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Finite-temperature regularization

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Abstract

We present a non-perturbative regularization scheme for Quantum Field Theories which amounts to an embedding of the original unregularized theory into a spacetime with an extra compactified dimension of length $L \sim \Lambda^{-1}$ (with Λ the ultraviolet cutoff), plus a doubling in the number of fields, which satisfy different periodicity conditions and have opposite Grassmann parity. The resulting regularized action may be interpreted, for the fermionic case, as corresponding to a finite-temperature theory with a supersymmetry, which is broken because of the boundary conditions. We test our proposal both in a perturbative calculation (the vacuum polarization graph for a D-dimensional fermionic theory) and in a non-perturbative one (the chiral anomaly).

1. Introduction

When a classical symmetry cannot be preserved by *any* regularization method, this manifests itself as an anomaly [1], an important phenomenon which is perhaps best understood in the context of the path integral method [2,3]. For the case of the chiral symmetry, the importance of the anomalies, should hardly need to be emphasized, since they put constraints for model-building, and they modify the naive predictions of current algebra [4].

A *non-perturbative* regularization (like the lattice) makes it possible to use approximation methods whose application to the usual perturbative context is problematic. On the other hand, a non-perturbative regularization, to be useful, should preserve as many symmetries as possible, and they should be realized on a *local* regularized action. In the case of theories with anomalies, the construction of a non-perturbative regularization is a non-trivial task since, by definition, a regularization introduces a major change in the short distance properties of the theory, precisely the region where theories with anomalies are most sensitive.

In this Letter we present a non-perturbative regularization method formulated in terms of a higher-dimensional extension of the model, at finite-temperature. The original fields have to be immersed in a spacetime with an extra finite dimension; for the fermionic case a supersymmetry is trivially implemented by a doubling in the number of

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degrees of freedom, adding to each physical field an unphysical partner which is Grassmann even, but has the same action as the fermion.[1] This supersymmetry is, however, broken because of the boundary conditions in the extra coordinate, whose length is of the order of the inverse of the UV momentum cutoff. Moreover, the non-anomalous symmetries of the theory are manifest.

The method may be understood as an implementation of the ideas developed in [5,6], namely, the use of an infinite number of Pauli–Villars regulators [7,8]. Our main point is to give a concrete realization of the regularized action in terms of an action, which should then have a compactified coordinate for the masses of the regulator fields to emerge naturally as the corresponding discrete momenta.

For the bosonic case, the extension requires the addition of a bosonic partner with a negative sign in the quadratic part of its Euclidean action, like in the standard Pauli–Villars method, when applied to a (bosonic) scalar field theory [8]. This means that, in the canonical quantization framework, the norm of the corresponding modes must be negative, to avoid negative energies. Besides, the regulator field has anti-periodic boundary conditions in the imaginary time direction.

The use of higher-dimensional representations for fermionic determinants is a well-known idea; indeed it is the basis of the celebrated 'overlap' formalism [9], based on Kaplan's idea [10] to represent a D-dimensional chiral determinant by a $D+1$ Dirac theory with a non-homogeneous mass. However, our motivation here is to find a representation of the *regularized* fermionic determinant in the continuum. The compact length for the extra dimension makes it also possible to use a Matsubara-like formalism as in finite-temperature QFT. In this setting, the finiteness of the regularized action results from the fact that at finite-temperature the UV divergencies are the same as for $T=0$, in combination with the property that the zero-temperature limit of the regularized theory is trivial: for the fermionic case, the supersymmetry is exact at $T=0$, while for the bosonic case there is no effective propagation in that limit.

The organization of this Letter is as follows: we first define the regularization by a study of the particular case of a Dirac operator determinant. We then extend the method to a bosonic QFT and perform a perturbative test on the method. The (non-perturbative) issue of chiral anomalies is also discussed.

