Vector Fields Tangent to Foliations. II. Handlebody Foliations

F. Wesley Wilson, Jr.

Department of Mathematics, University of Colorado, Boulder, Colorado 80309

Received January 12, 1976; revised January 5, 1977

In our previous paper on this topic [1], we investigated the questions of existence and homotopy uniqueness of nonsingular vector fields which are tangent to an orientable foliation \( \mathcal{F} \) on a closed, orientable manifold \( M \). We were especially interested in the case of foliations of codimension 1 on 3-manifolds. We obtained precise information about the geometry of vector fields which are tangent to the Reeb components of \( \mathcal{F} \).

In this paper, we shall restrict our attention to foliations of 3-manifolds which are compatible with the handlebody structure, the handlebody foliations, and we develop a procedure for computing these obstructions from a list of parameters which describe \( \mathcal{F} \). The procedure which we use is to view the tangent vector field \( \xi \) as having been constructed in stages. The first step is to specify \( \xi \) on the Reeb components; the results of our previous investigation provide constraints. Next, we try to extend \( \xi \) to one complementary leaf; the Poincaré Index Formula provides a constraint here. At this stage, we have essentially specified \( \xi \) on the 1-skeleton of a CW decomposition of \( M \), and since there is no obstruction to this extension (the fibers are arcwise connected) we know that this is always possible. However, our analysis has provided us with information regarding the structure which \( \xi \) must have. Since \( H^1(M, \pi_1(S^1)) \) contains the only obstructions to homotopies between tangent vector fields to \( \mathcal{F} \), it follows that if there is an extension of \( \xi \) to all of \( \mathcal{F} \), then its homotopy class is determined by the part which has already been specified. A sharper statement is valid: The homotopy class of any extension is determined by the homotopy class of its restriction to the Reeb components (Theorem 3.6). On the other hand, the obstructions to existence lie in \( H^2(M, \pi_1(S^1)) \), and so we expect to encounter the full obstruction to existence in this next step. There is a choice of 2-cells in \( M \) so that the complement of the 2-skeleton consists of a single open 3-cell. Evidently, if we can specify \( \xi \) on the boundary of this 3-cell, then there will be no difficulty with extending to its interior. From the geometry, we can describe the obstructions to the extension to the 2-skeleton explicitly in terms of the parameters which describe \( \mathcal{F} \). Finally, we discuss these obstructions in terms of the handlebody structure for \( M \). Additional information is available if \( \mathcal{F} \) is compatible with a Lickorish handlebody structure for \( M \).
These obstructions can be expressed in terms of linear diophantine equations, and the questions of existence and homotopy uniqueness of tangent vector fields to a specific foliation reduce to the questions of existence and uniqueness of solutions to these diophantine equations. This situation is examined in Section 6. The number theory becomes especially nice if we are dealing with a foliation which is compatible with a special Lickorish handlebody structure (Theorem 1.2) and in this case we can show that if \( H^1(M) \neq 0 \), then every Lickorish foliation fails to have a tangent vector field (Theorem 6.5). Thus every compact orientable 3-manifold with nonvanishing first Betti number admits a foliation for which every compact leaf is a torus and for which there is no nonsingular tangent vector field. It is not known when such a manifold can have a handlebody foliation which does admit a tangent vector field. On the other hand, every orientable homology 3-sphere has the property that every foliation has a unique tangent vector field. Thus the diophantine systems corresponding to foliations on these manifolds must have a unique solution. This may provide an avenue between the study of linear diophantine equations and the structure of homology 3-spheres.

In the course of this paper, we shall have cause to refer to results from our previous paper [1]. We shall use the prefix I or II on theorem and section numbers to identify these results; e.g., I.3 refers to Section 3 of [1], and II.1.2 refers to Theorem 2 of Section 1 of the present paper.

1. The Structure of a Handlebody Foliation

Every closed orientable 3-manifold \( M \) is a handlebody [2, 5]; i.e., \( M \) is diffeomorphic to a manifold which is constructed by removing a finite family of solid tori from the 3-sphere \( S^3 \), and smoothly reattaching them by some identification along their boundaries. This fact can be combined with a lemma of J. W. Alexander to obtain the following useful statement (cf. [6, Sect. 4]).

**Theorem 1.1.** Let \( M \) be a closed, orientable 3-manifold. Then there are disjoint solid tori \( U_1, \ldots, U_k \) in \( M \) and \( V_0, \ldots, V_n \) in \( S^3 \) such that

1. \( S^3 - \text{Int}(V_0) = D \times S^1 \), where \( D \) denotes the 2-disk,
2. \( V_1, \ldots, V_n \) are transverse to the product foliation of \( D \times S^1 \),
3. there is an orientation-preserving diffeomorphism

\[
h: S^3 - \bigcup_{i=1}^k \text{Int}(V_i) \to M - \bigcup_{i=1}^k \text{Int}(U_i),
\]

4. the diffeomorphism class of \( M \) is completely determined by the structure of the link \( \{V_i\}_{i=1}^k \) and the isotopy classes of the pasting homeomorphisms

\[
h_i = h \mid \partial V_i: \partial V_i \to \partial U_i, \quad i = 1, \ldots, k.
\]
Whenever a solid torus $V'$ is embedded in an orientable 3-manifold $M$, so that it is transverse to an existing foliation $\mathcal{F}$, then it is possible to build a new foliation on $M$ which coincides with $\mathcal{F}$ away from $V'$, and which has $V'$ as an additional Reeb component (cf. [3] or [6]). Roughly speaking, each leaf of $\mathcal{F}$ which intersects $\partial V'$ is cut along the circle(s) of intersection. These free edges of the leaf can be drawn around $V'$ in the direction of a longitude circle in such a way that it spirals to $\partial V$. The product foliation of $V' = D \times S^1$ is drawn around the inside of $V'$ in a similar manner, and $\partial V$ is added as a compact leaf. Every leaf on one side of $\partial V$ must spiral in the same direction, but the interior and exterior directions of spiraling may differ. After a longitude orientation has been chosen, then four different combinations of spiraling directions are possible. Combining this construction with the Handlebody Theorem, Lickorish [3] and S. P. Novikov and H. Zeischung (unpublished; cf. [6]) have shown that every 3-manifold can be foliated so that $V_0$ and the handles are the Reeb components. For a given handlebody structure, there are $4^d$ different foliations depending on the directions of spiraling which are chosen near the boundaries of the Reeb components. We shall call such a foliation a handlebody foliation, and we call it a simple handlebody foliation if it has the additional property that each $V_i$ links $V_0$ once.

In order to describe these foliations, we shall need to give precise descriptions of the homeomorphisms $h_i$. Each of these homeomorphisms is determined up to isotopy by $(h_i)_1: H_1(\partial V_i) \to H_1(\partial U_i)$, and once bases for these groups have been chosen, each homomorphism $(h_i)_1$ is described by a $2 \times 2$ unimodular matrix $A_i$. The orientation of $S^3$ induces a product orientation on $D \times S^1$. Choose longitudes $l_i$ for $\partial V_1, \ldots, \partial V_k$ so that $l_i$ does not link the axis of $V_i$. Since $V_1, \ldots, V_k$ are interior to $D \times S^1$ and transverse to the product foliation, we can orient the longitudes of $V_1, \ldots, V_k$ to agree with the $S^1$ orientation, and we can orient the meridians of $\partial V_1, \ldots, \partial V_k$ to be the positive boundary of a component of $V_i \cap D \times \{\theta\}$. Call these oriented generators $l_i$ and $m_i$. We chose a basis $l_i, m_i$ for $H_1(\partial U_i)$ by choosing $l_i$ to be a generator for $H_1(U_i)$, by requiring that $m_i$ to bound a disk in $U_i$, and by requiring that they be so oriented that $A_i$ has positive determinant and that the first nonzero element in the first row is positive. With these choices,

\[
(h_i)_1(\bar{l}_i) = a_i l_i + b_i m_i,
\]

\[
(h_i)_1(\bar{m}_i) = c_i l_i + d_i m_i,
\]

and

\[
A_i = \begin{pmatrix} a_i & b_i \\ c_i & d_i \end{pmatrix}, \quad A_i^{-1} = \begin{pmatrix} d_i & -b_i \\ -c_i & a_i \end{pmatrix}.
\]

We note that $l_i$ is vaguely defined, since any $l_i + km_i$ would do equally well. We shall comment on the effect of such variations at the appropriate time.

