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A comparative study of numerical integration based on Haar wavelets and hybrid functions

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ABSTRACT

A quadrature rule based on uniform Haar wavelets and hybrid functions is proposed to find approximate values of definite integrals. The wavelet-based algorithm can be easily extended to find numerical approximations for double, triple and improper integrals. The main advantage of this method is its efficiency and simple applicability. Error estimates of the proposed method alongside numerical examples are given to test the convergence and accuracy of the method.

1. Introduction

Numerical integration has many applications in science and engineering. In recent years the wavelet approach is becoming more popular in the field of numerical approximations. Different types of wavelets and approximating functions have been used in numerical approximations. Among them Haar wavelets [1] and hybrid functions [2] have gained popularity among researchers due to their useful properties. In most cases, the beauty of the wavelet approximation is overshadowed by the computational cost of the algorithm. Haar wavelets are the simplest orthonormal wavelet with compact support and they have been used in different numerical approximation problems. Chen and Hsiao [1] established an operation matrix of integration based on these wavelets and applied it to analyse lumped-parameter and distributed-parameter dynamic systems. In another paper, Chen and Hsiao [3] showed that the Haar wavelet operational matrix is the fastest among the orthogonal functions for solving identification and optimization problems of dynamic systems. Hsiao and Wang [4] proposed an algorithm based on Haar wavelets for solving nonlinear stiff systems and Hsiao [5] proposed an algorithm based on Haar wavelets for solving linear stiff systems. Lepik [6] applied Haar wavelets in solving differential equations. Lepik and Tamme [7,8] used Haar wavelets for solving linear and nonlinear integral equations. Maleknejad and Mirzaee [9] solved linear integral equations via Haar wavelets. Lepik [10,11] applied Haar wavelets in solving nonlinear integro–differential equations and partial differential equations.

Hybrid functions have faster convergence than Haar wavelets and they can model discontinuities in a better manner than Haar wavelets [12]. Another useful property of hybrid functions is a special product matrix and a related coefficient matrix with optimal order. The advantage of hybrid functions is that the orders of block-pulse functions and Legendre polynomials are adjustable to obtain highly accurate numerical solutions compared to the piecewise constant orthogonal function for the solution of integral equations [2]. Recently, hybrid functions have been successfully used for the numerical solution

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of ordinary differential equation as well as integral equations. Marzban and Razzghi [13] applied hybrid functions to find the numerical solution of a controlled duffing oscillator. [14] used hybrid functions for nonlinear initial-value problems. Marzban et al. [2] applied hybrid functions for solving Fredholm and Volterra integral equations of the second kind. Marzban and Razzghi [15] applied hybrid functions to find the optimal control of linear delay systems.

Motivated by the excellent performance of these methods, we will apply the same techniques for numerical integration. The organization of this paper is as follows. In Section 2, numerical integration using Haar wavelets is described and in Section 3 hybrid functions are used for numerical integration. Error analysis for Haar Wavelets is given in Section 4 and numerical results are reported in Section 5. Some conclusions are drawn in Section 6.

2. Numerical integration using Haar wavelets

2.1. Haar wavelets

The scaling function for the family of Haar wavelets defined on the interval $[a, b)$ is

$$h_1(x) = \begin{cases} 1 & \text{for } x \in [a, b) \\ 0 & \text{elsewhere.} \end{cases} \tag{1}$$

The mother wavelet for the Haar wavelets family is also defined on the interval $[a, b)$, and is given by

$$h_2(x) = \begin{cases} 1 & \text{for } x \in \left[a, \dfrac{a+b}{2}\right) \\ -1 & \text{for } x \in \left[\dfrac{a+b}{2}, b\right) \\ 0 & \text{elsewhere.} \end{cases} \tag{2}$$

All the other functions in the Haar wavelet family are defined on subintervals of $[a, b)$ and are generated from $h_2(x)$ by the operations of dilation and translation. Each function in the Haar wavelets family defined for $x \in [a, b)$ except the scaling function can be expressed as

$$h_i(x) = \begin{cases} 1 & \text{for } x \in [\alpha, \beta) \\ -1 & \text{for } x \in [\beta, \gamma) \\ 0 & \text{elsewhere,} \end{cases} \tag{3}$$

where

$$\alpha = a + (b-a)\frac{k}{m}, \qquad \beta = a + (b-a)\frac{k+0.5}{m}, \qquad \gamma = a + (b-a)\frac{k+1}{m}, \quad i = 3, 4, \dots, 2M. \tag{4}$$

The integer $m = 2^j$, where $j = 0, 1, \dots, J, J = 2^M$ and integer $k = 0, 1, \dots, m-1$. The integer j indicates the level of the wavelet and k is the translation parameter. The maximal level of resolution is the integer J. The relation between i, m and k is given by $i = m + k + 1$.