2. Fermionic theory

We begin with the particular case of a fermionic determinant in an external gauge field, which, as we shall see, serves both as motivation and test for the method. To this end, we first introduce the partition function $\mathcal{Z}_f(A)$ for the unregularized theory, which is defined by:

$$\mathcal{Z}_f(A)=\int \mathcal{D}\psi\,\mathcal{D}\bar{\psi}\,\exp\bigl[-S_f(\bar{\psi},\psi;A)\bigr], \tag{1}$$

where S_f denotes the unregularized Euclidean action for a massless Dirac field in D dimensions, i.e.,

$$S_f(\bar{\psi},\psi;A)=\int d^Dx\,\bar{\psi}\,\mathcal{D}_c\psi, \tag{2}$$

where $\mathcal{D}_c=\not\partial+\not A$, and the Hermitian γ matrices satisfy the relation $\{\gamma_\mu,\gamma_\nu\}=2\delta_{\mu\nu}$. The gauge connection A verifies

$$A_\mu=-A_\mu^\dagger\equiv\begin{cases} i\mathcal{A}_\mu, & \text{in the Abelian case,}\\ \mathcal{A}_\mu^a\tau_a, & \text{in the non-Abelian case,}\end{cases} \tag{3}$$

[1] Of course, this means that the unphysical field violates the spin-statistics theorem, as a Pauli–Villars regulator does.

where τ_a are (anti-Hermitian) generators of the Lie algebra of the non-Abelian gauge group, and both $\mathcal{A}_\mu$ and $\mathcal{A}_\mu^a$ are real. This implies the anti-Hermiticity of $\mathcal{D}_c$ in Euclidean space, a property that will be taken into account for the construction of the regularized theory.

To introduce the non-perturbative regularization, we define a 'regularized' Dirac operator $\mathcal{D}$, a function of the unregularized operator $\mathcal{D}_c$, through the equation

$$\frac{i\mathcal{D}}{M} = f\left(\frac{i\mathcal{D}_c}{M}\right), \tag{4}$$

where M is a constant with the dimensions of a mass, playing the role of an UV cutoff and the function f has to be chosen in order to tame the UV behaviour of the Dirac operator. Since the UV properties manifest themselves in the large eigenvalues of $\mathcal{D}_c$, when considered as a function of a real variable x, $f : \mathbb{R} \to \mathbb{R}$ should verify

$$x \to 0 \Rightarrow f(x) \sim x, \qquad x \to \pm\infty \Rightarrow f(x) \to \pm 1. \tag{5}$$

Note that the first relation amounts to requiring the regularized operator to behave like the unregularized one when the eigenvalues are small, while the second condition implies that $\mathcal{D}$ is bounded. Indeed, the spectrum of $\mathcal{D}$ is confined to the range $[-M, M]$. Besides, the function f should be one to one, in order to preserve some important properties of the spectrum, as the degeneracy of each eigenvalue. A convenient choice for f satisfying all of these constraints is:

$$\frac{i\mathcal{D}}{M} = \tanh\left(\frac{i\mathcal{D}_c}{M}\right), \tag{6}$$

which, of course, would yield a non-local theory if $\mathcal{D}$ were used to build a D-dimensional theory. Rather than working with this non-local expression, we shall instead consider an equivalent formulation of this regularized theory where the non-locality is traded *for the existence of a compactified extra dimension, plus an extra unphysical field*. A first step in that direction is to use an infinite product representation for the tanh function,

$$\tanh(x) = \xi x \prod_{n=1}^{\infty} \frac{x^2 + n^2\pi^2}{x^2 + (n - \frac{1}{2})^2\pi^2}, \tag{7}$$

where $\xi = \prod_{n=1}^{\infty}(1 - \frac{1}{2n})^2$. By the use of some straightforward algebra, we may insert (7) in (6) to write the regularized Dirac operator as:

$$\mathcal{D} = \xi \prod_{n=-\infty}^{+\infty} \left[\frac{\mathcal{D}_c + \pi n M}{\mathcal{D}_c + (n + \frac{1}{2})\pi M}\right], \tag{8}$$

a form which already suggests the use of a higher-dimensional representation and the introduction of fields of opposite statistics to obtain a local version of the determinant of $\mathcal{D}$. Indeed, we note that the regularized partition function $\mathcal{Z}_f^{\mathrm{reg}}$, defined in terms of $\mathcal{D}$, may be written as follows:

$$\mathcal{Z}_f^{\mathrm{reg}} = \int \mathcal{D}\psi\, \mathcal{D}\bar{\psi}\, \exp\left[-\int d^D x\, \bar{\psi}\, \mathcal{D}\psi\right] = \exp \sum_{-\infty}^{+\infty} \left\{\mathrm{Tr}\ln[\mathcal{D}_c + \pi n M] - \mathrm{Tr}\ln\left[\mathcal{D}_c + \left(n + \frac{1}{2}\right)\pi M\right]\right\}. \tag{9}$$