It will be useful to treat $V_0$ similarly. Let $l_0, m_0$ be the natural longitude
and meridian of $V_0$, and let $\bar{l}_0$, $\bar{m}_0$ be defined relative to the $D \times S^1$ structure. Then $l_0 = \bar{m}_0$ and $m_0 = -\bar{l}_0$; i.e. we are thinking of $D \times S^1$ as the set $S^2 \times S^1 - \text{Int}(V_0)$. This viewpoint will be useful throughout this paper.

We also want to introduce coefficients to describe the directions in which the noncompact leaves spiral toward the Reeb components. The direction of external spiraling is $\pm l_i$, and the direction of internal spiraling is $\pm k_i$ ($i = 0, \ldots, k$). Choose coefficients $r_i, s_i = \pm 1$ so that these directions of spiraling are $-r_i l_i$ and $-s_i k_i$, respectively. Also define $\sigma_i = r_i s_i$.

The Lickorish proof of the handlebody theorem is more elementary in that it depends constructively on certain geometric properties of 2-manifolds, rather than on general principles of differential topology. It is therefore not too surprising that it provides more detailed information about the handlebody structure. We shall now state a sharper version of the handlebody theorem which is the consequence of this approach.

**Theorem 1.2 (Lickorish).** Let $M$ be a closed, orientable 3-manifold. Then there are disjoint solid tori $U_1, \ldots, U_k$ in $M$, and $V_0, \ldots, V_k$ in $S^3$, such that

1. $S^3 - \text{Int}(V_0) = D \times S^1$;
2. $V_1, \ldots, V_k$ are transverse to the product foliation of $D \times S^1$ and each $V_i$ links $V_0$ once (in particular, each $V_i$ is unknotted in $S^3$);
3. for each $i, j = 1, \ldots, k$ the linking number $\lambda^i_j$ of $V_i$ and $V_j$ is 0 or $\pm 1$;
4. there is an orientation-preserving diffeomorphism

$$h: S^3 - \bigcup_{i=1}^k \text{Int}(V_i) \to M - \bigcup_{i=1}^k \text{Int}(U_i);$$

5. the pasting homeomorphisms $h_i = h \mid \partial V_i$ are described by matrices of the form

$$A_i = \begin{pmatrix} 1 & 0 \\ c_i & 1 \end{pmatrix},$$

where $c_i$ is called the twisting coefficient of $U_i$.

**Proof.** The first four conclusions are the consequence of [2, Theorem 2; 3, Theorem 2, Lemma 3]. The final conclusion follows by a close examination of the proofs of these theorems.

### 2. The Structure of a Tangent Vector Field Near the Reeb Components

A handlebody foliation $\mathcal{F}$ for $M$ is described by specifying $\{U_1, \ldots, U_k\}$, $\{V_0, \ldots, V_k\}$, $h: S^3 - \bigcup_{i=1}^k \text{Int}(V_i) \to M - \bigcup_{i=1}^k \text{Int}(U_i)$, and $(r_0, \ldots, r_k, s_0, \ldots, s_k)$. 
An essential part of the description of \( h \) is the set of pasting coefficients \( \{a_i, b_i, c_i, d_i\}_{i=-1}^0 \). Suppose that \( \xi \) is a vector field which is tangent to \( \mathcal{F} \) on its Reeb components. Then by I.4.1, it follows that \( \xi \mid U_i \) is described algebraically by \( \deg(\xi \mid \partial U_i) \cdot \sigma_i = s_i \) and \( \text{Ker}(\xi \mid \partial U_i) \cdot \mu_i = l_i + n_i m_i \) for some integer \( n_i \).

By I.2.1, the homotopy coefficients \( \{n_0, ..., n_k\} \) completely describe the homotopy class of the restriction of \( \xi \) to the Reeb components. On the other hand, I.2.1 also tells us that if \( \xi \) has such an algebraic description, then \( \xi \) can be described geometrically, and we can always assume that \( \xi \mid \partial U_i \) has the simplest possible geometry in its homotopy class; cf. the comment following I.1.5. Now the diffeomorphisms \( h_i^{-1}: \partial U_i \to \partial V_i \) carry \( \xi \mid \partial U_i \) and its family of trajectories onto \( \partial V_i \). Let \( \zeta \) denote this induced vector field on \( \bigcup_{i=0}^k \partial V_i \). It is clear how the geometric description of \( \xi \mid \partial U_i \) is translated into a geometric description of \( \zeta \mid \partial V_i \). By Reinhart's theorem (I.2.1), the location of the periodic solutions can be used to compute the algebraic description of \( \zeta \mid \partial V_i \).

**Lemma 2.1.** Let \( \mathcal{F} \) be a handlebody foliation for \( M \) as described above, and let \( \xi \) be a vector field which is tangent to \( \mathcal{F} \) along its Reeb components. Then \( \xi \) and \( h \) induce a vector field \( \zeta \) on the boundary of \( S^3 = \bigcup_{i=0}^k \text{Int}(V_i) \). The problem of extending \( \zeta \) to \( \mathcal{F} \) is identical (by \( h \)) to the problem of extending \( \xi \) to the smoothing of the product foliation of

\[
S^3 - \bigcup_{i=0}^k \text{Int}(V_i) = S^2 \times S^1 - \bigcup_{i=0}^k \text{Int}(V_i)
\]

in the directions \( \{r_0, ..., r_k\} \) near the boundary. The vector field \( \zeta \mid \partial V_i \) has the algebraic description

\[
\deg(\zeta \mid \partial V_i) \cdot \sigma_i = s_i,
\]

\[
\text{Ker}(\zeta \mid \partial V_i) \cdot \mu_i = (d_i - n_i\epsilon_i)\tilde{l}_i + (n_i a_i - b_i)\tilde{m}_i.
\]

The winding numbers of \( \zeta \) along \( \tilde{l}_i \) and \( \tilde{m}_i \) can be computed from this information (I.3.1), provided that we give an orientation to \( \partial V_i \). We want to do this in a way which is compatible with the foliation \( \mathcal{F} \). Since \( \mathcal{F} \) is transversely orientable, we can choose a normal field to \( \mathcal{F} \), and this normal field will induce the appropriate orientation on each leaf. By convention, we choose the normal field to agree with the orientation of \( S^1 \) on those portions of \( S^2 \times S^1 \) where \( \mathcal{F} \) agrees with the product foliation. This means that when \( r_i = +1 \), then the normal to \( \partial V_i \) is directed to the exterior of \( S^3 = S^1 - \bigcup_{i=0}^k \partial V_i \), and that when \( r_i = -1 \), then the normal to \( \partial V_i \) is directed to the interior.

