MATHEMATICS

RATIONAL QUADRILATERALS
SECOND COMMUNICATION

BY

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§ 1. *Introduction*

A polygon, of which the sides and diagonals have a rational length, will be called a rational polygon. In a former paper, published in these Proceedings [1] I developed a method to construct rational quadrilaterals. Our starting point is a rational triangle ABC; without loss of generality we can suppose the sides of this triangle to be integers. Next we construct points E, situated in the plane of the triangle, for which AE, BE and CE are rational. Such a point E, not situated on a side of the triangle, for which AE, BE and CE are rational, will be called a *convenient* point.

We have seen that the convenient points lie everywhere dense in the plane of the triangle and moreover that through any vertex an infinite number of straight lines exists that each contain an infinite number of convenient points.

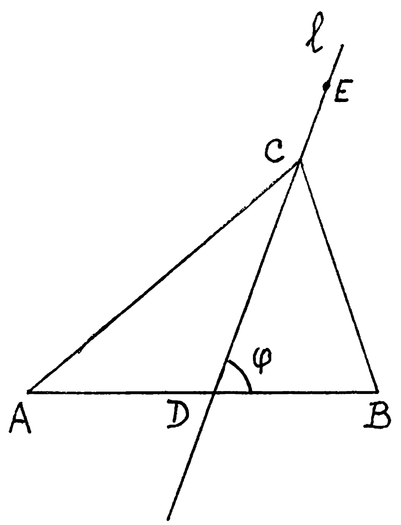

Fig. 1

In the following an arbitrary line through C, different from AC and BC, will be denoted by "a line l", l meets AB in D (figure 1). Let us assume that the line l contains an infinite number of convenient points.

For a convenient point E of l the lengths of AE, BE and CE are rational numbers, which we suppose to be written with the smallest positive denominator. The smallest common multiple of these denominators will be called the "denominator" of the convenient point considered.

We will show that the number of convenient points on l, of which the denominator H does not exceed a given natural number G, is finite. Next we will derive a lower bound for this number, that increases unlimitedly with G, namely

$$\frac{\log\log G}{\log 4} - K$$

where K represents a number, that is wholly determined by the position of A, B, C and l.

For the convenient points on l a law of distribution will be formed, by the aid of which we will prove the

Theorem. Given a circle Γ and a rational triangle ABC situated in the plane of Γ. For large G the number of convenient points in the interior of Γ with denominators $\leqq G$ has an order of magnitude larger than $\log\log G$.

Contrary to former investigations, that were restricted to qualitative results, here quantitative results are derived. It may be advisable to remark that these quantitative results are presumably very rough, because the order of magnitude of the lower bound, indicated above, is probably much lower than that of the considered function itself.

It is obvious that the number of points E, situated in the plane of the triangle ABC, for which BE and CE are rational, is countable, as for BE and CE only a countable number of values can be taken into consideration.

§ 2. *The cubic curve*

Let ω be the index of the triangles that occur in the figure and let E be a convenient point on the line l. The index ω is a squarefree natural number. Let α and β be the angles of the triangle ABC that are situated at A and B, α' and β' the corresponding angles of the triangle ABE. We now put (See [II], p. 42)

$$\operatorname{tg}\tfrac{1}{2}\alpha = \frac{x_1}{x_2}\sqrt{\omega}\,, \quad \operatorname{tg}\tfrac{1}{2}\beta = \frac{y_1}{y_2}\sqrt{\omega}\,,$$

$$\operatorname{tg}\tfrac{1}{2}\alpha' = |x|\sqrt{\omega}\,, \quad \operatorname{tg}\tfrac{1}{2}\beta' = |y|\sqrt{\omega}\,,$$

$$< CDB = \varphi\,, \quad \operatorname{tg}\tfrac{1}{2}\varphi = \frac{u_1}{u_2}\sqrt{\omega}\,,$$

where the separate pairs x_1, x_2; y_1, y_2; u_1, u_2 are relatively prime positive integers, x and y are rational numbers that are to be taken positive if

E lies at the same side of AB as C, and negative if E lies at the opposite side of AB.