It should be evident that the sum over n of the D-dimensional traces may also be interpreted as $(D + 1)$-dimensional traces for a finite-temperature theory, in the Matsubara formalism, and with translation invariance along the imaginary time coordinate. In this finite-temperature language, we have $\beta = 2/M$ (or $T = M/2$) and the two operator traces are associated with two fields with opposite (odd/even) Grassmann character which have to be integrated with opposite (periodic/anti-periodic) boundary conditions. Indeed, the two fields can be naturally associated to the Matsubara frequencies:

$$\omega_n^{(+)} = \frac{2\pi}{\beta} n = \pi n M, \qquad \omega_n^{(-)} = \frac{2\pi}{\beta}\left(n + \frac{1}{2}\right) = \pi\left(n + \frac{1}{2}\right) M, \tag{10}$$

where $\omega_n^{(+)}$ and $\omega_n^{(-)}$ correspond to periodic and anti-periodic boundary conditions, respectively. According to this interpretation, we may then write the partition function as a functional integral over two Dirac fields in $D+1$ dimensions, $\Psi^{(+)}(\tau, x)$ and $\Psi^{(-)}(\tau, x)$, associated to the two different types of boundary conditions (bc) and with opposite Grassmann character,

$\Psi^{(+)}(\tau, x)$: periodic bc, Grassmann odd,

$\Psi^{(-)}(\tau, x)$: anti-periodic bc, Grassmann even.

The resulting partition function reads

$$\mathcal{Z}_f^{\text{reg}} = \int \mathcal{D}\Psi^{(+)} \mathcal{D}\Psi^{(-)} \mathcal{D}\bar{\Psi}^{(+)} \mathcal{D}\bar{\Psi}^{(-)} \exp\left[-\mathcal{S}_f^{\text{reg}}\left(\bar{\Psi}^{(+)}, \bar{\Psi}^{(-)}, \Psi^{(+)}, \Psi^{(-)}; A\right)\right], \tag{11}$$

with

$$\mathcal{S}_f^{\text{reg}} = \int_{-1/M}^{+1/M} d\tau \int d^D x \left[\bar{\Psi}^{(+)}(-i\partial_\tau + \mathcal{D}_c)\Psi^{(+)} + \bar{\Psi}^{(-)}(-i\partial_\tau + \mathcal{D}_c)\Psi^{(-)}\right]. \tag{12}$$

Notice that the integration rules associated with $\Psi^{(+)}$ coincide with those to be imposed on ghosts in finite-temperature gauge theories. In this sense, one could think on conditions imposed to $\Psi^{(-)}$ as those corresponding to ghosts of ghosts.

Concerning action (12), it should be noted that it is not covariant when considered as a $(D+1)$-dimensional object. This is not a problem, of course, since real physics corresponds to D dimensions, where the theory is indeed invariant. Besides, the fact that there is a finite range for the imaginary time τ coordinate already breaks (explicitly) the symmetry between τ and the remaining D coordinates, denoted collectively by x. This breaking is a usual phenomenon in finite-temperature QFT, and may be traced to the fact that there is a preferential reference system, namely, the 'thermal bath'. However, had the original theory been defined on an even number of dimensions, i.e., $D = 2n$, the corresponding Hermitian chirality matrix γ_s ($\gamma_s = \gamma_5$ when $D = 4$) could have been used in order to obtain an equivalent $(D+1)$-invariant looking expression for the Lagrangian. Indeed, it is sufficient to note that, both for $\Psi^{(+)}$ or $\Psi^{(-)}$, we have

$$\bar{\Psi}(-i\partial_\tau + \mathcal{D}_c)\Psi = \bar{\Psi}(-i\gamma_s)i\gamma_s(-i\partial_\tau + \not{D})\Psi \equiv \tilde{\Psi}\Gamma_\alpha D_\alpha \Psi, \tag{13}$$

where, in the last line, the index α runs from 0 to D, and we shall alternatively use the notation τ or x^D for the extra coordinate, depending on the context. We have introduced the Dirac matrices Γ_α in $D+1$ dimensions in such a way that $\Gamma_\mu = i\gamma_s\gamma_\mu$, $\forall\mu$ such that $0 \leqslant \mu \leqslant D-1$, and $\Gamma_D \equiv \Gamma_\tau \equiv \gamma_s$.