**Lemma 2.2.** Let \( M, \mathcal{F}, \xi, \zeta \) be as above. Then for \( i = 0, ..., k \), we have

\[
W_z(\tilde{m}_i) = \sigma_i(d_i - n_i\epsilon_i),
\]

\[
W_z(\tilde{l}_i) = \sigma_i(b_i - n_i a_i).
\]
Proof. We take this opportunity to point out that there is a sign error in the formula for $W_i(\tilde{I})$ in 1.3.1. Using the correct formulas, and the longitude-meridian-normal framing, we have $W_i(\tilde{m}_i) = s_i(d_i - n_i e_i)$ and $W_i(\tilde{I}_i) = s_i(b_i - n_i a_i)$. When $r_i = +1$, then the longitude-meridian-normal framing is oriented to agree with the product orientation of $S^2 \times S^1$, and so these formulas give the correct winding numbers for $\partial V_i$. If $r_i = -1$, then the normal to the foliation is reversed along $\partial V_i$, and so the longitude-meridian-normal framing has opposite orientation to that of $S^3$. Thus a factor of $-1 (r_i)$ must be added to the formulas.

We note that if $l_i$ had been chosen instead to be $l_i + km_i$, this would not have altered formulas (2.1) and (2.2). For changing the $l_i$ will change both the coefficients of $A_i$ and the description of the periodic solutions of $\xi$ on $\partial U_i$.

After all, what is at issue is the location of the periodic solutions of $\xi$, and it does not matter what base for $\partial U_i$ was used in describing the periodic solutions of $\xi$ on $\partial U_i$ or in defining the matrix $A_i$. Since the properties of $\xi \mid U_i$ shall enter our considerations only through these formulas, it follows that the ambiguity in our choice of $l_i$ will not alter any of our results. On the other hand, such a change will affect the form of $A_i$, which may possibly be used to computational advantage.

We now turn to the problem of how $\xi$ extends to a neighborhood of the Reeb components. Since $\mathcal{V} = \bigcup_{i=0}^k \partial V_i$ has a closed collar neighborhood in $S^2 \times S^1 - \bigcup_{i=0}^k \text{Int}(V_i)$, and since $\mathcal{V}$ is a strong deformation retract of $\mathcal{F}$, all extensions of $\xi$ to $\mathcal{F} \mid \mathcal{N}$ are homotopic rel $\mathcal{V}$. But $\partial \mathcal{N}$ is also a strong deformation retract of $\mathcal{N}$; so a similar argument shows that if two extensions coincide on $\partial \mathcal{N}$, then they are also homotopic rel $\partial \mathcal{N}$. However, they are generally not homotopic rel $\partial \mathcal{N} \cup \mathcal{V}$. Since any collar for $\partial V_i$ can also be viewed as a collar for $\partial U_i$, and hence also for $U_i$, we can also view these extensions as extensions of $\xi$ from $\bigcup_{i=0}^k U_i$ to a collar neighborhood $\mathcal{N}$ of $\bigcup_{i=0}^k U_i$, and ask if any two extensions are homotopic rel($\partial \mathcal{N}$). The answer is again affirmative, and this follows from the following slightly more general lemma.

**Lemma 2.3.** If $\mathcal{N}$ is a solid torus of genus $k$ on which there is an orientable foliation $\mathcal{F}$ (not necessarily tangent at the boundary) and a pair of tangent vector fields to $\mathcal{F}$, $\xi_0$ and $\xi_1$, which coincide on the boundary, then $\xi_0$ is homotopic to $\xi_1$ rel $\partial \mathcal{N}$ through tangent vector fields to $\mathcal{F}$.

**Proof.** As in 1.1.1, the obstructions to homotopy lie in $H^k(\mathcal{N}, \partial \mathcal{N}; \pi_k(S^1))$, and in this case the obstruction groups all vanish.

**Corollary 2.4.** If $\mathcal{N}$ is a closed submanifold which is a collar neighborhood of the Reeb components of $\mathcal{F}$, and if $\xi_0$ and $\xi_1$ are tangent vector fields to $\mathcal{F} \mid \mathcal{N}$ which coincide on $\partial \mathcal{N}$, then $\xi_0$ is homotopic to $\xi_1$ rel $\partial \mathcal{N}$ through tangent vector fields to $\mathcal{F}$. 

3. The Obstruction to the Extension to One Leaf

By the discussion of the previous section, we can assume that we have chosen collars \( W_i \) for \( \partial V_i \) \((i = 0, \ldots, k)\) so that \( \partial W_i \) is transverse to \( \mathcal{F} \), and we can assume that we have a particular extension of \( \zeta \) to \( W^+ = \bigcup_{i=0}^k W_i \), which we shall also call \( \zeta \). Let \( L \) denote a leaf of \( \mathcal{F} \) which lies in the complement of the Reeb components. Then \( \zeta \) has been specified on all of \( L \) except for a relatively compact subset.

**Lemma 3.1.** Each leaf \( L \) of \( \mathcal{F} \) which lies in the complement of the Reeb components is the image of a one-to-one immersion of \( S^2 - \{ x, f = 0, \ldots, t + \sum_{i=0}^k t_i \} \), where \( t_i \) is the number of components of \( V_i \cap S^2 \times \{ \theta \} \) for any \( \theta \in S^1 \). The relatively compact subset of \( L \), on which \( \zeta \) has not been specified, is an embedded image of \( S^2 - \bigcup_{i=0}^k D_i \), where each \( D_i \) is a 2-disk.

**Proof.** The foliation \( \mathcal{F} \) was constructed by beginning with the product foliation for \( S^2 \times S^1 \) and cutting our wormholes \( \{ V_i \}_{i=0}^k \), which were to be the Reeb components. The leaves on the complement of these wormholes have the form described, and the spiraling process does not alter their topological type. The enlargement from \( V_i \) to \( W_i \) covers this spiraling.

**Corollary 3.2.** The problem of extending \( \zeta \) to \( L \) is equivalent to the problem of extending a vector field \( \zeta \) from a family of disjoint disks \( \{ D_i \}_{i=0}^k \) to all of \( S^2 \). The relative homotopy classes of these extension problems also correspond.

In examining this extension problem, we shall state our findings in terms of the parameters \( \{ a_i, b_i, c_i, d_i, r_i, s_i, t_i, n_i \} \) which describe \( \mathcal{F} \) and \( \zeta \). Let \( \mathcal{F} \) be a handlebody foliation for \( M \) with structural parameters \( \{ a_i, b_i, c_i, d_i, r_i, s_i, t_i, n_i \} \) and let \( \xi \) be a tangent vector field for the Reeb

**Lemma 3.3.** Let \( D_i \) be a disk in \( S^2 \) associated with the Reeb component \( V_i \). Then

\[
W_i(\partial D_i) = \sigma_i(d_i - n_i c_i).
\]

**Proof.** By II.2.2 and the definition of \( \tilde{m}_i \), we have

\[
\pm W_i(\partial D_i) = W_i(\tilde{m}_i) = \sigma_i(d_i - n_i c_i),
\]

where the sign is determined by reconciling the orientation of \( \partial D_i \), the orientation of \( \tilde{m}_i \), and the way that \( \zeta \), \( \partial V_i \) induces its extension to that portion of \( L \) which lies in \( W_i \). But since the winding number on every leaf is defined relative to the same normal field to \( \mathcal{F} \), and since \( \tilde{m}_i \) was oriented as the boundary of the meridian disk (in \( S^2 \)), it follows that \( \partial D_i \) and \( \tilde{m}_i \) are similarly oriented, and so the sign is positive.

