Discrete Mathematics 33 (1981) 323–325
North-Holland Publishing Company

COMMUNICATION

ON CROSS-BANDWIDTH

J. KAHN* and D.J. KLEITMAN*

Mathematics Department, Massachusetts Institute of Technology, Cambridge, MA 02139, USA

Communicated by R.L. Graham
Received 5 August 1980

The notion of cross-bandwidth is introduced, and it is shown that any graph that is suitably contractible to a k-connected graph has cross-bandwidth at least k. The contracted edges must induce in the original graph a subgraph of maximum degree at most one. This is used to resolve a conjecture of Erdös and Chinn on the bandwidth of certain graphs.

For G a graph on n vertices, and f an assignment of the integers from 1 to n to the vertices of G, let $M(f)$ be the maximum over edges (v, w) of $|f(v)-f(w)|$. The *bandwidth* of G is the minimum over f of $M(f)$.

Finding the bandwidth of a graph is in general an NP-complete problem. It can even be difficult to find the bandwidth of relatively simple graphs.

For example, Erdös and Chinn have raised the question, what is the bandwidth of the bipartite graph G_m, having $2m$ vertices u_i and v_i for $1 \leq i \leq m$, with an edge between u_i and v_j whenever $i \leq j$? One can easily find an f for G_m having $M(f)=m-1$ for $m \geq 2$. They then asked: is $m-1$ the bandwidth of G_m? The reason for this question, as pointed out to the authors by E. Milner, is that an affirmative answer—one is given below—implies that the sum of the bandwidths of G and $\bar{G}$ is at least $2\,([\frac{1}{2}|V(G)|]-1)$. (The reader may easily convince himself of this fact. The reverse inequality is easily seen to hold for $G=G_m$, so that the bound is in fact sharp.)

For G a graph let $\bar{G}$ and G^k denote respectively the complementary graph of G and the graph with vertex set $V(G)$ and v joined to w iff v and w are at distance at most k in G. Let P_n denote a path with n edges. Then $G_m=\overline{P_{2m}^{m-1}}$, and we have from R. Graham the more general question does: $\overline{P_n^k}$ have bandwidth $n-k-2$ if $1 \leq k \leq n-3$? (Again it is easy to show this is an upper bound.)

In this note we introduce the concept of *cross-bandwidth*, and show that the cross-bandwidth of a graph that can be contracted in a certain way to one that is "k-connected" is at least k. (A "k-connected" graph here will be one having at least $k+1$ vertices that remains connected after the removal of any $k-1$ of them.) A special case of this result is that the cross-bandwidth (a fortiori the

* Supported in part by ONR Contract #N00014-76-C-0366.

bandwidth) of $\overline{P_n^k}$ is at least $n-k-2$ for $\frac{1}{2}n-1 \leq k \leq n-3$, which provides an affirmative answer to the above questions for k in this range.

We also give an entirely different short proof that the bandwidth of $\overline{P_n^k}$ is at least $n-k-2$ $(1 \leq k \leq n-3)$.

The *cross-bandwidth* of a graph G on n vertices is defined as the minimum over $f: V(G) \to \{1, \ldots, n\}$ of $\bar{M}(f)$, where $\bar{M}(f)$ is the maximum of $|f(v)-f(w)|$ taken over edges (v, w) with

$$1 \leq f(v) \leq m, \quad m+1 \leq f(w) \leq 2m \qquad \text{if} \quad n=2m,$$

or

$$1 \leq f(v) \leq m, \quad m+1 \leq f(w) \leq 2m+1 \qquad \text{if} \quad n=2m+1.$$

(of course for $n=2m+1$ we get the same number by restricting $1 \leq f(v) \leq m+1$ and $m+2 \leq f(w) \leq 2m+1$). Thus differences across edges both of whose ends are at most $[\frac{1}{2}n]$, or both of whose ends are at least $[\frac{1}{2}n]+1$ do not count in cross-bandwidth, though they do in bandwidth. Complexity of computation of cross-bandwidth may be worth investigating.