If we further put

$$m = y_1 y_2 (x_2 u_1 - x_1 u_2)\ (\ \ \omega x_1 u_1 + x_2 u_2)$$
$$n = x_1 x_2 (y_2 u_1 + y_1 u_2)\ (-\omega y_1 u_1 + y_2 u_2),$$

then x and y satisfy the equation:

$$(1)\qquad n\left(\frac{1}{x} - \omega x\right) + m\left(\omega y - \frac{1}{y}\right) + \frac{m+n}{u_1 u_2}(\omega u_1^2 - u_2^2) = 0.$$

Clearly the numbers m and n are integers, that are determined by the position of A, B, C and l. If the position of A, B, C and l is given, (1) gives the equation of a cubic curve in the coordinates x and y. A point with rational coordinates will be called a rational point. Any rational point of the curve (1) for which $x>0$, $y>0$ gives rise to a convenient point on the line l. A rational point (x, y) with $0<x<u_1/u_2$, $0<y<u_2/\omega u_1$ yields a convenient point that lies at the same side of AB as C. If $x>u_1/u_2$, $y>u_2/\omega u_1$ then we consider instead of (x, y) the rational point $(-1/\omega x, -1/\omega y)$ of the curve, which yields a convenient point at the other side of AB. This is connected with the fact, that the equation (1) does not change if for x is substituted $-1/\omega x$ or $-1/\omega y$ for y.

§ 3. *Transformation of the cubic curve*

In the same way as in our earlier paper (see [I] p. 194) we apply to (1) a birational transformation. We here take the transformation in such a way that points (x, y) with positive x and y are mapped on points with positive X and Y, and moreover that the point $x=0$, $y=0$ is transformed into the point $X=0$, $Y=0$.

In the earlier formulae we replace to this end x by $-1/\omega x$. By this (1) is replaced by itself and the transformed equation remains the same as previously. The transformation formulae are

$$(2)\qquad \begin{cases} x = \dfrac{X(X+q)}{2n\omega u_1 u_2 (Y-pX)} \\[2ex] y = \dfrac{Y-pX}{2m\omega u_1 u_2 (X+q)} \end{cases}$$

where

$$(3)\qquad p=(m+n)(\omega u_1^2 - u_2^2)\ ,\ \ q = 4\omega n^2 u_1^2 u_2^2.$$

The transformed equation is

$$(4)\qquad Y^2 = X^3 + (N^2 - 2M)X^2 + M^2 X,$$

where

$$M = 4mn\omega u_1^2 u_2^2\ ,\ \ N = (m+n)(\omega u_1^2 + u_2^2).$$

The numbers M and N are integers.

We will call (4) the "normalized equation" and the curve that is represented by this equation the "normal curve".

The inverse transformation is

$$(5)\qquad \begin{cases} X = M\omega xy \\ Y = M\omega xy u_1 u_2 \left\{ n\left(\omega x + \dfrac{1}{x}\right) + m\left(\omega y + \dfrac{1}{y}\right)\right\}. \end{cases}$$

In the following we will suppose that the point D lies *between* A and B. Then we have

$$m > 0 \ , \ n > 0 \ , \ M > 0 \ , \ N > 0,$$

and then with a point (x, y) for which $x > 0$, $y > 0$ corresponds a point (X, Y) with $X > 0$, $Y > 0$.

§ 4. *In what cases has l an infinite number of convenient points?*

The line l has an infinite number of convenient points if the corresponding normal curve possesses a non-exceptional point. In our previous paper ([I] p. 196) we considered the rational point $P\,(X_2, Y_2)$ of the normal curve for which

$$X_2 = \left\{ \frac{2mn}{m-n} (\omega u_1^2 - u_2^2) \right\}^2.$$

There is some advantage in replacing P_2 by the rational point P_3, that is obtained by intersecting the line OP_2 with curve. We obtain for the abscissa X_3 of the latter point:

$$X_3 = \left\{ \frac{2\omega u_1^2 u_2^2 (m-n)}{\omega u_1^2 - u_2^2} \right\}^2.$$

The point P_3 can be exceptional only if $9X_3$ is an integer and that is if the number ϱ defined by

$$\varrho = \frac{6\omega u_1^2 u_2^2 (m-n)}{\omega u_1^2 - u_2^2}$$

is an integer. In the latter expression we substitute the values of m and n of § 2 and we obtain

$$\varrho = 6\omega u_1^2 u_2^2 \left\{ 2x_1x_2y_1y_2 + \frac{(x_2y_1 - x_1y_2)(\omega x_1y_1 + x_2y_2)\, u_1u_2}{\omega u_1^2 - u_2^2} \right\}.$$