Of course, $\Gamma_\alpha^\dagger = \Gamma_\alpha$, and

$$\{\Gamma_\alpha, \Gamma_\beta\} = 2\delta_{\alpha\beta}, \quad \forall\alpha, \beta = 0, 1, \ldots, D. \tag{14}$$

Also, the $\alpha = D$ component of A_α vanishes. In the last line of (13) we have used the notation $\tilde{\Psi} \equiv \bar{\Psi} i\gamma_s = \Psi^\dagger \Gamma_0$, i.e., $\tilde{\Psi}$ is the natural definition for $\bar{\Psi}$ in the $D+1$ covariant representation. With this in mind, we shall often write also $\bar{\Psi}$ (rather than $\tilde{\Psi}$) when working in the $D+1$ covariant representation, since the meaning of the bar should be clear from the context.

The changes to introduce when the original theory is massive are quite straightforward, since they stem from the replacement $\mathcal{D}_c \to \mathcal{D}_c + m$, where m is the fermion mass, in (4). However, when the $D+1$ covariant notation is used, we note that the physical mass m will arise as a constant gauge potential $A_\tau = im$ in the extra coordinate, namely,

$$\mathcal{S}_f^{\rm reg}(m) = \int d^{D+1}x \left[\bar{\Psi}^{(+)} \Gamma_\alpha D_\alpha(m) \Psi^{(+)} + \bar{\Psi}^{(-)} \Gamma_\alpha D_\alpha(m) \Psi^{(-)}\right], \tag{15}$$

where $d^{D+1}x \equiv d\tau\, d^D x$, and $D_\tau \equiv \partial_\tau + im$. The fact that the τ coordinate is compact implies that the constant m cannot be gauged away, unless a twist is introduced for the fermions. This twisting would carry, of course, the same physical content as the constant gauge field.

The regulated Lagrangian in $D+1$ dimensions is supersymmetric, in the sense that it is invariant under the global transformations:

$$\delta\Psi^{(+)} = i\xi\Psi^{(-)}, \qquad \delta\Psi^{(-)} = i\xi\Psi^{(+)}, \tag{16}$$

where ξ is a real Grassmann variable. One should expect that this symmetry is broken because of the different boundary conditions for the $(+)$ and $(-)$ fields.

We conclude this section with an alternative (also finite-temperature) representation for the regularized fermionic determinant. It should be evident that the logarithm of the regularized partition function (11), may also be written in terms of traces over anti-periodic fields, namely

$$\ln \mathcal{Z}_f^{\rm reg} = {\rm Tr}^{(-)} \ln\left[\Gamma_D\left(\partial_\tau + i\frac{\pi}{\beta}\right) + \Gamma_\mu D_\mu\right] - {\rm Tr}^{(-)} \ln[\Gamma_D \partial_\tau + \Gamma_\mu D_\mu], \tag{17}$$

where the $(-)$ over the traces indicates that for both fields it is anti-periodic, while the periodicity for the first term has been traded for the presence of a constant gauge field. This may also be written in the equivalent way:

$$\ln \mathcal{Z}_f^{\rm reg} = \int_0^{\pi/\beta} da \frac{\partial}{\partial a} {\rm Tr}^{(-)} \ln[\Gamma_\alpha D_\alpha(a)], \tag{18}$$

where

$$D_\mu(a) = D_\mu, \qquad D_D(a) = \partial_\tau + ia. \tag{19}$$

By taking the derivative with respect to a, in (18), we see that the equation becomes an integral over a of the thermal average of the conserved charge $Q(a)$, corresponding to the $(D+1)$-dimensional current for a (single) Dirac fermion. Namely,

$$\ln \mathcal{Z}_f^{\rm reg} = \int_0^{\pi/\beta} da\, {\rm Tr}^{(-)}\left[i\Gamma_\tau\left(\Gamma_\alpha D_\alpha(a)\right)^{-1}\right] = \int_0^{\pi/\beta} da\, \langle Q(a)\rangle, \tag{20}$$

where the charge $Q(a)$ depends of course on the value of the constant a and on the external field $A_\mu(x)$. When D is even, it is interesting to see that $\langle Q(a)\rangle$ may be decomposed into its parity breaking and parity conserving parts, yielding the same decomposition for the D-dimensional regularized effective action.

