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Embedding Rings with Krull Dimension in Artinian Rings

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1. INTRODUCTION

Any commutative noetherian ring embeds as a subring in an artinian ring, and for a long time it was an open question whether this was also true for non-commutative noetherian rings. While such an artinian embedding exists for many classes of noetherian rings, Dean and Stafford [2] showed that there are exceptions. They did this by providing an example of a noetherian algebra over the field of complex numbers that does not admit a faithful Sylvester rank function, yet according to one of the main results in Schofield's book [14], the existence of such a function is necessary and sufficient for an algebra over a field to admit an embedding into an artinian ring, indeed into a simple artinian ring. On the positive side, Blair and Small [1] used Schofield's Theorem to prove that a right noetherian k-algebra R can be embedded in an artinian ring S such that ${}_RS$ is flat, whenever R contains no nonzero right ideal with reduced rank zero. In [9] it was shown that for a noetherian k-algebra R this latter condition is necessary as well. If, furthermore, R has finite Gelfand–Kirillov dimension, then an artinian overring can be obtained by embedding R in a finite

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direct sum of irreducible rings, each of which has an artinian quotient ring. As this construction does not make use of Schofield's result, one naturally tries to carry it out when R is a noetherian ring, but is no longer assumed to be an algebra over a field. The main obstacle in doing so is that a symmetric dimension function (such as the Gelfand–Kirillov dimension) is not available for noetherian rings in general, making it difficult to establish the incomparability condition for certain sets of prime ideals. Thus, we are more or less forced to assume such a condition as one of the hypotheses. However, it turns out that the ring need not be noetherian, it is enough to assume that it has right Krull dimension.

To date, no example has been found of a noetherian ring with two comparable prime ideals that belong to the same clique. Whether cliques always satisfy the incomparability condition is a long-standing open question (see, for example, [5, Problem 11, p. 288]) for noetherian rings. Thus, our result (Corollary 8) that a noetherian ring R embeds in an artinian ring, provided it satisfies the incomparability condition for cliques and has no nonzero right ideals of reduced rank zero, seems best possible under the circumstances.

2. DEFINITIONS AND NOTATIONS

All rings considered are associative with unit element 1, modules are unitary. For standard terminology the reader is referred to [5, 13].

Let R be a ring with right Krull dimension $\mathrm{Kdim}(R_R)$. The notion of the *reduced rank* ρ, originally defined for finitely generated modules over noetherian rings, can be extended to right R-modules with Krull dimension, as was shown by Lenagan in [11]. Thus, ρ is a function that assigns to each right R-module M with Krull dimension a nonnegative integer $\rho(M)$, such that $\rho(M) = \rho(K) + \rho(L)$ for each short exact sequence $0 \to K \to M \to L \to 0$. Recall that $\rho(M) = 0$ if and only if for each $m \in M$ there exists an element c, regular modulo the nil radical N of R, such that $mc = 0$. Associated with ρ is a Gabriel filter (see, e.g., [15]) $\mathscr{F}$, consisting of all right ideals F of R with $\rho(R/F) = 0$. The *ρ-closure*, or *$\mathscr{F}$-closure*, or simply *closure* in M of a submodule N of a right R-module M is the submodule

$$\mathrm{cl}_{\rho, M}(N) = \mathrm{cl}(N) = \{m \in M | mF \subseteq N \text{ for some } F \in \mathscr{F}\}.$$

The submodule N is called *ρ-closed* or *closed* in M if $N = \mathrm{cl}(N)$. A right R-module M is called *ρ-torsionfree*, or *without ρ-torsion* if $\mathrm{cl}(0) = 0$. Note that a right R-module M with Krull dimension satisfies the ascending chain condition as well as the descending chain condition for closed submodules.