The number ϱ is an integer if and only if the number

$$\varrho_1 = 6\omega (x_2y_1 - x_1y_2)(\omega x_1y_1 + x_2y_2) \cdot \frac{u_1^3 u_2^3}{\omega u_1^2 - u_2^2}$$

is an integer. We can write ϱ_1 in the form

$$(6)\qquad \sigma \cdot \frac{u_1^3 u_2^3}{\omega u_1^2 - u_2^2}$$

where

$$\sigma = 6\omega(x_2y_1 - x_1y_2)(\omega x_1y_1 + x_2y_2).$$

The number σ is an integer, dependent on the position of A, B and C but independent of the line l. We now have the

Theorem. It is always possible to find two positive relatively prime integers u_1 and u_2 in such a way that

1) u_1/u_2 is arbitrarily near any positive real number $A \neq 1/\sqrt{\omega}$,

2) the expression (6) does not represent an integer.

Proof. (See fig. 2). We consider the part δ_1 of the (u_1, u_2) plane for which $u_1 > 0$, $u_2 > 0$, $\omega u_1^2 - u_2^2 > |\sigma|$ and the part δ_2 of the (u_1, u_2) plane for which $u_1 > 0$, $u_2 > 0$, $u_2^2 - \omega u_1^2 > |\sigma|$. We take an arbitrary positive number ε such that the halflines with the equations $u_1/u_2 = A + \varepsilon$, $u_1 > 0$ and $u_1/u_2 = A - \varepsilon$, $u_1 > 0$ intersect both with δ_1 or both with δ_2. This is

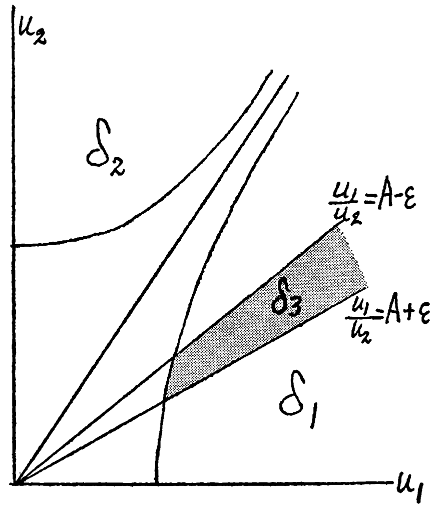

Fig. 2

always possible as $A \neq 1/\sqrt{\omega}$. We suppose for instance that both lines intersect with δ_1. We consider that part of δ_1 that lies between the two halflines and call this δ_3. Now we choose the point (u_1, u_2) in δ_3 in such a manner that u_2 is relatively prime with u_1 and with ω, we attain this for instance by choosing for u_2 a prime number greater than ω and by choosing for u_1 a positive integer, that is not a multiple of u_2. Then the numerator and the denominator of the fraction

$$\frac{u_1^3u_2^3}{\omega u_1^2 - u_2^2}$$

cannot be divided by a factor of u_1 nor by a factor of u_2 and so this fraction cannot be reduced. Moreover (6) is not an integer because $|\omega u_1^2 - u_2^2| > |\sigma|$.

From the above it follows: An infinite number of lines through the point C exists such that each of them possesses an infinite number of convenient points, these lines are everywhere dense in the plane of the triangle ABC.

§ 5. *An upper bound for the denominator of a convenient point*

Let (x, y) be a rational point of the curve (1) with $x>0, y>0$ and $xy\neq 1/\omega$. For the lengths of AE, BE and CE we find

$$\overline{AE} = \frac{y(1+\omega x^2)}{(x+y)\,|1-\omega xy|} \cdot \overline{AB},$$

$$\overline{BE} = \frac{x(1+\omega y^2)}{(x+y)\,|1-\omega xy|} \cdot \overline{AB},$$

$$\overline{CE} = \frac{xy(\omega u_1^2+u_2^2)}{u_1u_2(x+y)\,|1-\omega xy|} \cdot \overline{AB}.$$

With the rational point (x, y) of (1) corresponds a rational point of the normal curve, that can be written in homogeneous coordinates (X, Y, Z), where X, Y and Z are positive integers without a common factor greater than 1. Applying the formulae (2) we obtain expressions in X, Y and Z for the lengths of AE, BE and CE. These expressions are quotients of homogeneous polynomials in X, Y and Z with integer coefficients that depend on x_1, x_2, y_1, y_2, u_1, u_2 and ω. The denominators of these expressions appear to be the same, namely

(7) $$|X-MZ| \cdot \{mX(X+\omega qZ)^2+nZ(Y+pX)^2\}.$$

Here p and q have the values mentioned in (3). The expression (7) represents an integer that is an upper bound for the number H as defined in the introduction.