When $M \to \infty$ ($\beta \to 0$), we see that $\langle Q(a)\rangle$ becomes just the charge corresponding to the Chern–Simons current in $D+1$ dimensions, thus reproducing the well-known relation between gauge topological terms in even and odd dimensions. It is amusing to see that this regularization somehow preserves a relation like that, even when the theory is regulated.

When the original theory is massive, the only change is in the integration range for the a integral in (20), i.e.,

$$\ln \mathcal{Z}_f^{\rm reg}(m) = \int_m^{\pi/\beta+m} da\, \langle Q(a)\rangle. \tag{21}$$

3. Bosonic theory

Let us adapt here the previously introduced idea to the case of a bosonic action. We take as the essential property of the method to preserve by such a generalization, not the functional relation between two Dirac operators, but rather the finite-temperature interpretation. Indeed, the latter is a framework that can be implemented for any model, regardless of the spin and internal symmetry group of the fields, and its precise realization may be inferred from the Pauli–Villars method, by introducing the proper generalization.

To be more precise, we consider a real scalar field φ in D Euclidean dimensions, described by the action:

$$S_D = \int d^D x \left[\frac{1}{2}\partial_\mu\varphi\partial_\mu\varphi + \frac{1}{2}m^2\varphi^2 + V(\varphi)\right]. \tag{22}$$

Guided by the fermionic case, we introduce now two scalar fields $\Phi^{(\pm)}(\tau, x)$ and a finite-temperature action (with $\beta = 2/M$) in $D+1$ dimensions, with periodic boundary conditions for the field $\Phi^{(+)}(\tau, x)$:

$$\Phi^{(+)}(\tau, x) = \beta^{-1/2}\sum_{n=-\infty}^{+\infty}\Phi_n^{(+)}(x)e^{i\omega_n^{(+)}\tau}, \tag{23}$$

where $\omega_n^{(+)} = \frac{2\pi n}{\beta}$ and

$$\Phi_n^{(+)}(x) = \beta^{-1/2}\int_{-\beta/2}^{+\beta/2} d\tau\, \Phi^{(+)}(\tau, x)e^{-i\omega_n^{(+)}\tau} \tag{24}$$

and anti-periodic boundary conditions for $\Phi^{(-)}$.

In particular, the zero mode of $\Phi^{(+)}(\tau, x)$ is assumed to be proportional to $\varphi(x)$, i.e.,

$$\Phi_0^{(+)}(x) = \beta^{-1/2}\varphi(x). \tag{25}$$

On the other hand, inspired by the Pauli–Villars method in its scalar field version [8], we may introduce the regularized action $\mathcal{S}_{D+1}$ as follows:

$$\mathcal{S}_{D+1}\left[\Phi^{(+)}, \Phi^{(-)}\right] = \int_{-1/M}^{+1/M} d\tau \int d^D x \left\{\frac{1}{2}\left[\partial\Phi^{(+)}\partial\Phi^{(+)} + m^2\Phi^{(+)}\Phi^{(+)}\right] - \frac{1}{2}\left[\partial\Phi^{(-)}\partial\Phi^{(-)} + m^2\Phi^{(-)}\Phi^{(-)}\right] + V\left(\Phi^{(+)} + \Phi^{(-)}\right)\right\}. \tag{26}$$

Since there is no free propagator connecting $\Phi^{(+)}$ to $\Phi^{(-)}$, an application of the Wick theorem to a given Green's function yields the result that any diagram in the perturbative expansion can be built in the following way: replace in every diagram of the unregularized theory the free propagator by the sum of the propagator for $\Phi^{(+)}$ and the propagator for $\Phi^{(-)}$ (which differ in a global sign and in their boundary conditions). Thus the 'regularized propagator' G^{reg} is

$$D^{\text{reg}} = \left\langle\Phi^{(+)}\Phi^{(+)}\right\rangle + \left\langle\Phi^{(-)}\Phi^{(-)}\right\rangle. \tag{27}$$

Taking into account the different boundary conditions and the minus sign in front of the $(-)$ action, we see that in momentum space the sum of the propagators becomes, at large momenta, exponentially damped.