If X is a subset of the ring R, and M and W are right and left R-modules, respectively, then

$$l_M(X) = \text{annihilator of } X \text{ in } M = \{m \in M \mid mX = 0\},$$
$$r_W(X) = \text{annihilator of } X \text{ in } W = \{w \in W \mid Xw = 0\}.$$

If X is a subset of the right R-module M, then $r_R(X)$ is the right annihilator of X in R, and similarly $l_R(X)$ denotes its left annihilator if M is a left R-module. The subscripts will be deleted if there is no danger of ambiguity. A prime ideal P of R is *associated* with the right R-module M if there exists a submodule $0 \neq N \subseteq M$ such that $P = r(N')$ for all submodules $0 \neq N' \subseteq N$.

$$\begin{aligned} \operatorname{Ass}(M) &= \text{set of associated primes of the } R\text{-module } M \\ \operatorname{Spec}(R) &= \text{set of all prime ideals of } R \\ \operatorname{minSpec}(R) &= \text{set of minimal primes of } R \\ N = \mathrm{N}(R) &= \text{nil radical of } R = \textstyle\bigcap_{P \in \operatorname{Spec}(R)} P \\ \mathrm{E}(M) &= \text{injective envelope of the module } M. \end{aligned}$$

If I is an ideal of the ring R, then

$$\begin{aligned} \mathscr{C}'(I) &= \{c \in R \mid cx \in I \text{ implies that } x \in I\}, \\ {}'\mathscr{C}(I) &= \{c \in R \mid xc \in I \text{ implies that } x \in I\}, \\ \mathscr{C}(I) &= {}'\mathscr{C}(I) \cap \mathscr{C}'(I). \end{aligned}$$

An *affiliated series* of a right R-module M is a sequence of submodules $0 = M_0 \subset M_1 \subset \cdots \subset M_{n-1} \subset M_n = M$, together with a set of prime ideals $\{P_1, \dots, P_n\}$ called *affiliated primes* such that each P_i is maximal in $\operatorname{Ass}(M/M_{i-1})$ and $M_i/M_{i-1} = l_{M/M_{i-1}}(P_i)$. The set of all affiliated primes of M, that is, the set of prime ideals that appear in some affiliated series of M, is denoted by $\operatorname{Aff}(M)$.

Let R be a ring, and let $Q, P \in \operatorname{Spec}(R)$. Then Q is *linked to* P (*via* A), denoted by $Q \rightsquigarrow P$, if A is an ideal with $QP \subseteq A \subset Q \cap P$, such that $Q \cap P/A$ is torsionfree as a right R/P-module (that is, no nonzero element is annihilated by an element in $\mathscr{C}(P)$) and fully faithful (that is, has no nonzero unfaithful submodules) as a left R/Q-module.

The *graph of links* or the *link graph* of R is the directed graph whose vertices are the elements of $\operatorname{Spec}(R)$, with an arrow from Q to P whenever $Q \rightsquigarrow P$. The connected components of this graph are called *cliques*. For $P \in \operatorname{Spec}(R)$, the (unique) clique containing P is denoted by $\operatorname{Cliq}(P)$.

A subset X of $\operatorname{Spec}(R)$ is said to be *right link closed* if whenever $P \in X$ and $Q \rightsquigarrow P$, it follows that $Q \in X$. The *right link closure* $\Omega^r(X)$ of a set X of primes is the smallest right link closed subset of $\operatorname{Spec}(R)$ containing X. Note that for a prime ideal P, $\Omega^r(P)$ consists of all primes $Q \in \operatorname{Spec}(R)$ for which there is a sequence of prime ideals $Q = Q_1, Q_2, \dots, Q_n = P$ with $Q_i \rightsquigarrow Q_{i+1}$ for $i = 1, \dots, n-1$. Obviously, $\Omega^r(P) \subseteq \operatorname{Cliq}(P)$.

A set X of prime ideals is said to satisfy the *incomparability condition* if $P \subseteq Q$ for $P, Q \in X$ implies that $P = Q$.

3. ρ-TORSIONFREE MODULES

Let R be a ring with right Krull dimension, and let M be a nonzero right R-module. By [6, Theorem 7.1], R satisfies the maximum condition for prime ideals, and $\operatorname{Ass}(M) \neq \varnothing$ by [6, Theorem 8.3]. Thus, one can attempt to construct an affiliated series for M in the usual way, but the construction may not reach the top in finitely many steps. Our first proposition shows that no problem arises for the case of an injective right R-module E with $\operatorname{cl}_{\rho, E}(0) = 0$, since it turns out that E has finite length as a module over its endomorphism ring. Note that this latter fact follows from a general result about Δ-modules: A quasi-injective right R-module M (where R is any ring) is a Δ-module (that is, R has DCC for annihilators of subsets of M), if and only if M has finite length as a module over $\operatorname{End}_R(M)$ (see, e.g., [3, Proposition 8.1, p. 31]). Note also that for the case $E = \operatorname{E}(R_R)$, Proposition 1 below specializes to Theorem 2.2 of [4].