**Theorem 3.4.** Let \( \mathcal{F} \) be a handlebody foliation for \( M \) with structural parameters \( \{ a_i, b_i, c_i, d_i, r_i, s_i, t_i, n_i \} \) and let \( \xi \) be a tangent vector field for the Reeb
components of $\mathcal{F}$ with homotopy coefficients $\{n_i\}_{i=0}^t$. Then $\zeta$ can be extended to one leaf in the complement of the Reeb components if and only if

$$-\sigma_0 n_0 := (t - 1) + \sum_{i=1}^k t_i(d_i - n_i c_i).$$

Whenever extension is possible, there is a one-to-one correspondence between the set of homotopy classes of extensions rel $W$ and the set of sequences of $t$ integers $\{k_j\}_{j=1}^t$.

Proof. The problems of extending $\xi$ to $L$ is equivalent to the problem of extending the associated vector field $\xi$ from $\bigcup_{j=0}^t D_j$ to all of $S^2$. By I.3.3, $W_\xi(\partial D_j) = \sigma_i(d_i - n_i c_i)$ when $D_j$ corresponds to $V_j$. By the Poincaré Index Formula (e.g., I.3.2) $\xi$ has a nonsingular extension if and only if

$$2 = \sum_{j=0}^t [1 + W_\xi(\partial D_j)] = (t + 1) + \sigma_0 n_0 + \sum_{i=1}^k \sigma_i t_i(d_i - n_i c_i).$$

We express this result in the form used in the statement of the theorem because we shall want to think of $\{n_1, \ldots, n_t\}$ as independent variables.

If $\xi_0$ and $\xi_1$ are two extensions, then they are homotopic rel $\bigcup_{j=0}^t D_j$ if and only if their winding difference is zero (I.3.3). In the present case,

$$D(\xi_0, \xi_1, \ast) \in \text{Hom} \left[ H_1\left(S^2, \bigcup_{j=0}^t D_j\right) ; Z \right] \cong H^1\left(S^2, \bigcup_{j=0}^t D_j\right),$$

which is free on $t$ generators.

This theorem generalizes I.4.3, since we can obtain that result by setting $t = 0$, and $\sigma_0 = \pm 1$ for $\mathcal{A}_\pm$. Then $n_0 = -\sigma_0$.

The final statement of this theorem is disturbing, since it seems to indicate that in order to solve the full extension problem for $\xi$, we must examine the extendibility of infinitely many extensions of $\xi$ to $L$. Fortunately, all of these different extensions to $L$ behave similarly with respect to the rest of the extension problem.

Before proving this last assertion, it is useful to list representative examples of the various possible extensions. If $\xi_0$ and $\xi_1$ are two such extensions, then as in the proof of the theorem, their homotopy difference is measured by the winding difference homomorphism $D(\xi_0, \xi_1 ; \ast) : H_1(S^2, \bigcup_{j=0}^t D_j) \rightarrow Z$. A convenient basis for this homology group is a family of disjoint simple curves $\{\gamma_j\}_{j=1}^t$ where each $\gamma_j$ has its initial point in $\partial D_0$ and its terminal point in $\partial D_t$. Of course, there is nothing canonical about the choice of such a family. To get representatives of every possible homotopy class of extension, we begin with an arbitrarily chosen extension $\xi_0$, and produce a modification $\xi_1$ such that $D(\xi_0, \xi_1 ; \gamma_j) = k_j$ for any preassigned sequence $\{k_1, \ldots, k_t\}$. For $j = 1, \ldots, t$, choose a collar $A_j$ for $D_j$ so that in the coordinates $S^1 \times I = A_j$, the curve
segment $\gamma_j \cap A_j$ looks radial. We construct $\xi_1$ by modifying $\zeta_0$ in these collars. There are two constructions needed, depending on whether or not $m = W_\xi (\partial D_j)$ is zero.

If $m = 0$, then we can assume (by homotopy) that $\zeta_0$ has a thick band of periodic solutions in $A_j$ (concentric with $D_j$). We define $\xi_1 \mid A_j$ to coincide with $\zeta_0$ except on the interior of this band, and on the interior of this band, $\xi_1$ has $\mid k_j \mid$ Reinhart bands (cf. I.2) (this requires inserting $2 \mid k_j \mid$ periodic solutions and regions where the vector field reverses direction appropriately to give the desired winding difference). Then $D(\zeta_0, \xi_1 ; \gamma_3) = k_j$.

If $m \neq 0$, then we can assume (by homotopy) that $\zeta_0 \mid A_j$ is periodic of period $2\pi/m$ in the $S^1$ variable. We define a twist diffeomorphism (a la Lickorish [3]) $h^{l,m} : A_j \to A_j$, where $l = k_j$ and the twist is through $-2\pi l/m$ radians, rather than being necessarily integral (Lickorish defines a twist to be a homeomorphism which wraps radial lines some integral number of times around the annulus; begging smooth extendibility at $\partial A_j$, our twist would send $(p, \theta) \to (p, \theta + \rho 2\pi l/m)$). We define $\xi_1 = D h^{l,m}(\zeta_0)$. Then $D(\zeta_0, \xi_1 ; \gamma_3) = l = k_j$.

The set of examples which we have just described represents every possible homotopy class extension of $\xi$ from $W$ to $L$; so in dealing with the problem of extending from $W \cup L$ to the rest of $\mathcal{F}$, we shall always assume that we are dealing with one of these examples. Let $X_i$ be a collar for $W_i$ so that $\partial X_i$ is transverse to $\mathcal{F}$ and so that $\partial X_i \cap L = \bigcup \partial A_j$, where the union is taken over those indices $j$ which correspond to $W_i$. Since $X_i$ is a collar of $W_i$, any extension $\tilde{\xi}$ of $\zeta_0$ to all of $\mathcal{F}$ can be chosen (by homotopy) so that $\tilde{\xi} \mid \partial X_i$ and $\tilde{\xi} \mid \partial W_i$ are equal when compared via the radial homeomorphism of $\partial X_i \to \partial W_i$ (assume that $\zeta_0$ has this property on each $A_j$).

**Lemma 3.5.** There is a Compatibility Condition which $\zeta_0$ must satisfy before there is any possibility that $\zeta_0$ can be extended to all of $\mathcal{F}$.

**Proof.** If $\zeta_0$ has been specified on $\partial W_i$, one annulus $A_1$, and on $\partial X_i$, then there is a unique extension $\tilde{\xi}$ of $\zeta_0$ to the rest of $X_i$. This unique extension defines $\tilde{\xi} \mid A_j$ for all other $A_j$'s in $X_i$, and these values may or may not be compatible with $\zeta_0 \mid A_j$. Indeed they are compatible if and only if

$$D(\zeta_0, \tilde{\xi} ; \gamma_j \cap A_j) = 0.$$ 

The detailed description of the Compatibility Condition is most easily given after several additional concepts have been introduced. We postpone this description until Section 4.