The Haar wavelet functions are orthogonal to each other because

$$\int_a^b h_j(x)h_k(x)\,\mathrm{d}x = \begin{cases} (b-a)2^{-j} & \text{when } j = k \\ 0 & \text{when } j \neq k. \end{cases} \tag{5}$$

Thus any function $f(x)$ which is square integrable in the interval (a, b) can be expressed as an infinite sum of Haar wavelets

$$f(x) = \sum_{i=1}^{\infty} a_i h_i(x). \tag{6}$$

The above series terminates at finite terms if $f(x)$ is piecewise constant or can be approximated as piecewise constant during each subinterval.

2.2. Method of numerical integration based on Haar wavelets

In this section we consider numerical integration for single, double and triple integrals using Haar wavelets.

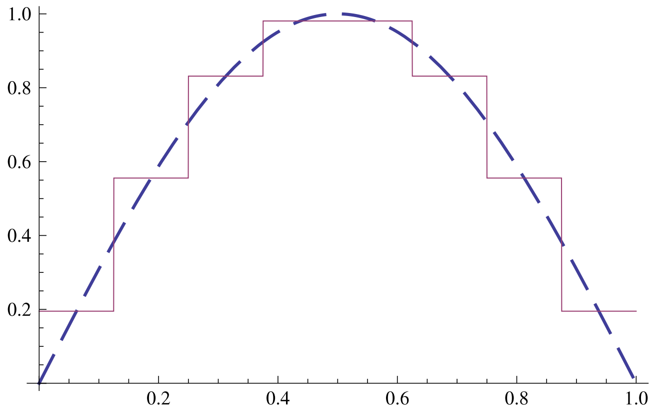

Fig. 1. Haar approximation of the function $f(x) = \sin(\pi x)$ for $M = 4$: - - - exact curve, — Haar curve.

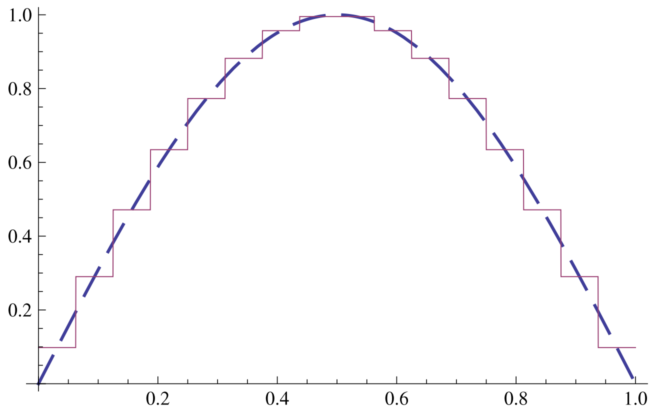

Fig. 2. Haar approximation of the function $f(x) = \sin(\pi x)$ for $M = 8$: - - - exact curve, — Haar curve.

2.2.1. Numerical technique for single integrals

We consider the integral

$$\int_a^b f(x)\,\mathrm{d}x \tag{7}$$

over the interval $[a, b]$. The function $f(x)$ can be approximated using Haar wavelets as

$$f(x) \approx \sum_{i=1}^{2M} a_i h_i(x). \tag{8}$$

The behavior of Haar wavelets to approximate $\sin(\pi x)$ is shown in Figs. 1 and 2. The Haar wavelets approximation converges rapidly to the exact function by increasing the value of M.

Lemma 1. *The approximate value of the integral is*

$$\int_a^b f(x)\,\mathrm{d}x \approx a_1(b-a). \tag{9}$$

Proof. Since

$$\int_a^b h_i(x)\,\mathrm{d}x = 0, \quad i = 2, 3, \ldots, \tag{10}$$

and

$$\int_a^b h_1(x)\,\mathrm{d}x = b - a, \tag{11}$$

$$\int_a^b f(x)\,\mathrm{d}x \approx \sum_{i=1}^{2M} a_i \int_a^b h_i(x)\,\mathrm{d}x = a_1(b-a). \quad \square \tag{12}$$

It is clear from Eq. (12) that Haar approximation involves only one coefficient in the evaluation of the definite integral. To calculate the Haar coefficient a_1 we consider the nodal points

$$x_k = a + (b-a)\frac{k-0.5}{2M}, \quad k = 1, 2, \ldots, 2M. \tag{13}$$

The discretized form of (8) can be written as

$$f(x_k) = \sum_{i=1}^{2M} a_i h_i(x_k), \quad k = 1, 2, \ldots, 2M. \tag{14}$$

The second advantage of Haar wavelet approximation is that we do not need to solve the above system which is computationally expensive for large values of M. The next lemma gives us an easy formula with which to calculate the value of the Haar coefficient a_1.