§ 6. *The function $N(G)$*

We consider a line l with an infinite number of convenient points. For any convenient point E of l the lengths of AE, BE and CE are rational numbers with denominators that do not exceed the number H. We have the

Theorem. For any given natural number G there are only a finite number of convenient points on l for which the "denominator" H is smaller than G.

Proof. Let us suppose that for a certain number G an infinite number of convenient points exists for which $H<G$. By multiplying all distances by $G!$ we obtain a figure with an infinite number of points $E_1, E_2, E_3, \ldots$ such that the lengths of AE_n and CE_n $(n=1, 2, 3, \ldots)$ are integers (fig. 3).

In the triangle AE_nE_{n+1} we have

(8) $$AE_n+E_nE_{n+1}-AE_{n+1}\geqq 1.$$

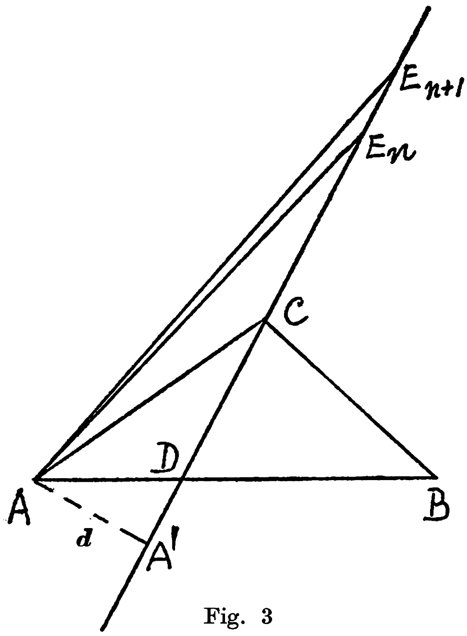

Fig. 3

Let A' be the projection of A on l, we put $AA'=d$, $A'E_n=x_n, A'E_{n+1}=x_{n+1}$. Then the left hand side of (8) can be written in the form

$$(\sqrt{d^2+x_n{}^2}-x_n)+(x_{n+1}-\sqrt{d^2+x_{n+1}^2}).$$

If $n \to \infty$ the expressions between the brackets tend to zero and a contradiction arises.

So for any given line l we can define a function $N(G)$ that gives the number of convenient points of which the denominator does not exceed the natural number G.

§ 7. *The "weight" of a rational point of the normal curve*

By the "weight" of a rational point $P(X, Y, Z)$ of the normal curve we mean:

$$\gamma(P) = \max \{|X|, |Y|, |Z|\}.$$

Here X, Y and Z are supposed to be integers without a common factor greater than 1.

From two different rational points A and B we can derive a third rational point on the normal curve by intersecting the line AB with the curve. We will call this the "construction by chords". For the weight of the point C obtained we have the theorem:

Theorem. Let C be a rational point of the normal curve that is obtained by the "construction by chords" from the rational points A and B of the curve. Then we have the inequality

$$\gamma(C) \leqq \{2\gamma(A)\cdot\gamma(B)\}^4.$$

Proof. Putting $A=(X_1, Y_1, Z_1)$, $B=(X_2, Y_2, Z_2)$, $C=(X_3, Y_3, Z_3)$ we have

$$\begin{aligned} X_3 &= (X_1Y_2-X_2Y_1)^2\,(X_1Z_2-X_2Z_1)\,Z_1Z_2,\\ Y_3 &= (X_1Y_2-X_2Y_1)\,\{X_1X_2(X_1Z_2-X_2Z_1)^2+\\ &\qquad +(X_1Y_2-X_2Y_1)\,(Y_1Z_2-Y_2Z_1)\,Z_1Z_2\},\\ Z_3 &= X_1X_2(X_1Z_2-X_2Z_1)^3, \end{aligned}$$

from which the required inequality is easily derived.