**Theorem 3.6.** Let $\mathcal{F}$ be a handlebody foliation, and suppose that $\tilde{\xi}_0$ and $\tilde{\xi}_1$ are vector fields tangent to $\mathcal{F}$ whose restrictions to the Reeb components of $\mathcal{F}$ are homotopic (same homotopy coefficients). Then $\tilde{\xi}_0$ and $\tilde{\xi}_1$ are homotopic as tangent vector fields to $\mathcal{F}$. 
Proof. Let $X = \bigcup_{i=0}^{k} X_i$, and let $\zeta_0 = \bar{\zeta}_0 \mid (X \cup L)$, $\alpha = 0, 1$. We can assume (by homotopy) that $\zeta_0$ and $\zeta_1$ agree on $\partial X \cup W \cup L$. (Since $\zeta_0$ and $\zeta_1$ must both satisfy the Compatibility Condition, it follows that for each $i$, $D(\zeta_0, \zeta_1 ; \gamma_j)$ has the same value for every $j$ associated to $W_i$.) Now the complement of $\partial X \cup (L - X)$ consists of several open solid tori (the interiors of the $X_i$) and one other region which is an open solid torus of genus $t$. By II.2.3, $\bar{\zeta}_0$ and $\bar{\zeta}_1$ are homotopic rel $\partial X \cup (L - X)$.

4. Vertical Structure of $\mathcal{F}$ and $\zeta$

Before we can compute the obstruction to further extension, it is necessary to know what is happening in the $S^1$-direction of $S^2 \times S^1 - W = S^3 - W$. There are two features of the vertical structure to which we shall pay special attention:

I. What is the structure of the link $\bar{W} = \bigcup_{i=1}^{k} W_i$?

II. How is $\zeta$ revolved as one moves vertically along a longitude $l_i$ of $\partial W_i$ which is parallel to $l_i$?

To answer (I), it is necessary to introduce notation which allows us to describe the link more precisely. Let $t_i = \tau$ denote the number of times that $W_i$ links $W_a$, i.e., the number of components of $W_i \cap L$. Enumerate these components as $\{D_{ij}\}_{j=1}^{t_i}$ and adopt the index convention that $j$ is always reduced mod $\tau$. Also, let $W_{ij}$ denote the portion of $W_i$ between $D_{ij}$ and $D_{ij+1}$. By an isotopy, we can assume that for those $i$ with $t_i > 1$, there are disjoint disks $D_i$ in $L$, such that the smaller disks $\{D_{ij}\}_{j=1}^{t_i}$ are symmetrically placed interior subsets of $D_i$, and we shall assume that the $j$-indexing corresponds to both the order of progression of the sets $D_{ij}$ along $W_i$ and the order of placement (angular variable) in $D_i$.

Preparation Theorem 4.1. There is an isotopy of $D \times S^1$ which alters the link $\bar{W}$ so that it has the following additional properties:

1. Each $W_i$ is transverse to the product foliation.

2. There is a finite family of closed subintervals $\{I_a\}$ in $S^1$ so that on the complement of $\bigcup_a D \times I_a$, $\bar{W}$ coincides with $\bigcup_j (D_{ij} \times S^1)$.

3. If $I_a$ is one of these special intervals, then either (A) there is a pair of index pairs $(i', j'), (i'', j'')$ and a disk $D_a \subset D - \bigcup \{D_{ij} \mid i \neq i' \text{ or } i'' \text{ or } j \neq j' \text{ or } j''\}$ which is a neighborhood of $D_{i',j'}$ and $D_{i'',j''}$ so that $\bar{W} \cap D_a \times I_a$ is the image of $D_{i',j'} \times I_a$ and $D_{i'',j''} \times I_a$ after $D_a$ has been twisted once (positively or negatively) over the interval $I_a$, or else (B) there is a special interval $I_\beta$ such that if $t_i > 1$, then $\bar{W} \cap (I_\beta \times \bigcup D_{ij})$ is the image of $(\bigcup D_{ij}) \times I_\beta$ after $D_i$ has been twisted positively by $2\pi/t_i$ over the length of $I_\beta$.
Proof. Let \( \{\theta^1_{a_i} = 0 \subset S^1 \) denote the set of all levels which contain crossing points of the link. There is no loss in assuming that the crossing points are finite in number, that they are separated from \( L \), and that there is at most one crossing pair per level. Choose \( \epsilon > 0 \) to be smaller than one third the minimum separation between the points \( \theta_s \). Let \( I_s = (\theta_s - \epsilon, \theta_s + \epsilon) \subset S^1 \). Beginning at \( L \), there is an obvious level-preserving isotopy of \( D \times S^1 \) which carries the link into \( \bigcup_{i,j} D_{ij} \times [0, 2\pi] \), except for the crossings of type (A). Let \( \theta_{i,j} \) be chosen so that \( \theta_s - \epsilon < \theta_{i,j} < 2\pi \), and choose an interval \( I_{ij} \subset [\theta_{i,j}, 2\pi] \). Twist each \( D_{ij} \) over \( I_{ij} \) by an amount \( 2\pi/t_i \) if \( t_i > 1 \).

We note that what we have essentially done is to turn the link \( W \) into a special form of braid, and the above theorem is essentially Alexander's theorem which asserts that every knot is the closure of some braid. With the link in this form, we can discuss the self-linking of the components, i.e., the linking number \( \lambda^{i,j}_{l,i} \) of \( W_{i,j} \) and \( W_{l,i} \), where \( 1 \leq j \leq l \), \( 1 \leq l \leq t \). The list of coefficients \( \{\lambda_{ik}^{ij}\} \) describes the self-linking of this particular braid, but since two different braids may give rise to the same knot, and since there may be geometric linking which is not detected by these coefficients, they are not a topological invariant of the knot \( W \), and they do not contain full information about the knotting of \( W \). Nonetheless, we shall see that they contain the information about \( W \) which is necessary for our problem.

The second goal of this section is to give a meaningful definition for \( W(C) \) and to compute its value. At present \( W(C) \) has meaning only along curves which are tangent to \( F \). If \( C \) is transverse to \( F \), then \( \zeta \mid C \) defines a directed line bundle over \( C \). Let \( C' \) denote a nonzero cross section in this bundle. Define \( W(C') \) to be the linking number of \( C' \) and \( C \). For example, if \( C \) is the axis of the Ree component \( U_i \), then \( W(C) = n_i \).

**Lemma 4.2.** \( W(C) = W(C') = \sigma_i (b_i - n_i a_i) \).

**Proof.** Let \( Z \) denote a cylinder in \( W \) whose boundary is \( \bar{l}_i \cup l_i \), and let \( \pi_t (0 \leq t \leq 1) \) denote a family of smooth curves which are transverse to \( F \), which fiber \( Z \) smoothly, and which satisfy \( Z_0 = \bar{l}_i \) and \( Z_1 = l_i \). The foliation \( F \) provides a 2-plane bundle over each \( Z_t \). When \( t = 0 \), this is the bundle in which the rotation of \( \zeta \) about \( Z_0 = \bar{l}_i \) is defined, and when \( t = 1 \), this is the bundle \( T(\partial V_i) \mid l_i \), i.e., the bundle in which \( W(Z) \) is defined. Each of these winding numbers is referenced with respect to the normal field to \( F \), which is compatible defined for every \( t \). Consequently, \( W(Z) \) is defined for every \( t \) and varies continuously with \( t \). Since its values are always integers, it follows that it is constant as a function of \( t \).