PROPOSITION 1. *Let R be a ring with right Krull dimension, let E be an injective right R-module, let $H = \operatorname{End}_R(E)$, and let J denote the Jacobson radical of H. Then the following are equivalent.*

(i) $\operatorname{cl}_{\rho, E}(0) = 0$.

(ii) ${}_H E$ *has finite length* $\lambda \leq \rho(R_R)$, *and* $\operatorname{Ass}(E_R) \subseteq \operatorname{minSpec}(R)$.

(iii) ${}_H E$ *is noetherian, and* $\operatorname{Ass}(E_R) \subseteq \operatorname{minSpec}(R)$.

If any of the above hold, then J is nilpotent of nilpotency index $\leq \lambda$. Furthermore, if E_R has finite uniform dimension, then H/J is semisimple artinian, so H is semiprimary.

Proof. Note that $\rho(R_R) < \infty$, so the additivity of the reduced rank implies the ascending and descending chain conditions for ρ-closed right ideals.

(i) → (ii). Let N be a submodule of ${}_HE$. Since annihilators in R of subsets of N are closed right ideals of R, it follows that

$$r_R(N) = \bigcap_{x \in N} r(x) = \bigcap_{i=1}^{n} r(x_i)$$

for finitely many elements $x_i \in N$. Given any element $y \in l_E(r_R(N))$, the R-homomorphism ϕ: $(x_1, x_2, \dots, x_n)r \to yr$, $r \in R$, from

$$(x_1, x_2, \dots, x_n)R \subseteq x_1R \oplus x_2R \oplus \cdots \oplus x_nR \subseteq E^n$$

into E can be extended to an element of $\mathrm{Hom}_R(E^n, E) \cong H^n$. Thus, there exist $h_1, h_2, \dots, h_n \in H$ with

$$(h_1x_1 + h_2x_2 + \cdots + h_nx_n)r = \phi\big((x_1, x_2, \dots, x_n)r\big) = yr, \qquad \text{for all } r \in R.$$

In particular, $y = h_1x_1 + h_2x_2 + \cdots + h_nx_n$, proving that $l_E(r_R(N)) = Hx_1 + \cdots + Hx_n = N$. Thus, if $N \subset M \subseteq_H E$, then the closed right ideal $r_R(N)$ properly contains the closed right ideal $r_R(M)$, so any ascending or descending chain of submodules of ${}_HE$ can have at most $\rho(R_R)$ proper inclusions. It is clear that any $P \in \mathrm{Ass}(E_R)$ must be a minimal prime, since non-minimal primes are in $\mathscr{F}$.

(ii) → (iii). This is trivial.

(iii) → (i). Assume that $K = \mathrm{cl}_{\rho, E}(0) \neq 0$. Note that K is an H-R-sub-bimodule of ${}_HE_R$. Since ${}_HK$ is finitely generated, $r_R(K) \in \mathscr{F}$. Since $r_R(K) \subseteq P$ for some $P \in \mathrm{Ass}(E)$, this would imply that a minimal prime belongs to $\mathscr{F}$, which is impossible.

It is clear that $J^\lambda E = 0$, where λ denotes the composition length of ${}_HE$. Finally, if E_R has finite uniform dimension, then it is well known that H/J is semisimple artinian (see, for example, [10, Proposition 2, p. 103]).

In [9, Theorem 3.5], it was shown that for a right noetherian ring R whose right ideals are finitely annihilated, the condition $\mathrm{cl}_{\rho, R_R}(0) = 0$ is equivalent to $\mathrm{Ass}(R_R) \subseteq \mathrm{minSpec}(R)$. Using the above proposition, this can be generalized as follows.