By drawing the tangent in a rational point A of the normal curve and intersecting this line with the curve we obtain the tangential point of A, this also is a rational point. We have the

Theorem. Let A be a rational point of the normal curve (3) and let B be the tangential point of A, we then have the inequality

$$\gamma(B)\leqq\Omega\cdot\{\gamma(A)\}^6, \tag{9}$$

where Ω is a positive constant, depending on the coefficients of the curve.

Proof. Writing the equation of the curve in the form

$$Y^2Z=X^3+KX^2Z+LXZ^2$$

and putting $A=(X_0, Y_0, Z_0)$, $B=(X_1, Y_1, Z_1)$ we obtain

$$\left\{\begin{aligned} X_1 &= 2Y_0Z_0(X_0^2-LZ_0^2)^2,\\ Y_1 &= (LZ_0^2-X_0^2)(X_0^4+2KX_0^3Z+6LX_0^2Z_0^2+2KLX_0Z_0^3+L^2Z_0^4),\\ Z_1 &= 8Y_0^3Z_0^3. \end{aligned}\right. \tag{10}$$

The right hand sides of the expressions (10) are of the sixth degree in X, Y and Z from which the inequality (9) follows.

Finally we give an upper bound for the denominator of a convenient point. Using the weight of the point we obtain by (7)

$$H\leqq(M+1)\,\{m(1+\omega^2q^2)^2+n(1+p)^2\}\,\gamma^4.$$

We can write this as

$$H\leqq\theta\gamma^4 \tag{11}$$

where θ is a natural number, dependent on $x_1, x_2, y_1, y_2, u_1, u_2$ and ω.

§ 8. *Introduction of elliptic parameters on the normal curve*

By a translation parallel to the axis of X we bring the normalized equation

$$Y^2=X^3+KX^2+LX$$

into the form

$$\eta^2=\xi^3+A\xi+B \tag{12}$$

where

$$Y=\eta\ ,\ X=\xi-\tfrac{1}{3}K.$$

In (12) we introduce elliptic functions by putting

$$\eta = \tfrac{1}{2}P'(v) \quad , \quad \xi = P(v),$$

here P is the Weierstrassian elliptic function.

If we denote the differentiation with respect to the elliptic parameter by a dash we have

$$\xi' = 2\eta \quad ; \quad X' = 2Y.$$

In our earlier paper ([I], § 3) we saw that the normal curve consisted of an oval situated in the region $X<0$ and an infinite branch in the region $X \geqq 0$. For the infinite branch we can choose for the elliptic argument a real number between 0 and $2\omega_1$, where $2\omega_1$ is the real period of the elliptic function. The parameters of points on the oval can be written $\lambda\omega_1+\omega_3$ with $-1<\lambda<1$, here $2\omega_3$ is the purely imaginary period.

If three points of a cubic curve are collinear, the sum of their elliptic arguments is a constant modulo a period. For the normal curve this constant is zero.

On a cubic curve a finite number of rational points exists with the arguments

$$v_1, v_2, v_3, \dots, v_R$$

in such a way that the argument of any rational point of the curve can be expressed in the form

$$v = m_1 v_1 + m_2 v_2 + \dots + m_R v_R,$$

where $m_1, m_2, \dots, m_R$ are integers. If the number R is as small as possible, we have a system of base-points on the cubic, then R is the rank of the curve. On the normal curve it is always possible to choose a basis in such a manner that $v_1, v_2, \dots, v_{R-1}$ are real and that $v_R = \lambda\omega_1 + \omega_3$ with $-1<\lambda<1$. We then obtain all rational points on the infinite branch by taking $m_R = 0$ and letting the numbers $m_1, m_2, \dots, m_{R-1}$ run through all possible integer values.

In the case that the normal curve possesses an infinite number of rational points there are an infinite number of these points on the infinite branch and so at least one of the numbers $v_1, v_2, \dots, v_{R-1}$ has an irrational proportion to ω_1.

If we arrange the numbers

$$m_1 v_1 + m_2 v_2 + \dots + m_{R-1} v_{R-1}$$

in a sequence, first taking all cases in which the sum

$$|m_1| + |m_2| + \dots + |m_{R-1}|$$

is equal to 1, next all cases in which this sum is equal to 2, etc., then according to a theorem of number-theory the sequence, formed in this manner is uniformly distributed modulo $2\omega_1$. (See for a similar theorem the paper [III] of Kuipers and Scheelbeek.) In this way all rational points of the infinite branch are arranged in a sequence so that the elliptic arguments are uniformly distributed modulo $2\omega_1$.