We shall now use this structure to give a precise description of the Compatibility Condition (II.3.5). The setting is that \( X_i \) is a collar for \( W_i \) such that \( \partial X_i \) is transverse to \( F \), \( (A_j \mid j \in Z_t(t_i)) \) is a family of annuli in \( X_i \) whose union is \( L \cap X_i \) for some leaf \( L \) of \( F \), and \( \zeta \) is a vector field which has been specified.
on $\partial W_i \cup \partial X_i \cup L$. Let $I_i$ and $\hat{I}_i$ be parallel longitudes in $\partial W_i$ and $\partial X_i$, respectively. A necessary condition for $\zeta$ to have an extension to $\mathcal{F} | X_i$ is that $W_\zeta(I_i) = W_\zeta(\hat{I}_i)$ (indeed, in II.3.5, we have assumed $\zeta | I_i = \zeta | \hat{I}_i$ if $I_i$ and $\hat{I}_i$ are identified by the chosen projection of $\partial X_i$ onto $\partial W_i$). Let $\{I_{ij} : j \in \mathbb{Z}/(t_i)\}$ and $\{\hat{I}_{ij} : j \in \mathbb{Z}/(t_i)\}$ be the families of subarcs of $I_i$ and $\hat{I}_i$, respectively, such that $I_{ij}(\hat{I}_{ij})$ lies between $A_j$ and $A_{j+1}$ (mod $t_i$). Let $\{\gamma_j : j \in \mathbb{Z}/(t_i)\}$ be a family of arcs such that $\gamma_j \subset A_j$, $\gamma_j$ joins $I_i \cap A_j$ to $\hat{I}_i \cap A_j$, and $\lambda_j = \gamma_j - I_{ij} - \gamma_{ij+1} - \hat{I}_{ij}$ does not link $W_i$.

**Lemma 4.3.** A necessary condition for $\zeta$ to have an extension to $X_i$ is that $W_\zeta(\lambda_j) = 0$ for each $j$.

**Proof.** Since $\lambda_j$ does not link $W_i$, there is a disk $D_j \subset X_i$ such that $\partial D_j = \lambda_j$. If $\zeta$ has an extension to $X_i$, then it has an extension to $D_j$, and so $W_\zeta(\lambda_j) = 0$.

**Corollary 4.4.** A necessary condition for $\zeta$ to have an extension to $X_i$ is that for each $j$,

$$0 = W_\zeta(\gamma_j) - W_\zeta(\gamma_{j+1}) + W_\zeta(I_{ij}) - W_\zeta(\hat{I}_{ij}).$$

**Corollary 4.5.** The Compatibility Condition which $\zeta$ must satisfy in order that it have an extension to $X_i$ is that for each $j$,

$$W_\zeta(\gamma_j) = W_\zeta(\gamma_{j+1}) + \sum_{k=1}^{j-1} [W_\zeta(I_{ik}) - W_\zeta(\hat{I}_{ik})].$$

In particular, when $\zeta | \partial X_i$ and $\zeta | \partial W_i$ are identical under horizontal projection, then the requirement is that for each $j$, $W_\zeta(\gamma_j) = W_\zeta(\gamma_{j+1})$.

5. The Full Obstruction Class

We shall now complete our description of $M$ as a CW-complex. We have treated $L \cup W$ as the 1-skeleton of $M$. Actually, there are some essential 2-cells in $L$, but since we have an extension of $\zeta$ to $L$, they are of no great consequence. Observe that $N = S^3 - L \cup W$ is an open solid torus of genus $t$. In the next lemma, we construct $t$ 2-cells $\{B_{ij} : i = 1, \ldots, k; j = 1, \ldots, t_i\}$ in $N$ such that $N - \bigcup B_{ij}$ is an open 3-cell. These cells constitute the remainder of the desired CW decomposition for $M$. Our problem then reduces to finding an extension of $\zeta$ to $\bigcup B_{ij}$. Given such an extension, there is a unique extension to the remaining 3-cell.

**Lemma 5.1.** There is a disjoint family of open 2-disks $\{B_{ij} : i = 1, \ldots, k; j = 1, \ldots, t_i\}$ in $N$ such that
1. each $B_{ij}$ is transverse to $\mathcal{F}$;

2. $\partial B_{ij} = \gamma_{ij} + l_{ij} - \eta_{ij} - l_a$, where $\{\gamma_{ij}\}$ and $\{\eta_{ij}\}$ are disjoint families of arcs in $L - W$ such that $\gamma_{ij}$ and $\eta_{ij}$ both originate at $l_a \cap D_i$, $\gamma_{ij}$ terminates at $l_{ij} \cap D_{ij}$, and $\eta_{ij}$ terminates at $l_{ij} \cap D_{ij+1}$;

3. $\tilde{\gamma}_{ij} \equiv \gamma_{ij} \cap \sum_{k=1}^{h-1} \lambda_{ik}^j \partial D_{kj} + \sum_{k=h+1}^{h+t} \lambda_{ik}^j \partial D_{ki} + \psi_j$, where $\psi_j$ is an arc in $D_i$ from $D_{ii} + t_{ij+1}$ and $\lambda_{ik}^j = \sum_r \lambda_{ik}^j$.

Proof. Let $\{\gamma_{ij}\}$ be any disjoint family of arcs in $L$ which satisfies the requirements of the lemma. Define $\{B_{ij}\}$ to be the family of loci traced out by lifting $\{\gamma_{ij}\}$ vertically through the levels of $S^3 - W = S^2 \times S - W$ where the lift is accomplished by means of the isotopies which were described in II.4.1. We note that in the course of an isotopy of $I_a \times D_i$, it is necessary for $D_{ij}$ to rotate in the counterdirection in order to maintain the nonlinking relationship. Thus the operation induced on $\gamma_{ij}$ as it passes through the $I_a$-levels is that it is stretched (positively or negatively) around $D_{ij}$ and $D_{ij+1}$ and then it is stretched back around $I_{ij}$ (negatively or positively, respectively). Thus the homotopy difference between the lifts of $\gamma_{ij}$ to the initial level of $I_a$ and to the terminal level of $I_a$ is precisely $\pm \partial D_{ij}$. Since these lifts are achieved by an isotopy at each stage, it follows that the loci $(B_{ij})$ are disjoint. It remains to describe the change in these curves as we lift through $I_B$. The disk $D_i$ is twisted by $2\pi/\ell_i$, and in order to keep $l_i$ from linking $l_i$, it is necessary for $D_{ij}$ to be counterrotated by an equal amount. Thus the homotopy difference between the lifts of $\gamma_{ij}$ to the initial and terminal levels of $I_a$ is an arc $\delta_i$, which is the locus of the point $l_{ij} \cap D_{ij}$ under the isotopy of $I_a$.

For each $(i,j)$, there is an oriented framing $\Pi$ for $B_{ij}$ such that the first axis $\Pi^{(1)}$ is tangent to $\mathcal{F} \cap B_{ij}$, the second axis $\Pi^{(2)}$ is tangent to $\mathcal{F}$ and transverse to $B_{ij}$, and the third axis $\Pi^{(3)}$ is tangent to $B_{ij}$ and transverse to $\mathcal{F}$ in the same sense as the globally defined orienting normal field for $\mathcal{F}$. Expressing $\xi | \partial B_{ij}$ in terms of its $\Pi$-coordinates, we obtain a mapping $\partial B_{ij} \rightarrow S^1$, and $\xi | \partial B_{ij}$ has an extension to $\mathcal{F} | B_{ij}$ if and only if this mapping has degree zero, i.e., if and only if

$$0 = W_\ell(\partial B_{ij}) = W_\ell(l_{ij}) - W_\ell(l_a) \div W_\ell(\gamma_{ij}) - W_\ell(\eta_{ij}).$$