COROLLARY 2. *Let R be a ring with right Krull dimension. Then the following statements are equivalent.*

(i) $\mathrm{Ass}(R_R) \subseteq \mathrm{minSpec}(R)$, *and right ideals of R are finitely annihilated.*

(ii) $\mathrm{cl}_{\rho, R_R}(0) = 0$.

Proof. (i) → (ii). Assume that $K = \mathrm{cl}_{\rho, R_R}(0) \neq 0$, so $r(K)$, being the intersection of finitely many right ideals in $\mathscr{F}$, would be in $\mathscr{F}$. As above, this is impossible, because it would give a minimal prime ideal in $\mathscr{F}$.

(ii) → (i). Let $E = \mathrm{E}(R_R)$, $H = \mathrm{End}_R(E)$, and note that $\mathrm{cl}_{\rho, E}(0) = 0$, as R_R is essential in E. By Proposition 1, $\mathrm{Ass}(E) \subseteq \mathrm{minSpec}(R)$, and ${}_H E$ is noetherian. Thus, for any right ideal A of R,

$$HA = Ha_1 + \cdots + Ha_n, \qquad a_i \in A.$$

Hence $r_R(A) = r_R(HA) = r(a_1) \cap \cdots \cap r(a_n)$, so A is finitely annihilated.

As a consequence of Proposition 1, given a ring R with right Krull dimension, a ρ-torsionfree injective right R-module E with endomorphism ring H has an affiliated series, whose members, being H-R-bimodules, have finite length as left H-modules. Although we shall not make use of this fact later on, it turns out that the sets of primes arising from any two affiliated series of E are identical. This is a consequence of the following lemma, which is a slight generalization of [5, Lemma 7.20].

LEMMA 3. *Let R be a ring with right Krull dimension, let ${}_H B_R$ be a bimodule such that ${}_H B$ has finite length, and let*

$$0 = V_0 \subset V_1 \subset \cdots \subset V_{i-1} \subset V_i \subset \cdots \subset V_{n-1} \subset V_n = B$$

be a sequence of sub-bimodules with $P_i = r(V_i/V_{i-1}) = \mathrm{Ass}(V_i/V_{i-1})$ for all $1 \leq i \leq n$. Then $\{P_1, \dots, P_n\} = \mathrm{Aff}(B_R)$.

Proof. Let $0 = W_0 \subset \cdots \subset W_{j-1} \subset W_j \subset \cdots \subset W_m = B$ be a bimodule composition series of B with $Q_j = r(W_j/W_{j-1}) = \mathrm{Ass}(W_j/W_{j-1})$. By the Jordan−Hölder Theorem, it suffices to show that $\{P_1, \dots, P_n\} = \{Q_1, \dots, Q_m\}$, and it may be assumed that the composition series is a refinement of the series of the V_i's. Furthermore, we may reduce to the case $n = 1$. Obviously, $P_1 = r(V_1) = r(B) \subseteq r(W_j) = Q_j$ for all j. Assume that $P_1 \subset Q_j$ for some j, and let $c \in Q_j \cap \mathscr{C}(P_1)$. Since ${}_H B$ is noetherian, B_R is torsionfree as an R/P_1-module, so right multiplication by c induces an H-isomorphism, whence $W_j \cong W_j c \subseteq W_{j-1}$, which is impossible in a module of finite length.

The next result is the key for obtaining an embedding of a ρ-torsionfree ring R with right Krull dimension into a right artinian ring, because it describes the set of regular elements of R in terms of the affiliated primes of $\mathrm{E}(R_R)$.

THEOREM 4. *Let R be a ring with right Krull dimension, and let ${}_H B_R$ be a bimodule such that ${}_H B$ has finite length and $\mathrm{cl}_{\rho, B_R}(0) = 0$. Let*

$$0 = V_0 \subset V_1 \subset \cdots \subset V_{i-1} \subset V_i \subset \cdots \subset V_{n-1} \subset V_n = B$$

be a sequence of sub-bimodules with $P_i = r(V_i/V_{i-1}) = \mathrm{Ass}(V_i/V_{i-1})$ *for all* $1 \le i \le n$. *Then*

(i) $(V_i/V_{i-1})c = V_i/V_{i-1}$ *for any* $c \in \mathscr{C}(P_i)$, $i = 1, \dots, n$.