§ 9. *A special sequence of rational points on the normal curve*

We suppose that the values of $x_1, x_2, y_1, y_2, u_1, u_2$ and ω are such, that the curve (1) has an infinite number of rational points. In that case the normal curve also has an infinite number of rational points and so this curve possesses at least one rational point P_1 situated on the infinite branch, with an elliptic argument α that has an irrational proportion to $2\omega_1$. In the case that the point P_3 mentioned in § 4 is a non-exceptional point, we can take the latter for the point P_1.

The point P_1' with the argument $-\alpha$ is a rational point too. We say that P_1' is obtained from P using the "construction by reflection". The point 2α also is a rational point, for this point is the tangential-point of the point $-\alpha$. (For reasons of brevity we denote a point with the argument ε by "the point ε".) Applying the "construction by chords" on the points $-\alpha$ and -2α we obtain the point 3α; the points $-\alpha$ and -3α yield the point 4α etc.

Proceeding in this manner we obtain the sequence of rational points

$$P_1, P_2, P_3, \dots$$

with the arguments

$$\alpha, 2\alpha, 3\alpha, \dots .$$

These points are all situated on the infinite branch and their arguments are uniformly distributed modulo $2\omega_1$.

We derive an upper bound for the weight γ_n of the point P_n with the argument $n\alpha$. Applying a theorem of § 7 we have

$$\gamma_3 \leqq (2\gamma_1)^4\, \gamma_2{}^4$$

$$\gamma_4 \leqq (2\gamma_1)^4\, \gamma_3{}^4 \leqq (2\gamma_1)^4 \cdot (2\gamma_1)^{4^2} \cdot \gamma_2{}^{4^2}.$$

Continuing in this manner we obtain

$$\gamma_n \leqslant (2\gamma_1)^4\, (2\gamma_1)^{4^2} \cdot \dots \cdot (2\gamma_1)^{4^{n-2}} \cdot \gamma_2{}^{4^{n-2}}.$$

From this it follows

$$\gamma_n \leqslant (2\gamma_1)^{4^{n-1}} \cdot \gamma_2{}^{4^{n-2}} \leqslant (2\gamma_1\gamma_2) \cdot {}^{4^{n-1}}.$$

Applying (9) we have

$$\gamma_2 \leqslant \Omega\gamma_1{}^6$$

and so

$$\gamma_n \leqslant K_1{}^{4^{n-1}}$$

where

$$K_1 = 2\Omega\gamma_1{}^7.$$

From the sequence of rational points $P_1, P_2, P_3, \dots$ we form a second sequence in this manner: If for $P_n(X, Y)$, $Y > 0$ then we take $Q_n = P_n$. If for P_n, $Y < 0$ then we take $Q_n = (X, -Y)$. The arguments of the sequence

$Q_1, Q_2, Q_3, \ldots$ are uniformly distributed modulo ω_1. All points Q_n are different because the proportion $\alpha/2\omega_1$ is irrational.

Let the point Q_n have a denominator H_n. Then applying (11) we obtain

$$H_n \leqslant \theta\gamma_n{}^4 \leqslant \theta \cdot K_1{}^{4^n}.$$

By means of this inequality we find a lower bound of the function $N(G)$. Every point Q_n gives rise to a rational point (x_n, y_n) of the curve (1) with $x_n > 0$, $y_n > 0$ and finally to a convenient point E_n on the line l. Two different rational points of the normal curve yield different convenient points of l.

Let G be a given natural number. Then $H_n \leqq G$ if

$$\theta \cdot K_1{}^{4^n} < G$$

from which it follows for $G > \theta$ and $K_1 > 1$:

$$n < \frac{\log\log (G/\theta) - \log\log K_1}{\log 4}. \tag{13}$$

If the number n satisfies this inequality, E_n is a convenient point with a denominator that is at most equal to G. So the number of such convenient points is at least equal to the right hand side of (13) diminished by 1.

So we obtain

$$N(G) \geqq \frac{\log\log (G/\theta) - \log\log K_1}{\log 4} - 1$$

and consequently

$$N(G) \geqq \frac{\log\log G}{\log 4} - K_2$$

where $G > 1$ and K_2 represents a positive constant that only depends on the position of A, B, C and l.