As we inspect these terms, we discover that $W_\ell(l_a)$ is an integer and that the other terms are usually not integers. Moreover, the value of $W_\ell(l_{ij})$ depends not only on $\xi | \partial W_i$, but on the properties of $\xi | W_{ij}$, and these properties are not invariant under homotopies of $\xi | W$. Thus we must determine how homotopies of $\xi | W$ influence the values of $W_\ell(\partial B_{ij})$. For this, it is necessary to examine how homotopies of $\xi | W$ influence $\xi | (W \cup L)$, and this examination can be conducted by using an analysis which is quite similar to the analysis which was used in II.4 to study the Compatibility Condition. First of all, to understand what is happening geometrically, observe that if $\xi | \partial W_k$ is altered by a homotopy, then $\xi | L$ must be altered in a compensating way,
and this alteration can be confined to a family of annuli in $L$ which are contiguous to $\partial W_i$. Since the index $i$ will be fixed for the remainder of this discussion, and out of respect for the problems of the typesetter, we introduce the notation $\tau = t_i$ for this section. Now we can choose $\zeta : \partial W_i$ such that for $j = 1, \ldots, \tau - 1,$

$$0 = W_\zeta(l_{ij}) - W_\zeta(l_0) + W_\zeta(\gamma_{ij}) - W_\zeta(\gamma_{ij}),$$

i.e.,

$$W_\zeta(l_{ij}) = W_\zeta(l_0) + W_\zeta(\gamma_{ij}) - W_\zeta(\gamma_{ij}).$$

Now by II.4.2,

$$\sigma_i(b_i - n_i a_i) = W_\zeta(l_i) = \sum_{j=1}^{\tau - 1} W_\zeta(l_{ij})$$

and so we conclude that

$$W_\zeta(l_{ij}) = \sigma_i(b_i - n_i a_i) - (\tau - 1) W_\zeta(l_0) + \sum_{j=1}^{\tau - 1} [W_\zeta(\gamma_{ij}) - W_\zeta(\gamma_{ij})].$$

Combining this formula with the formula for the extendibility of $\zeta$ to $B_i, \tau$, we obtain the next result.

**Lemma 5.2.** A necessary and sufficient condition for $\zeta : (W \cup L)$ to have an extension to $\{B_1, j = 1, \ldots, \tau\}$ is that

$$\sigma_i(b_i - n_i a_i) - \tau W_\zeta(l_0) - \sum_{j=1}^{\tau - 1} [W_\zeta(\gamma_{ij}) - W_\zeta(\gamma_{ij})].$$

Now II.5.1(3) provides a homotopy description of $\gamma_{ij} - \gamma_{ij}$ which allows us to compute the value of the right-hand side. The only problem with this approach is that we do not yet understand $\sum_{j=1}^{\tau} W_\zeta(\gamma_{ij}) = W_\zeta(\sum_{j=1}^{\tau} \gamma_{ij})$. But we observe, from the proof of II.5.1, that this composite arc is just the projection of $l_i$ into $D_{i1}$, and since $l_i$ does not link $W_i$, it follows that this projection bounds a portion of $D_i - \bigcup D_{ij}$, and so $W_\zeta(\sum_{j=1}^{\tau} \gamma_{ij}) = 0$. This completes the proof of the main result of this section.

**Theorem 5.3.** A necessary and sufficient condition for $\zeta : (W \cup L)$ to have an extension to $\{B_{ij}\}$ is that for $i = 1, \ldots, k,$

$$\sigma_i(b_i - n_i a_i) = \sigma_i t_i + \sum_{h \neq i} \lambda_{ij}^i \sigma_h (d_h - n_h c_h) + \sum_{h \neq i} \lambda_{ij}^i \sigma_j (d_i - n_i c_i).$$

6. Interpretation

We conclude this investigation by giving an interpretation of the meaning of these obstruction results. By II.3.6, if $\zeta$ is specified on the Reeb components,
then its homotopy coefficients \( \{n_0, \ldots, n_k\} \) determine its extendibility and the homotopy class of any possible extension. By II.3.4, \( n_0 \) is determined by \( \{n_1, \ldots, n_k\} \). Finally, II.5.3 gives a necessary and sufficient condition on \( \{n_1, \ldots, n_k\} \) for \( \zeta \) to have an extension. All of the other coefficients in these formulas are parameters which describe the foliation. Observe that II.5.3 can be viewed as a linear diophantine system of equations for \( \{n_1, \ldots, n_k\} \), that any solution to this system describes a globally defined tangent vector field for \( \tilde{\mathcal{F}} \), and that distinct solutions represent tangent vector fields for \( \tilde{\mathcal{F}} \) which are not homotopic.

Define matrices \( \Lambda = \text{diag}(a_h) \), \( C = \text{diag}(c_h) \), \( \Sigma = \text{diag}(\sigma_h) \) and \( A = (\lambda_h^i) \) (where \( \lambda_h^i \) is the linking number which is defined in Theorem 1.2), and let \( b, d, n, t \) denote the column vectors with respective elements \( b_i, d_i, n_i, \) and \( t_i \). Then the system II.5.3 can be written as

\[
(AC - \Lambda) \Sigma n = \sigma_0 t - \Sigma b + A \Sigma d.
\]

**Lemma 6.2.** A necessary and sufficient condition for \( \mathcal{F} \) to have at most one tangent vector field is that \( AC - \Lambda \) be nonsingular.

**Proof.** By [4, p. 37], nonsingularity is the criterion for uniqueness of solutions to linear diophantine problems.

We gain additional insight into this last result by computing the Mayer-Vietoris description of the handlebody \( \mathcal{M} \). Let \( W = \bigcup_{i=1}^{k} W_i \), and let \( N = \mathcal{M} - W \cong S^3 - W \). Then we have the exact Mayer Vietoris homology sequence

\[
0 \to H_2(M) \to H_2(\partial N) \oplus H_4(N) \oplus H_4(W) \to H_4(M) \to 0.
\]

where \( \varphi = (\varphi_1, \varphi_2) \) and \( \varphi_1, \varphi_2 \) are inclusions. Since \( \partial N = \partial W = \bigcup_{i=1}^{k} \partial W_i \), we have a homology basis \( \{l_i, m_i\}_{i=1}^{k} \). \( H_4(N) \) is generated by \( \{\bar{m}_i\}_{i=1}^{k} \) and \( l_i = \sum_{h=1}^{k} \lambda_h^i \bar{m}_i \). Also, \( H_4(W) \) is generated by \( \{l_i = d_j l_i - b_i \bar{m}_i\}_{i=1}^{k} \). Thus

\[
\varphi_1(l_i) = \sum \lambda_h^i \bar{m}_i, \quad \varphi_2(l_i) = a_i l_i - b_i m_i = a_i l_i,
\]

\[
\varphi_1(m_i) = \bar{m}_i, \quad \varphi_2(m_i) = c_i l_i - b_i m_i = c_i l_i,
\]

and so the matrix for \( \varphi \) with respect to this basis is

\[
\begin{pmatrix}
I & A \\
C & A
\end{pmatrix},
\]

which is row equivalent to (with no change in determinant)

\[
\begin{pmatrix}
I & A \\
0 & A - CA
\end{pmatrix},
\]

whose determinant is \( \det(AC - \Lambda) \). Since \( A, C, \Lambda \) are all symmetric, \( \det(A - CA) = \det(A - AC) \).
LEMMA 6.3. A necessary and sufficient condition for $H_2(M) = 0$ is that $\det(A - AC) \neq 0$. A necessary and sufficient condition for $H_1(M) = 0$ is that $\det(A - AC) = \pm 1$.

COROLLARY 6.4. A necessary and sufficient condition for $\mathcal{F}$ to have unique tangent vector fields (when they exist) is that $H^1(M) = 0$. (We knew sufficiency from obstruction theory!)