(ii) $V_i c = V_i$ *for any* $c \in \bigcap_{j=1}^{i} \mathscr{C}(P_j)$, $i = 1, \dots, n$.

(iii) $\mathscr{C}(r_R(V_i)) = \bigcap_{j=1}^{i} \mathscr{C}(P_j)$, $i = 1, \dots, n$.

Proof. (i) Set $X_i = V_i/V_{i-1}$, and observe that as an R/P_i-module X_i is torsionfree, since ${}_H X_i$ is finitely generated and since P_i is the unique associated prime of X_i. Thus, for any $c \in \mathscr{C}(P_i)$, the map $x \to xc$, $x \in X_i$, is a left H-module isomorphism from X_i onto $X_i c \subseteq X_i$, so $X_i c = X_i$, as ${}_H X_i$ is artinian.

(ii) Proceed by induction on i, the case $i = 1$ being a consequence of (i). Assume that the statement is true for $i \ge 1$, and let $c \in \bigcap_{j=1}^{i+1} \mathscr{C}(P_j)$. Then

$$\frac{V_{i+1}}{V_i} = \frac{V_{i+1}}{V_i} c = \frac{V_{i+1}c + V_i}{V_i}$$

by (i), so

$$V_{i+1} = V_{i+1}c + V_i = V_{i+1}c + V_i c = V_{i+1}c.$$

(iii) Let $c \in \mathscr{C}(r(V_i))$. Then as right $R/r(V_i)$-modules and hence also as right R-modules

$$\frac{R}{r(V_i)} \cong [c + r(V_i)] \frac{R}{r(V_i)} = \frac{cR + r(V_i)}{r(V_i)},$$

whence $\rho(R/cR + r(V_i)) = 0$, and consequently also $\rho(R/cR + r(V_j)) = 0$ for any $j \le i$, as $r(V_i) \subseteq r(V_j)$. Since B is ρ-torsionfree, the map $v \to vc$, $v \in V_j$, is a left H-module isomorphism from V_j onto $V_j c \subseteq V_j$ for any $j \le i$. Since ${}_H V_j$ is artinian, $V_j c = V_j$ follows. Assume now that $cx \in P_j$ for some $x \in R$. Then

$$0 = \frac{V_j}{V_{j-1}} cx = \frac{V_j c + V_{j-1}}{V_{j-1}} x = \frac{V_j}{V_{j-1}} x,$$

so $x \in P_j$. Thus $c \in \mathscr{C}(P_j)$ for all $j \le i$, whence $\mathscr{C}(r(V_i)) \subseteq \bigcap_{j=1}^{i} \mathscr{C}(P_j)$. For the reverse inclusion, let $c \in \bigcap_{j=1}^{i} \mathscr{C}(P_j)$, and assume that $V_i cx = 0$ for some $x \in R$. As $V_i c = V_i$ by (ii), it follows that $x \in r(V_i)$, hence $c \in \mathscr{C}'(r(V_i))$. To see that $c \in {}'\mathscr{C}(r(V_i))$ as well, assume that $V_i xc = 0$ for some $x \notin r(V_i)$. Then $V_i x \subset V_j \setminus V_{j-1}$ for some $1 \le j \le i$. Since $(V_i x + V_{j-1}/V_{j-1})c = 0$, this contradicts the fact that V_j/V_{j-1} has no $\mathscr{C}(P_j)$-torsion elements.

4. ARTINIAN EMBEDDINGS

The purpose of this section is to prove that a ring R with right Krull dimension and $\mathrm{cl}_{\rho, R_R}(0) = 0$ embeds in a right artinian ring, provided that the right link closure of each $P \in \mathrm{Ass}(R_R)$ satisfies the incomparability condition. This will be achieved by identifying a finite number of factor rings of R that have artinian quotient rings, and by subsequently embedding R into a direct sum of these quotient rings.

We begin with a generalization of Jategaonkar's Main Lemma [7, 6.1.3], adapting a proof given by Lenagan and Warfield [12, Theorem 1.2] for modules over noetherian rings.