§ 10. *A law of distribution belonging to the sequence $E_1, E_2, E_3, \ldots$*

Differentiating the equations (5) with respect to the elliptic parameter and in connection with the relation $X' = 2Y$ we obtain the equation

$$x'y + xy' = 2u_1u_2xy\left\{n\left(\omega x + \frac{1}{x}\right) + m\left(\omega y + \frac{1}{y}\right)\right\}. \tag{14}$$

Differentiating (1) we obtain

$$nx'\left(\omega + \frac{1}{x^2}\right) = my'\left(\omega + \frac{1}{y^2}\right). \tag{15}$$

Solving x' and y' from (14) and (15) we obtain

$$x' = 2mu_1u_2xy\left(\omega + \frac{1}{y^2}\right),$$

$$y' = 2nu_1u_2xy\left(\omega + \frac{1}{x^2}\right).$$

Let λ represent the length of ED. In the triangle AED we compute λ, expressed in m, ω, x, u_1, u_2:

$$\lambda = \frac{mx(\omega u_1^2 + u_2^2)}{(u_1 - xu_2)(\omega u_1 x + u_2)}.$$

From this it follows that

$$\lambda' = \frac{d\lambda}{dx} \cdot x' = \frac{2u_1^2 u_2^2 (1+\omega x^2)(1+\omega y^2)\lambda^2}{xy(\omega u_1^2 + u_2^2)}.$$

and from this we derive

$$\lambda' = \frac{4u_1 u_2 \sqrt{\omega}}{\sin \varphi} \cdot r_1 r_2$$

where r_1 and r_2 represent the lengths of AE and BD. So we have

$$\frac{d\lambda}{dv} = \frac{4u_1 u_2 \sqrt{\omega}}{\sin \varphi} \cdot r_1 r_2.$$

As the convenient points $E_1, E_2, E_3, \ldots$ are uniformly distributed with respect to the elliptic parameter v, the number of points E_n on a small segment $d\lambda$ of the line l for $n \to \infty$ is proportional to

$$\frac{1}{r_1 r_2} d\lambda.$$

So we have for $\Lambda > 0$ the law of distribution

$$f(\Lambda) = \frac{\int_0^\Lambda \frac{1}{r_1 r_2} d\lambda}{\int_0^\infty \frac{1}{r_1 r_2} d\lambda}.$$

Here r_1 and r_2 are to be considered as functions of λ, the length of DE (fig. 4). Now the following theorem is proved:

Theorem. Let P_1 be a non-exceptional point of the normal-curve with the real elliptic argument α. Starting from P_1 we derive (for instance

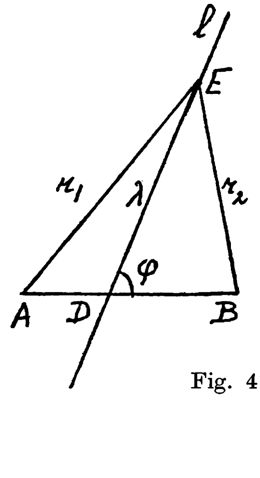

Fig. 4

by using the constructions by chords, by tangents and by reflection) the sequence of rational points P_n, the argument of P_n being $n\alpha$. To every point P_n one and only one convenient point E_n on l belongs. Now we consider the points $E_1, E_2, E_3, \ldots, E_k$. Let E_0 be an arbitrary point on l ($E_0 \neq D$) and let Λ be the length of E_0D. If Σ_1 is the number of points E_i ($i=1, 2, \ldots, k$) situated on the segment E_0D and if Σ_2 is the number of points E_i that are situated on the same side of AB as E_0, then

$$\lim_{k\to\infty} \frac{\Sigma_1}{\Sigma_2} = f(\Lambda).$$

§ 11. *The number of convenient points inside a given circle Γ*

Theorem. For large G the number of convenient points inside a given circle Γ, with denominators $\leqq G$ has an order of magnitude greater than $\log\log G$.

Proof. We may suppose that Γ does not intersect the sides of the triangle nor their produced parts, because otherwise Γ can be replaced by a circle, situated in the interior of Γ that satisfies this condition.