Namely,

$$D^{\mathrm{reg}}(k) \sim \exp\big[-\beta|k|\big]. \tag{28}$$

This explains the finiteness of the theory; we also need to be sure that the unregularized theory is recovered when $M \to \infty$. This property is indeed also true, since in this limit the anti-periodic fields $\Phi^{(-)}$ become infinitely massive, as well as all the non-zero modes of $\Phi^{(+)}$. Thus the theory is dimensionally reduced to the zero Matsubara frequency mode, which, by construction, is tantamount to $\varphi(x)$, and of course has the original action S_D.

4. A perturbative test

Let us now apply the regularization method to the 1-loop D-dimensional vacuum polarization. The regularized vacuum polarization function $\widetilde{\Pi}_{\mu\nu}$ can be written as the difference between the contributions corresponding to periodic and anti-periodic boundary conditions. Besides, the gauge field is independent of the τ coordinate, thus

$$\widetilde{\Pi}_{\mu\nu}(k) = \int \frac{d^D p}{(2\pi)^D} \big[t^{(+)}_{\mu\nu}(p,k) - t^{(-)}_{\mu\nu}(p,k)\big], \tag{29}$$

where

$$t^{(\pm)}_{\mu\nu}(p,k) = \sum_{n=-\infty}^{+\infty} \mathrm{tr}\bigg[\frac{1}{i\not{p} + \omega^{(\pm)}_n}\gamma_\mu \frac{1}{i(\not{p}+\not{k}) + \omega^{(\pm)}_n}\gamma_\nu\bigg] \tag{30}$$

and we have used the non-covariant expression for the Dirac matrices. Of course, since there is no physical gauge field component corresponding to $\alpha = D$, we shall not evaluate the components of $\widetilde{\Pi}_{\alpha\beta}$ involving that index. A similar remark holds true for the independence of the external field on τ, what allows us to set all the external Matsubara frequencies equal to zero.

Evaluating the Dirac traces for $t^{(\pm)}_{\mu\nu}$, we see that

$$t^{(\pm)}_{\mu\nu}(p,k) = \mathrm{tr}(I) \sum_{n=-\infty}^{+\infty} \frac{-p_\mu(p+k)_\nu - p_\nu(p+k)_\mu + [p\cdot(p+k) + \omega^{(\pm)}_n]\delta_{\mu\nu}}{(p^2 + (\omega^{(\pm)}_n)^2)[(p+k)^2 + (\omega^{(\pm)}_n)^2]}, \tag{31}$$

where $\mathrm{tr}(I)$ denotes the trace over the identity matrix in Dirac space.

It should be remembered, before integrating out the momenta p, that the regularization works only if both the $(+)$ and $(-)$ contributions are taken into account inside the integrand, i.e., the integral cannot be distributed (as in the Pauli–Villars method). This requires the knowledge of sums over frequencies which are not the standard ones of the Matsubara formalism, but rather objects defined as follows:

$$\sigma(f) \equiv \sum_{n=-\infty}^{+\infty} \bigg[f(n) - f\bigg(n+\frac{1}{2}\bigg)\bigg]. \tag{32}$$

By the same kind of trick applied in the Matsubara formalism, we may express the series (32) as a complex contour integral,

$$\sigma(f) = \frac{1}{i}\oint_{\mathcal{C}} dz\, \frac{f(z)}{\sin(2\pi z)}, \tag{33}$$

where $\mathcal{C}$ is the curve shown in Fig. 1.

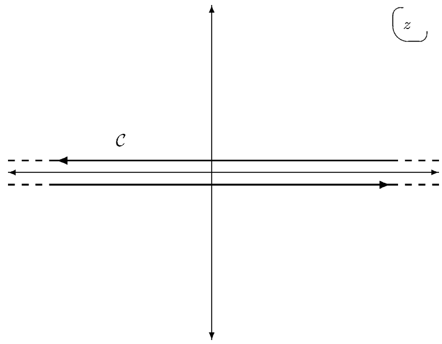

Fig. 1. The curve C used in the integral (33).