We now prove

Theorem 1. *Suppose there exists a contraction c of G onto a k-connected G' for which $|c^{-1}(x)| \leq 2$ for all $x \in V(G')$. Then G has cross-bandwidth at least k. (By a contraction of a graph G we mean a surjective map $C: V(G) \to V(H)$, (H another graph) satisfying* (i) *for $v \in V(H): c^{-1}(v)$ is connected in G, and* (ii) *$(v', w') \in E(H)$ iff there exists $(v, w) \in E(G)$ with $c(v)=v'$ and $c(w)=w'$.)*

Proof. The result is easily verified for $k=1$ or 2. For larger k we induct. Let $f: V(G) \to \{1, \ldots, |V(G)|\}$ satisfy $\bar{M}(f)=$ cross-bandwidth of G.

We will have to enumerate a number of cases, the procedure in each being as follows: delete a specified set W of vertices forming a graph G_1. Define $f': V(G_1)' \to \{1, 2, \ldots, |V(G) \setminus W|\}$ so that for $v, w \in V(G_1)$, $f'(v) < f'(w)$ iff $f(v) < f(w)$. The result then follows by applying the inductive hypothesis to G_1. In the various cases below we just specify W.

Suppose $|G|$ is even, equal to $2n$, and let $f: V(G) \to \{1, \ldots, 2n\}$ attain the cross-bandwidth of G. Let $v=f^{-1}(n)$, $x=f^{-1}(n+1)$, $\{v, w\}=c^{-1}(c(v))$, $\{x, y\}=c^{-1}(c(x))$. If $v=w$ take $W=\{v\}$. If $x=y$, take $W=\{x\}$. If $f(w)>n$ take $W=\{v, w\}$. If $f(y)<n$ take $W=\{x, y\}$. If none of these, take $W=\{v, w, x, y\}$.

Now suppose $|G|$ is odd, equal to $2n+1$, and let $f: V(G) \to \{1, 2, \ldots, 2n+1\}$ satisfy $\bar{M}(f)=$ cross-bandwidth of G. Let $v=f^{-1}(n+1)$, and $\{v, w\}=c^{-1}(c(v))$. There are three cases: $v=w$, $f(w)<n+1$, $f(w)>n+1$, but in each case we can induct with $W=\{v, w\}$.

We may now deduce that $\overline{P_n^k}$ has cross-bandwidth at least $n-k-2$ by observing that $\overline{P_n^k}$ or one of its subgraphs can always be contracted (in the appropriate way) to an $(n-k-2)$-connected graph, and of course this gives the result on bandwidth as well. There is, however, an entirely different approach that can be applied to $\overline{P_n^k}$, which we include for its elegance, and for the fact that it handles the $k<\frac{1}{2}n-1$ cases.

Theorem 2. *The bandwidth of* $\overline{P_n^k}$ *is* $n-k-2$ *for* $1 \leqslant k \leqslant n-3$.

Proof. Let $P_n=(u_1,\ldots,u_n)$. For each $j=1,\ldots,n-k-1$, $\{u_1,\ldots,u_j\} \cup \{u_{j+k+1},\ldots,u_n\}$ induces a complete bipartite subgraph of $\overline{P_n^k}$. If any f is to have $M(f)<n-k-2$, the smallest two and largest two of the f values taken by these vertices must all be taken in the same block of the bipartition. If we change the value of j by one, we remove a vertex from one block of the bipartition and add one to the other, so that if the smallest two and largest two values are again all in one block, it must be the same block as before. But this means they must be in the same block for all j, which is obviously absurd, since for $j=1$ there is only one vertex in the first block, and for $j=n-k-1$ only one in the second. This proves that the bandwidth of $\overline{P_n^k}$ is at least $n-k-2$. The reverse inequality is well known.

Somewhat longer proofs of this result have been obtained by F. Chung and R. Graham and by E.C. Milner and N. Sauer.

References

P. Erdös, private communication.
R. Graham, private communication.
E.C. Milner, private communication.