^0) = (\sigma, (n+1))^0$ and $\ell(\tau^1) = (\tau, (n+1))^1$. Induction on s completes the proof. □

Theorem 4.5. *Two consecutive permutations in $\mathcal{C}_n$ differ in at most five positions.*

Table 4
The Gray-code list $\mathcal{C}_5(321)$ for the set $\mathfrak{S}_5(321)$ given by relation (4.2) and with succession function χ in Section 4.1

12345	45123	**34152**	**13245**	**24513**
12534	41523	31254	23145	**24153**
51234	**41235**	35124	**23514**	21354
15234	12453	31524	**23154**	25134
12354	**12435**	**31245**	23451	21534
14253	31425	13425	**23415**	**21345**
14523	**31452**	**13452**	21435	
14235	34125	13254	**21453**	
41253	**34512**	13524	24135	

Permutations are listed column-wise in 14 groups; each group contains the sons of a same permutation in $\mathfrak{S}_4(321)$, see Fig. 1 b. In bold are permutations with direction *down* and the others with direction *up*.

Algorithm 2 Pseudocode for generating permutations avoiding a set T of regular patterns characterized by the succession function $\chi(i, k)$. After the initialization of π by the length 1 permutation [1], the call of Gen_Avoid(1, 2) produces $\mathfrak{S}_n(T)$. Its ordered version, as described in Section 4, produces distance-5 Gray codes.

```
procedure Gen_Avoid(size, k)
  if size = n then
    Print(π)
  else
    size := size + 1
    π := [π, size]
    Gen_Avoid(size, χ(1, k))
    for i := 1 to k − 1 do
      π := (size − i + 1, size − i) ∘ π
      Gen_Avoid(size, χ(i + 1, k))
    end for
    for i := k − 1 to 1 by −1 do
      π := (size − i + 1, size − i) ∘ π
    end for
  end if
end procedure
```

Proof. Let σ^i and τ^j be two consecutive elements of $\mathcal{C}_n$. If there is a $\pi^m \in \mathcal{C}_{n-1}$ such that $\sigma^i, \tau^j \in \phi(\pi^m)$, then, by Lemma 4.2, σ and τ differ in at most three positions. Otherwise, let π^m be the closest common ancestor of σ^i and τ^j in the generating tree, that is, π is the longest permutation such that there exists a direction $m \in \{0, 1\}$ with $\sigma^i, \tau^j \in \phi(\phi(\dots\phi(\pi^m)\dots))$. In this case, there exist α^a and β^b successive elements in $\phi(\pi^m)$ (so that α and β differ in at most three positions) and an $s \geq 1$ such that $\sigma^i = r^s(\alpha^a)$ and $\tau^j = \ell^s(\beta^b)$.

If $s = 1$, then σ and τ are obtained from α and β by the insertion of their largest element in the first or second active site, according to a and b; in these cases σ and τ differ in at most five positions. (Actually, if $a = b$, then σ and τ differ as α and β, that is, in at most three positions.)

If $s > 1$, by Lemma 4.3, $\text{dir}(r(\alpha^a)) = \cdots = \text{dir}(r^s(\alpha^a)) = 0$ and $\text{dir}(\ell(\beta^b)) = \cdots = \text{dir}(\ell^s(\beta^b)) = 1$. Since $r(\alpha^a)$ and $\ell(\beta^b)$ differ in at most five positions, by Lemma 4.4, so are σ and τ. $\square$

The first and last permutations in $\mathcal{C}_n$ have distance two, so $\mathcal{C}_n$ is a circular Gray code, see Table 4. The generating algorithm Gen_Avoid sketched in the beginning of this section and presented in Algorithm 2 can be easily modified to generate the list $\mathcal{C}_n(T)$ for any set of regular patterns: it is enough to change appropriately the order among its successive recursive calls by endowing each permutation with a direction as described above; see also Fig. 1.