PROPOSITION 5. *Let R be a ring with right Krull dimension, and let ${}_H M_R$ be a bimodule that satisfies the descending chain condition for sub-bimodules. Assume that M_R has an affiliated series*

$$0 = M_0 \subset M_1 \subset M_2 \subset \cdots \subset M_{n-1} \subset M_n = M$$

with affiliated primes $P_i = r(M_i/M_{i-1})$, such that each M_i/M_{i-1} is torsion-free as a right R/P_i-module. Let i be the least integer such that there exists a submodule M' of M_R with $M' \not\subseteq M_{n-1}$ and $M'P_n \subseteq M_i$. Then

(i) *If $i = 0$, then $P_n = r(M')$.*

(ii) *If $i > 0$, then $P_n \rightsquigarrow P_i$.*

Proof. Note that each M_i is an H-R-sub-bimodule of M. Proceed by induction on n, the result being trivial for $n = 1$. Let $n > 1$, and assume that the result holds for any H-R-bimodule with DCC on sub-bimodules that has an affiliated series of the above kind of length $< n$. If $i = 0$, then

$$P_n \subseteq r(M') \subseteq r(M' + M_{n-1}/M_{n-1}) = P_n,$$

so $r(M') = P_n$. If $i > 1$, then, using the inductive hypothesis on the affiliated series

$$0 \subset M_2/M_1 \subset M_3/M_1 \subset \cdots \subset M_{n-1}/M_1 \subset M/M_1$$

of M/M_1 of length $n - 1$, it follows that $P_n \rightsquigarrow P_i$. This leaves the case when $i = 1$, so $0 \neq M'P_n \subseteq M_1$. Let X be a right R-submodule of M', such that HX is minimal in the set $\{HN | N \subseteq M'_R, N \not\subseteq M_{n-1}\}$. Replacing M' by X, if needed, it may thus be assumed that $r(N) = r(M')$ for all submodules N of M' with $N \not\subseteq M_{n-1}$. Set $A = r(M')$, and observe that

$$A = r(M') \subseteq r(M' + M_{n-1}/M_{n-1}) \cap r(M' \cap M_1) = P_n \cap P_1.$$

Assume that $A = P_1 \cap P_n$. Now $P_1 \not\subseteq P_n$, for otherwise $A = P_1$, so $M' \subseteq l_M(P_1) = M_1 \subseteq M_{n-1}$, a contradiction. Consequently, $M'P_1 \not\subseteq M_{n-1}$, hence $r(M'P_1) = r(M')$. However, as $M'P_1P_n \subseteq M'(P_1 \cap P_n) = M'A = 0$, this contradicts $M'P_n \neq 0$. Thus $P_nP_1 \subseteq A \subset P_n \cap P_1$, and we proceed to show that $P_n \rightsquigarrow P_1$ via A.

If $P_n \cap P_1/A$ is not fully faithful as a left R/P_n-module, then there exists an ideal B with $A \subset B \subseteq P_n \cap P_1$ and an ideal $X \supset P_n$ such that $XB \subseteq A$, so $M'XB = 0$. Now, $M'X \not\subseteq M_{n-1}$, for otherwise $X \subseteq r(M' + M_{n-1}/M_{n-1}) = P_n$. Thus $A = r(M') = r(M'X) \supseteq B$, which gives a contradiction.

If $P_n \cap P_1/A$ is not torsionfree as a right R/P_1-module, then there exists an element $b \notin A$, $b \in P_n \cap P_1$, and $c \in \mathscr{C}(P_1)$, such that $bc \in A$, so $M'bc = 0$. As $0 \neq M'b \subseteq M'P_n \subseteq M_1$, this contradicts the hypothesis that M_1 is torsionfree as a right R/P_1-module.

Remark. It is obvious from the proof that the above proposition remains true, if the descending chain condition for H-R-sub-bimodules of M is replaced by the condition that R satisfies the maximum condition for annihilators of subfactors of M_R. Thus it holds, for example, if R is assumed to be right noetherian. If R is noetherian, then, furthermore, it need not be assumed that each M_i/M_{i-1} is torsionfree as a right R/P_i-module (see [12, Theorem 1.2]).

THEOREM 6. *Let R be a ring with right Krull dimension, and let $_HB_R$ be a bimodule such that $_HB$ has finite length and* $\mathrm{cl}_{\rho, B_R}(0) = 0$. *Then*

(i) $\mathrm{Aff}(B) \subseteq \Omega^r(\mathrm{Ass}(B))$.