Lemma 2. *The solution of the system (14) for a_1 is*

$$a_1 = \frac{1}{2M}\sum_{k=1}^{2M} f(x_k). \tag{15}$$

Proof. We prove the result by induction on J, where $M = 2^J$. For $J = 0$, we have $M = 1$, and the linear system in this case is

$$\begin{aligned} f(x_1) &= a_1 + a_2 \\ f(x_2) &= a_1 - a_2, \end{aligned} \tag{16}$$

which has solution

$$a_1 = \frac{1}{2}[f(a_1) + f(a_2)]. \tag{17}$$

Therefore, the lemma is true for $J = 0$.

Next assume that the lemma is true for $J = n-1,\ n = 1, 2, \ldots$ and consider the linear system with $J = n$. For $J = n$, we have $M = 2^n$ and the linear system has 2^{n+1} equations involving 2^{n+1} variables. From this system we obtain a new system by adding consecutive equations, first and second, third and fourth, fifth and sixth, and so on. This new system has 2^n equations involving 2^n variables. Replacing $2a_k$ by a'_k and $f(x_{2k-1}) + f(x_{2k})$ by $g(x_k)$ in this system, we obtain a system similar to system (14), and so we can apply an induction hypothesis to this system. Thus we have

$$a'_1 = \frac{1}{2.2^{n-1}}\sum_{k=1}^{2.2^{n-1}} g(x_k). \tag{18}$$

Substituting back the values of a'_1 and $g(x_k)$, we obtain

$$a_1 = \frac{1}{2.2^n}\sum_{i=1}^{2.2^n} f(x_k), \tag{19}$$

and so the lemma is true for $J = n$. Hence, by induction, the lemma is true for all $J = 0, 1, \ldots$. $\square$

Hence, using the quadrature method with Haar wavelets we obtain the following formula for numerical integration:

$$\int_a^b f(x)\,\mathrm{d}x \approx \frac{b-a}{2M}\sum_{i=1}^{2M} f(x_k) = \frac{b-a}{2M}\sum_{k=1}^{2M} f\left(a + \frac{(b-a)(k-0.5)}{2M}\right). \tag{20}$$

2.2.2. Numerical technique for double and triple integrals

We derive a similar formula for double and triple integrals. The method can be extended to higher integrals as well. Consider the double integral

$$\int_c^d \int_a^b f(x, y)\,\mathrm{d}x\,\mathrm{d}y. \tag{21}$$

The function $f(x, y)$ can be approximated using Haar wavelets as

$$f(x, y) \approx \sum_{j=1}^{2M} \sum_{i=1}^{2M} a_{ij} h_i(x) h_j(y). \tag{22}$$

Lemma 3. *The approximate value of the integral is*

$$\int_c^d \int_a^b f(x, y)\,\mathrm{d}x\,\mathrm{d}y \approx a_{11}(b-a)(d-c). \tag{23}$$

Proof. The proof of this lemma is similar to the proof of Lemma 1. □

As in the case of the single integral, we define the points

$$x_k = a + (b-a)\frac{k-0.5}{2M}, \quad k = 1, 2, \ldots, 2M \tag{24}$$

and

$$y_l = c + (d-c)\frac{l-0.5}{2M}, \quad l = 1, 2, \ldots, 2M. \tag{25}$$

Substituting these points in (22), we obtain

$$f(x_k, y_l) = \sum_{j=1}^{2M} \sum_{i=1}^{2M} a_{ij} h_i(x_k) h_j(y_l), \quad k = 1, 2, \ldots, 2M,\ l = 1, 2, \ldots, 2M. \tag{26}$$

Lemma 4. *The solution of the system (26) for a_{11} is*

$$a_{11} = \frac{1}{4M^2} \sum_{l=1}^{2M} \sum_{k=1}^{2M} f(x_k, y_l). \tag{27}$$