Proof. By II.6.2 and II.6.3, the condition is that $H_2(M) = 0$. By Poincaré duality, $H_2(M) \cong H_1^!(M)$.

We observe that if $H_1(M) = 0$, then $M$ is a homology sphere and so there are no obstructions to existence. In this case, we also note (II.6.2) that $\det(A - AC) = \pm 1$ and so the diophantine system is soluble. On the other hand, there will certainly be cases where $\det(A - AC) \neq \pm 1$ and yet a solution is possible.

Certain geometric cases deserve special mention. If each Reeb component is unknotted then the diagonal elements of $A$ are all zero. In particular, this is the case for simple handlebody foliations. If $\mathcal{F}$ is a Lickorish foliation (II.2.1), then it is simple ($t = 1$) and $A = I$, $b = 0$, $d = 1$.

THEOREM 6.5. If $\mathcal{F}$ is a Lickorish foliation, then the homotopy coefficients for tangent vector fields to $\mathcal{F}$ satisfy

$$(AC - I) \Sigma n = \sigma_0 1 + A \Sigma 1.$$

Moreover, if $H^1(M) \neq 0$, then this equation fails to have a solution for every choice of $\Sigma$; i.e., nonuniqueness implies nonexistence.

Proof. The equation for the homotopy coefficients is merely the restatement of II.5.3 with the special parameters of the Lickorish handlebody structure. If $0 \neq H^1(M) \cong H_2(M)$, then by the Universal Coefficient Theorem, $H^2(M)$ contains a direct summand which is isomorphic to $H_2(M) \cong \text{Hom}(H_2(M), Z)$. Now $H_2(M)$ is free and is generated by the 2-cells $\{B_i\}_{i=1}^k$ of our CW decomposition for $M$ (for a Lickorish handlebody structure, $t_i = 1$). Hence there is some $B_i$ which represents a nontrivial element of $H_2(M)$. Suppose that we can find a choice of $\xi \mid W \cup L$ such that $W_\xi(\partial B_i) \neq 0$. Then $W_\xi$ represents a nontrivial element in $\text{Hom}(H_2(M), Z)$, i.e., a nontrivial cohomology class. A different choice of $\xi \mid W \cup L$ amounts to a change of $\xi$ on the 1-skeleton and such a change will alter $W_\xi$ only by a coboundary; i.e., a better choice of $\xi$ cannot trivialize a nontrivial cohomology class. It remains to show that for some choice of homotopy coefficients, $W_\xi(\partial B_i) \neq 0$. By II.5.3,

$$W_\xi(\partial B_i) = \sigma_i a_i n_i \div \sigma_0 + \sum_{e=1}^{k} \sigma_e \lambda^e_b (1 - c_e n_e).$$
and so we seek \( \{n_1, \ldots, n_k\} \) to violate
\[
\sigma_i a_i n_i - \sum \sigma_h \lambda_h^i e_h n_h = -\sigma_0 - \sum \sigma_h \lambda_h^i.
\]
Now the only way for this equation to be satisfied for every choice of \( \{n_1, \ldots, n_k\} \) is for the coefficients of each \( n_h \) to be zero and for the right-hand side to be zero. But for a Lickorish handlebody structure, \( a_i = 1 \) and so we can violate this equation by choosing \( n_h = 0 \) for \( h \neq i \) and by choosing \( n_i \) judiciously.

Remark. We have given this proof in the above awkward form for a purpose. It would have been neater to substitute \( a_i = 1 \) when we first gave the formula for \( W_2(\partial B_1) \), but this would have suppressed the significance of the coefficient \( a_i \). One must wonder whether or not \( H^i(M) \neq 0 \) always implies that \( \mathcal{F} \) cannot have a tangent vector field. We note that if in the general case some \( B_{ij} (t_i > 1) \) represents a nontrivial element of \( H_2(M) \), then we can obtain a nontrivial cohomology class \( W_\zeta \) by simply choosing \( \zeta \mid W \cup L \) to violate the Compatibility Condition. On the other hand, if for every generator \( B_{ij} \) of \( H_2(M) \), we have \( t_i = 1, a_i = 0, \lambda_i^j e_h = 0 \), then neither trick works and the question remains unresolved.

**Corollary 6.6.** Let \( M \) be furnished with a Lickorish handlebody structure, and suppose that for some compatible foliation \( \mathcal{F} \) (choice of \( \Sigma \)) there is a tangent vector field. Then every compatible foliation (choice of \( \Sigma \)) has at most one tangent vector field.

**Proof.** If there is a tangent vector field, then \( H^i(M) = 0 \) and so there is no obstruction to homotopies between tangent vector fields for any foliation of \( M \).

Another way of stating this last result is that \( \det(\Lambda) \neq 0 \). If every compatible foliation to some Lickorish handlebody structure has a tangent vector field, then we obtain a sharper result.

**Theorem 6.7.** Suppose that \( M \) is given with a specified Lickorish handlebody structure with \( k \) handles and suppose that for every compatible foliation (choice of \( \Sigma \)) there exists a tangent vector field. Then \( \det(\Lambda) \) divides \( 2^k \), and if \( k \) is odd, then \( \det(\Lambda) < 2^k \).

**Proof.** By 6.5, \( \Lambda \neq 0 \) is nonsingular, and so the equation
\[
(\Lambda) \Sigma n = 1 + A \Sigma I
\]
has a unique solution \( \Sigma n \) for every choice of \( \Sigma \). Then
\[
(\Lambda)(\Sigma n - \Sigma' n') = A(\Sigma - \Sigma') I = 2Av,
\]
where \( v \) is a vector with elements 0 or \( \pm 1 \). Thus we conclude that for every integer vector \( \mu \),

\[
(AC - I)\mu = 2A\mu
\]

has a solution; i.e., \( 2(AC - I)^{-1}A \) is an integer matrix.

**Lemma 6.8.** \( 2(AC - I)^{-1} \) is an integral matrix.

**Proof.** \( (AC - I)[2(AC - I)^{-1} AC - 2I] = 2I. \)

**Corollary 6.9.** \( \det(AC - I) \) divides \( 2^k \).

**Proof.** \( 2^k = \det 2(AC - I)^{-1}(AC - I) = \det 2(AC - I)^{-1} \det(AC - I). \)

To prove the last statement of the theorem, we observe that by the Smith Normal Form [4], \( 2(AC - I)^{-1} = UDV \), where \( U \) and \( V \) are unimodular and \( D \) is diagonal. Consequently, \( AC - I = 2V^{-1}D^{-1}U^{-1} \). Since \( D \) and \( 2D^{-1} \) must be integer matrices, it follows that each has only diagonal elements \( \pm 1 \) or \( \pm 2 \). If every diagonal element of \( 2D^{-1} \) is \( \pm 2 \), then \( (AC - I)n = V^{-1}(2D^{-1})U^{-1}n \) must always be an even vector. But by hypothesis, there is a vector \( n \) such that \( (AC - I)n = 1 + A1 \). The following lemma completes the proof of the theorem.

**Lemma 6.10.** If \( A \) is a symmetric \( k \times k \) matrix with even trace, and if each row-sum of \( A \) is odd, then \( k \) is even.

**Proof.** For any symmetric matrix, the trace and the sum of all elements have the same parity. Thus the sum of the row-sums has the same parity as the trace. If \( k \) is odd and if each row-sum is odd, then the trace must be odd, contrary to hypothesis.

**References**