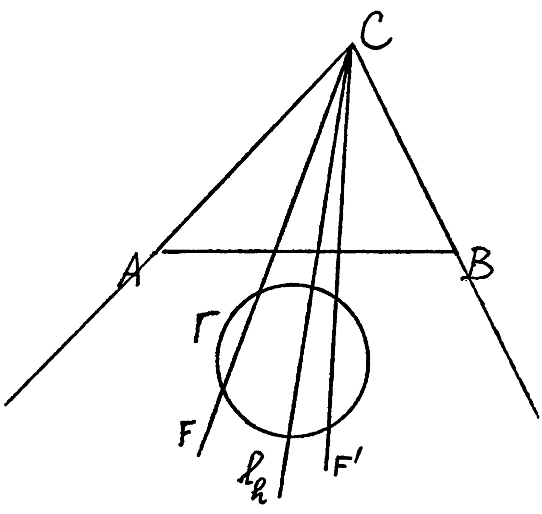

Fig. 5

Suppose that the situation is as drawn in fig. 5 (for other positions of Γ the proof is similar). Let CF and CF' be two halflines, issuing from C, that both intersect the circle Γ.

In the following c_1, c_2, c_3, and c_4 will represent conveniently chosen positive numbers that are fully determined if A, B, C, Γ, CF and CF' are given.

From every halfline, issuing from C and within the angle FCF' a chord $\geqq c_1$ is cut off by the circle. Let H be an arbitrary positive integer (not to be confused with the "denominator" H as used in § 6).

According to § 4 the angle FCF' contains H halflines, issuing from C, $l_1, l_2, l_3, \ldots, l_H$, that each contain an infinite number of convenient points.

Let l_h be one of these halflines and let α_h be a corresponding elliptic argument that has an irrational proportion to $\omega_{1,h}$, where $2\omega_{1,h}$ is the corresponding real period of the elliptic function. Let $U_{k,h}(h=1, 2, 3, \ldots, H)$ be the number of points with arguments $q\alpha_h(1 \leqq q \leqq k)$ that are situated on the chord that is cut off from the halfline l_h. According to the law of distribution we have for $h=1, 2, 3, \ldots, H$:

$$\lim_{k\to\infty} k^{-1}U_{k,h} = \int_{\mu_h}^{\nu_h} \frac{d\lambda}{r_1 r_2} : \left\{ \int_0^{\infty} \frac{d\lambda}{r_1 r_2} \right\}_h .$$

Here μ_h and ν_h $(\mu_h < \nu_h)$ are the values of λ that correspond with the intersection points of l_h with the circle $(h=1, 2, 3, \ldots, H)$.

It will be clear that if we consider all possible halflines in the interior of the angle FCF', a *lower* bound $c_2 > 0$ exists for the first integral and an *upper* bound $c_3 > 0$ for the second one, and thus

$$\lim_{k\to\infty} k^{-1}U_{k,h} \geqq \frac{c_2}{c_3} = c_4,$$

from which we derive for $k > k_h$:

$$U_{k,h} \geqq \tfrac{1}{2} c_4 k$$

where k_h is a conveniently chosen positive number, dependent on h.

In this manner we obtain $U_{k,h}$ convenient points on l_h, that are situated in the interior of Γ. Applying the method of § 9 we see that the denominator of any of these points does not exceed

$$K_h^{4^k}$$

here K_h is a constant, dependent on h $(h=1, 2, 3, \ldots, H)$. If we take now

$$K = \max_{h=1,2,\ldots,H} \{K_h\}$$

then the denominators of all convenient points considered in the interior of Γ are at most equal to

$$K^{4^k}$$

where K depends on the number H. Thus if k is the greatest integer for which

$$K^{4^k} \leqq G$$

we then obtain in this way

$$\sum_{h=1}^{H} U_{k,h}$$

convenient points inside Γ with denominators $\leqq G$.

We have

$$K^{4^{k+1}} > G$$

and consequently

$$k > \frac{\log\log G - \log\log K}{\log 4} - 1.$$

The number of convenient points inside Γ with denominator $\leqq G$ is for $k \geqq \max_{1 \leqslant h \leqslant H} k_h$ greater than or equal to $\frac{1}{2}c_4 Hk$, and as for a sufficiently large number G we have

$$\tfrac{1}{2}c_4 Hk > \tfrac{1}{2}c_4 H \left(\frac{\log\log G - \log\log K}{\log 4} - 1\right) > \frac{c_4}{4\log 4} \cdot H \log\log G,$$

where H is an arbitrary natural number, we see that the number of convenient points with denominator $\leqq G$ that are situated in the interior of Γ, has an order of magnitude exceeding $\log\log G$.

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