Performing the sums over frequencies according to this rule, we end up with an expression for $\widetilde{\Pi}_{\mu\nu}(k)$, which may be written as follows:

$$\widetilde{\Pi}_{\mu\nu}(k) = \beta\,\mathrm{tr}(I)\int \frac{d^D p}{(2\pi)^D}\frac{1}{(p+k)^2-p^2} \times \left\{\left[\frac{1}{|p|\sinh(\beta|p|)} - \frac{1}{|p+k|\sinh(\beta|p+k|)}\right]\left[-p_\mu(p+k)_\nu - p_\nu(p+k)_\mu + p\cdot(p+k)\delta_{\mu\nu}\right] - \left[\frac{|p|}{\sinh(\beta|p|)} - \frac{|p+k|}{\sinh(\beta|p+k|)}\right]\delta_{\mu\nu}\right\}, \tag{34}$$

where the UV convergence of the integral over p is evident (the symbol $|p|$ denotes the D-dimensional Euclidean norm).

The finite-temperature regularization is gauge invariant for gauge transformations in the D-dimensional space. This implies the (D-dimensional) transversality of the $\widetilde{\Pi}_{\mu\nu}$ tensor, $k_\mu\widetilde{\Pi}_{\mu\nu} = 0$. We may then write $\widetilde{\Pi}_{\mu\nu}(k)$ as follows:

$$\widetilde{\Pi}_{\mu\nu}(k) = \widetilde{\Pi}(k)\left(\delta_{\mu\nu} - \frac{k_\mu k_\nu}{k^2}\right). \tag{35}$$

Taking into account (35) and (34), and performing shifts in the integration variables, we note that

$$\widetilde{\Pi}(k) = \frac{2\beta\,\mathrm{tr}(I)}{D-1}\int \frac{d^D p}{(2\pi)^D}\frac{(D-2)(|p| + k\cdot\hat{p}) - Dp}{(k^2 + 2k\cdot p)\sinh(\beta|p|)}, \tag{36}$$

which is a finite scalar function of k for any value of D. The UV behaviour of the integrand is of course the same as in [6]. It should be clear from our starting point (6), that convergence will be achieved for all orders, and not just for the diagram quadratic in the external field.

5. The chiral anomaly

We shall restrict ourselves to the case of a massless fermion in an even number of dimensions, in order to observe the emergence of the chiral anomalies in this context. Since the finite-temperature regularization introduces an extra dimension into the game, it is not at all obvious whether the original chiral symmetry is still meaningful or not. It is however, easy to see that there is, indeed, a symmetry transformation which corresponds to the chiral transformations when the regulator is removed. They are however non-local on a scale of the order of M, and may be written as follows:

$$\delta\Psi^{(\pm)}(\tau,x) = i\alpha\Gamma_D(\Gamma_\mu D_\mu - \Gamma_D\partial_D)^{-1}\Gamma_\nu D_\nu\Psi^{(\pm)}(\tau,x),$$
$$\delta\bar{\Psi}^{(\pm)}(\tau,x) = i\alpha\bar{\Psi}^{(\pm)}(\tau,x)\Gamma_\nu D_\nu(\Gamma_\mu D_\mu - \Gamma_D\partial_D)^{-1}\Gamma_D. \tag{37}$$

When $M\to\infty$, the fields are dimensionally reduced and τ-independent. The transformations reduce of course to the standard ones, since the $\partial_D=\partial_\tau$ operator yields zero when acting on a dimensionally reduced field. When M is finite, they have of course a more complicated-looking expression, but nevertheless they leave the regularized action invariant.

There is an effect on the integration measure: we have a super-Jacobian, which is non-vanishing because of the different boundary conditions (otherwise, the $(+)$ and $(-)$ contributions would cancel). Indeed, under the transformations (37), the functional integration measure

$$\mathcal{D}\mu \equiv \mathcal{D}\Psi^{(+)}\mathcal{D}\Psi^{(-)}\mathcal{D}\bar{\Psi}^{(+)}\mathcal{D}\bar{\Psi}^{(-)} \tag{38}$$

transforms as follows:

$$\mathcal{D}\mu \to \mathcal{D}\mu\mathcal{J}, \tag{39}$$

where

$$\ln\mathcal{J} = -i\alpha\big\{\mathrm{Tr}\big[\Gamma_D(\Gamma_\mu D_\mu - \Gamma_D\partial_D)^{-1}\Gamma_\nu D_\nu\big]^{(+)} + \mathrm{Tr}\big[\Gamma_\nu D_\nu(\Gamma_\mu D_\mu - \Gamma_D\partial_D)^{-1}\Gamma_D\big]^{(-)} - (+)\to(-)\big\}, \tag{40}$$

where the $(\pm)$ indicates whether the corresponding trace has to be taken on the space of symmetric or anti-symmetric functions. A simple algebra shows that:

$$\ln\mathcal{J} = -i\alpha\big\{\mathrm{Tr}\big[\Gamma_D(\Gamma_\nu D_\nu)^2(\Gamma_\mu D_\mu - \Gamma_D\partial_D)^{-1}\big]^{(+)} - (+)\to(-)\big\}, \tag{41}$$

and, since the gauge fields are independent of τ, we may evaluate the trace over the τ coordinate. This sum over frequencies yields:

$$\ln\mathcal{J} = -2i\alpha\,\mathrm{Tr}\left[\Gamma_D\frac{\beta\Gamma_\mu D_\mu}{\sinh(\beta\Gamma_\mu D_\mu)}\right]. \tag{42}$$

This expression may be conveniently written in a D-dimensional notation, as follows:

$$\ln\mathcal{J} = -2i\alpha\,\mathrm{Tr}\left[\gamma_5\varphi\left(\frac{\not{D}}{\Lambda}\right)\right], \tag{43}$$

where:

$$\varphi(x) = \frac{x}{\sinh(x)} \tag{44}$$

and $\Lambda\equiv M/2$.

Now, the function $\varphi(x)$ verifies:

$$\varphi(0)=1, \qquad \lim_{x\to\infty}\varphi(x) = \lim_{x\to\infty}\varphi'(x) = \cdots = \lim_{x\to\infty}\varphi^{(k)}(x) = \cdots = 0, \tag{45}$$

the conditions to be imposed to regulating functions $\varphi(x)$ in order to give a φ-independent answer [2]. Then, formula (43) is nothing but

$$\ln\mathcal{J} = -2i\alpha\,\mathrm{Tr}\,\gamma_5|_{\mathrm{reg}} \tag{46}$$

which then yields to the known result for the chiral anomaly [2,11]

$$\langle\partial_\mu j_5^\mu(x)\rangle = \left.\frac{\delta\ln\mathcal{J}}{\delta\alpha(x)}\right|_{\alpha=0} = 2i\,\mathrm{Tr}\,\gamma_5|_{\mathrm{reg}}. \tag{47}$$

We see that the choice of $\tanh(i\mathcal{D}_c/M)$ to construct $\mathcal{D}$ through the defining equation (4), which in turn allowed (written as an infinite product) to the definition of the $(D+1)$-dimensional finite theory corresponds, at the level of regularized quantities, to the choice of a regulator which is close (but does not coincide) with the usual adopted heat-kernel $\exp(-\mathcal{D}_c^2/M^2)$ or zeta-function $\zeta(\mathcal{D}_c, s)$ regulators.

6. Conclusions and summary

In summary, we have introduced a higher-dimensional representation for regularized Dirac and scalar field theories which preserves the symmetries of the original system, in spite of the fact that the regularization is non-perturbative. The length of the extra coordinate may be related to a fictitious 'temperature', although the resulting finite-temperature QFT has unphysical features, a property that indeed should be expected from any *regularized* theory. This representation has the virtue that it automatically leads to the regulator masses one should introduce when using an infinite number of Pauli–Villars fields, as in [6]. Moreover, it also leads to a natural extension of the regularization to scalar field theories.

Since the method leads to a regularized *action*, it allows for the study of the realization of symmetries and their corresponding anomalies. In this respect, we have discussed for the fermionic case, the interplay between the symmetries of the original theory and the supersymmetry associated to the doubling of fields in the regulated theory. We have studied in detail the case of chiral anomalies, showing that the regularized action is invariant under transformations which tend to the usual chiral ones when the regulator is removed. At the quantum level, there is a non-trivial anomalous Jacobian which, as it was to be expected, is already regulated. It should be remarked that the symmetries of the regularized theory are non-local, a property that should be expected since it holds also for the standard perturbatively regularized theories [12,13].

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