(ii) *$R/r_R(B)$ has a right artinian, classical right quotient ring if and only if $\Omega^r(\mathrm{Ass}(B))$ satisfies the incomparability condition. In this case,* $\mathrm{Aff}(B) = \Omega^r(\mathrm{Ass}(B)) = \{P \mid P/r(B) \in \mathrm{minSpec}(R/r(B))\}$.

Proof. Given any affiliated series

$$0 = V_0 \subset V_1 \subset \cdots \subset V_{i-1} \subset V_i \subset \cdots \subset V_n = B$$

of B_R with affiliated primes $P_i = r(V_i/V_{i-1})$, each V_i/V_{i-1} is torsionfree as a right R/P_i-module, so Proposition 5 can be applied to each of the H-R-bimodules V_i. Since $P_1 \in \mathrm{Ass}(B)$, trivially $P_1 \in \Omega^r(\mathrm{Ass}(B))$. Assume that for $i > 1$ it has been established that $\{P_1, \ldots, P_{i-1}\} \subseteq \Omega^r(\mathrm{Ass}(B))$. By Proposition 5, either $P_i = r(M')$ for some submodule $M' \not\subseteq V_{i-1}$ or $P_i \rightsquigarrow P_j$ for some $j < i$. In the first case, $P_i \subseteq Q$ for some $Q \in \mathrm{Ass}(B)$, and hence $P_i = Q$, as $\mathrm{Ass}(B) \subseteq \mathrm{minSpec}(R)$, due to the hypothesis that $\mathrm{cl}_{\rho, B_R}(0) = 0$, so $P_i \in \mathrm{Ass}(B) \subseteq \Omega^r(\mathrm{Ass}(B))$. In the second case, $P_i \in \Omega^r(\mathrm{Ass}(B))$ by the inductive hypothesis. Thus (i) has been established.

For (ii), assume without loss of generality that $r_R(B) = 0$. Taking the above affiliated series for B, it follows that $BP_nP_{n-1} \cdots P_1 = 0$, so that

$$\operatorname{minSpec}(R) \subseteq \{P_1, \dots, P_n\} \subseteq \operatorname{Aff}(B) \subseteq \Omega^r(\operatorname{Ass}(B)),$$

so equality holds throughout, if the incomparability condition is assumed for the latter set. Thus $\mathscr{C}(N) = \bigcap_{i=1}^n \mathscr{C}(P_i)$. By Theorem 4, the second set equals $\mathscr{C}(0)$, so $\mathscr{C}(N) = \mathscr{C}(0)$, and R has a right artinian, classical right quotient ring by [13, Theorem 4.1.4]. For the converse, recall that $\operatorname{Ass}(B) \subseteq \operatorname{minSpec}(R)$, so in order to establish the incomparability condition for $\Omega^r(\operatorname{Ass}(B))$ it is sufficient to show that if $Q \rightsquigarrow P$, and if P is a minimal prime, then Q is also minimal. Assume that the link is via A, and that Q is not minimal. Then Q contains an element $c \in \mathscr{C}(N) = \mathscr{C}(0)$. Since $c(Q \cap P) \subseteq A \subseteq Q \cap P$, and since $c(Q \cap P) \simeq Q \cap P$ as right R-modules, it follows that $\rho(Q \cap P/c(Q \cap P)) = 0$, and hence that $\rho(Q \cap P/A) = 0$. As $Q \cap P/A$ has no $\mathscr{C}(P)$-torsion, hence no $\mathscr{C}(N)$-torsion, this is a contradiction.