4.1. Several well-known classes of regular patterns

Below we give several classes of regular patterns together with the χ function. For each class, a recursive construction is given in the corresponding reference(s); it is based (often implicitly) on the distribution of active sites of the permutations belonging to the class. It is routine to express these recursive constructions in terms of χ functions and check the regularity of each class.
Classes given by counting sequences:

(i) 2^{n-1} [4].
$T = \{321, 312\}$, $\chi_T(i,k) = 2$

(ii) Pell numbers [4].
$T = \{321, 3412, 4123\}$, $\chi_T(i,k) = \begin{cases} 3 & \text{if } i = 1 \\ 2 & \text{otherwise} \end{cases}$

(iii) even-index Fibonacci numbers [4].
- $T = \{321, 3412\}$, $\chi_T(i,k) = \begin{cases} k+1 & \text{if } i = 1 \\ 2 & \text{otherwise} \end{cases}$
- $T = \{321, 4123\}$, $\chi_T(i,k) = \begin{cases} 3 & \text{if } i = 1 \\ i & \text{otherwise} \end{cases}$

(iv) Catalan numbers [17,25].
- $T = \{312\}$, $\chi_T(i,k) = i + 1$
- $T = \{321\}$, $\chi_T(i,k) = \begin{cases} k+1 & \text{if } i = 1 \\ i & \text{otherwise} \end{cases}$

(v) Schröder numbers [11].
- $T = \{4321, 4312\}$, $\chi_T(i,k) = \begin{cases} k+1 & \text{if } i = 1 \text{ or } i = 2 \\ i & \text{otherwise} \end{cases}$
- $T = \{4231, 4132\}$, $\chi_T(i,k) = \begin{cases} k+1 & \text{if } i = 1 \text{ or } i = k \\ i+1 & \text{otherwise} \end{cases}$
- $T = \{4123, 4213\}$, $\chi_T(i,k) = \begin{cases} k+1 & \text{if } i = k-1 \text{ or } i = k \\ i+2 & \text{otherwise} \end{cases}$

(vi) central binomial coefficients $\binom{2n-2}{n-1}$ [11].
- $T = \{4321, 4231, 4312, 4132\}$, $\chi_T(i,k) = \begin{cases} k+1 & \text{if } i = 1 \\ 3 & \text{if } i = 2 \\ i & \text{otherwise} \end{cases}$
- $T = \{4231, 4132, 4213, 4123\}$, $\chi_T(i,k) = \begin{cases} 3 & \text{if } i = 1 \\ i+1 & \text{otherwise.} \end{cases}$

Variable length patterns:

(a) $T = \{321, (p+1)12\ldots p\}$, $\chi_T(i,k) = \begin{cases} k+1 & \text{if } i = 1 \text{ and } k < p \\ p & \text{if } i = 1 \text{ and } k = p \\ i & \text{otherwise.} \end{cases}$
See for instance [7,4]. If $p = 2$, then we retrieve the case (i) above; $p = 3$ corresponds to $T = \{321, 4123\}$ in case (iii); and $p = \infty$ corresponds to $T = \{321\}$ in case (iv).

(b) $T = \{321, 3412, (p+1)12\ldots p\}$, $\chi_T(i,k) = \begin{cases} k+1 & \text{if } i = 1 \text{ and } k < p \\ p & \text{if } i = 1 \text{ and } k = p \\ 2 & \text{otherwise.} \end{cases}$
See for instance [4]. If $p = 2$, then we retrieve the case (i) above; if $p = 3$, the case (ii); and $p = \infty$ corresponds to $T = \{321, 3412\}$ in case (iii).

(c) $T = \cup_{\tau \in \mathfrak{S}_{p-1}} \{(p+1)\tau p\}$.
$\chi_T(i,k) = \begin{cases} k+1 & \text{if } k < p \text{ or } i > k-p+1 \\ i+p-1 & \text{otherwise.} \end{cases}$
See [3,15,16]. If $p = 2$, then we retrieve the case $T = \{312\}$ in point (iv) above; and $p = 3$ corresponds to $T = \{4123, 4213\}$ in point (v).

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