COROLLARY 7. *Let R be a ring with right Krull dimension such that $\operatorname{cl}_{\rho, R_R}(0) = 0$. If $\Omega^r(P)$ satisfies the incomparability condition for every $P \in \operatorname{Ass}(R_R)$, then R embeds in a right artinian ring.*

Proof. Let $E = E(R_R)$, and let $\operatorname{Ass}(E_R) = \operatorname{Ass}(R_R) = \{P_1, P_2, \dots, P_n\}$. Then $E = E_1 \oplus E_2 \oplus \cdots \oplus E_n$, each E_i being an injective right R-module with $\operatorname{Ass}(E_i) = \{P_i\}$. Now, $\operatorname{cl}_{\rho, E_i}(0) = 0$ and E_i has finite length over $H_i = \operatorname{End}_R(E_i)$ by Proposition 1. Since $\Omega^r(\operatorname{Ass}(E_i)) = \Omega^r(P_i)$ is assumed to satisfy the incomparability condition, each of the rings $R/r_R(E_i)$ has a right artinian right quotient ring by the preceding theorem. The result follows, since $\bigcap_{i=1}^n r_R(E_i) = r_R(E) \subseteq r_R(R) = 0$.

For many classes of noetherian rings the incomparability condition has been established for cliques. This is usually done using a symmetric dimension function available for the particular class, for example, the Gelfand–Kirillov dimension (provided it is finite) in the case of noetherian algebras, or the classical Krull dimension in the case of noetherian rings satisfying the second layer condition (see, e.g., [5, Corollary 12.6]). To date, no example is known of a noetherian ring with a clique that contains comparable primes, so the following corollary seems quite general.

COROLLARY 8. *Let R be a noetherian ring with $\operatorname{cl}_{\rho, R_R}(0) = 0$. If $\operatorname{Cliq}(P)$ satisfies the incomparability condition for each $P \in \operatorname{Ass}(R_R)$, then R is a subring of an artinian ring.*

Proof. Observe that $\Omega^r(P) \subseteq \operatorname{Cliq}(P)$ for each $P \in \operatorname{Ass}(R_R)$, and decompose $\operatorname{E}(R_R)$ as in the proof of the preceding corollary. The claim now follows from Theorem 6, taking into account that for a noetherian ring $\mathscr{C}(N) = \mathscr{C}(0)$ implies the existence of a two-sided artinian quotient ring.

In general, one cannot expect the embeddability of a noetherian ring R into an artinian ring S to force R to be ρ-torsionfree. However, this does happen when ${}_RS$ is flat, as is the case when S is an artinian quotient ring of R.

PROPOSITION 9. *Let R be a right noetherian subring of a right artinian ring S such that ${}_RS$ is flat. Then* $\mathrm{cl}_{\rho, R_R}(0) = 0$.

Proof. Defining $\lambda(M) = \mathrm{length}(M \otimes_R S)$ for a finitely generated right R-module M produces an additive rank function. Let A be a right ideal of R with $\rho(A_R) = 0$. Then $\lambda(A) = 0$ by [8, Lemma 1.3], hence $A \otimes_R S = 0$. As ${}_RS$ is flat, $A \otimes_R S \cong AS$ canonically, so $A = 0$.

However, the following example shows that a noetherian ring R may have ρ-torsion even when it is "close" to having an artinian quotient ring, in the sense that there are ideals I_j, $1 \le j \le n$, with $\bigcap_{j=1}^{n} I_j = 0$, such that each of the rings R/I_j has an artinian quotient ring.

EXAMPLE. Let Z denote the ring of integers, let $Z[x]$ be the polynomial ring in one variable x, and let $M = Z[x]/(x)$, viewed as a Z-$Z[x]$-bimodule. In the ring $R = \begin{pmatrix} Z[x] & 0 \\ M & Z \end{pmatrix}$, $I_1 \cap I_2 = 0$, where $I_1 = \begin{pmatrix} (x) & 0 \\ 0 & 0 \end{pmatrix}$ and $I_2 = \begin{pmatrix} 0 & 0 \\ M & Z \end{pmatrix}$. Note that $R/I_1 \cong \begin{pmatrix} Z & 0 \\ Z & Z \end{pmatrix}$ and $R/I_2 \cong Z[x]$, so both R/I_1 and R/I_2 have artinian classical quotient rings. Now, consider the prime ideals $P = \begin{pmatrix} (x) & 0 \\ M & Z \end{pmatrix} \supset Q = \begin{pmatrix} 0 & 0 \\ M & Z \end{pmatrix}$, and note that $\begin{pmatrix} 0 & 0 \\ M & 0 \end{pmatrix} P = 0$. Thus $\mathrm{cl}_{\rho, R_R}(0) \neq 0$.

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