Proof. The proof of this lemma is similar to the proof of Lemma 2. □

Therefore, the formula for approximating double integrals using Haar wavelets is

$$\int_c^d \int_a^b f(x, y)\,\mathrm{d}x\,\mathrm{d}y \approx \frac{(b-a)(d-c)}{4M^2} \sum_{l=1}^{2M} \sum_{k=1}^{2M} f\left(a + (b-a)\frac{k-0.5}{2M}, c + (d-c)\frac{l-0.5}{2M}\right). \tag{28}$$

This formula can be extended to triple integrals, and is given by

$$\int_e^h \int_c^d \int_a^b f(x, y, z)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z \approx \frac{(h-e)(d-c)(b-a)}{8M^3} \sum_{l=1}^{2M} \sum_{k=1}^{2M} \sum_{j=1}^{2M} f(x_j, y_k, z_l), \tag{29}$$

where

$$x_j = a + (b-a)\frac{j-0.5}{2M}, \quad j = 1, 2, \ldots, 2M, \tag{30}$$

$$y_k = c + (d-c)\frac{k-0.5}{2M}, \quad k = 1, 2, \ldots, 2M, \tag{31}$$

$$z_l = e + (h-e)\frac{l-0.5}{2M}, \quad l = 1, 2, \ldots, 2M. \tag{32}$$

3. Numerical integration using hybrid functions

3.1. Hybrid functions

The orthogonal set of hybrid functions $\psi_{ij}(x)$, $i = 1, 2, \ldots, n$ and $j = 0, 1, \ldots, m-1$ is defined on the interval [0, 1) as

$$\psi_{ij}(x) = \begin{cases} L_j(2nt - 2i + 1), & \text{for } x \in \left[\dfrac{i-1}{n}, \dfrac{i}{n}\right) \\ 0, & \text{otherwise,} \end{cases} \tag{33}$$

where n and m are the orders of the block-pulse functions and Legendre polynomials, respectively. The notation ψ is used for hybrid functions to distinguish them from Haar wavelets. The Legendre polynomials can be calculated recursively as

$$L_0(x) = 1, \qquad L_1(x) = x, \tag{34}$$

$$L_{k+1}(t) = \left(\frac{2k+1}{k+1}\right) x L_k(x) - \left(\frac{k}{k+1}\right) L_{k-1}(x), \quad k = 1, 2, 3, \ldots. \tag{35}$$

Any function $f(x)$ which is square integrable in the interval [0, 1) can be expressed as

$$f(x) = \sum_{i=1}^{\infty} \sum_{j=0}^{\infty} c_{ij} \psi_{ij}(x), \quad i = 1, 2, \ldots, \infty,\ j = 0, 1, \ldots, \infty,\ x \in [0, 1). \tag{36}$$

However, if the function $f(x)$ is piecewise constant or may be approximated as piecewise constant, then we can approximate $f(x)$ as

$$f(x) \approx \sum_{i=1}^{n} \sum_{j=0}^{m-1} c_{ij} \psi_{ij}(x). \tag{37}$$

3.2. Method of numerical integration based on hybrid functions

In this section we consider numerical integration for single and double integrals using hybrid functions.

3.2.1. Numerical technique for single integrals

We consider the definite integral

$$\int_0^1 f(x)\,\mathrm{d}x. \tag{38}$$

If the limits of integration are different, then by a suitable substitution these limits can be changed to 0 and 1. Using (37), the approximate value of the integral is given by

$$\int_0^1 \sum_{i=1}^{n} \sum_{j=0}^{m-1} c_{ij} \psi_{ij}(x) = \frac{1}{n} \sum_{i=1}^{n} c_{i0}. \tag{39}$$

In order to calculate the coefficients c_{i0} of hybrid functions we consider the nodal points

$$x_k = \frac{2k-1}{2mn}, \quad k = 1, 2, \ldots, mn. \tag{40}$$

Substituting these points in (37), we obtain

$$f(x_k) = \sum_{i=1}^{n} \sum_{j=0}^{m-1} c_{ij} \psi_{ij}(x_k), \quad k = 1, 2, \ldots, mn. \tag{41}$$

We can calculate the hybrid coefficients $c_{i0},\ i = 1, 2, \ldots, n$ from the above system of equations. Note that we need to calculate only n coefficients.

The fast convergence of hybrid functions for $\sin(\pi x)$ is shown in Figs. 3 and 4. These clearly show an improved convergent behavior in comparison with Haar functions.

For $m = 1$, the coefficients c_{i0} are given as

$$c_{i0} = f\left(\frac{2i-1}{2n}\right), \tag{42}$$

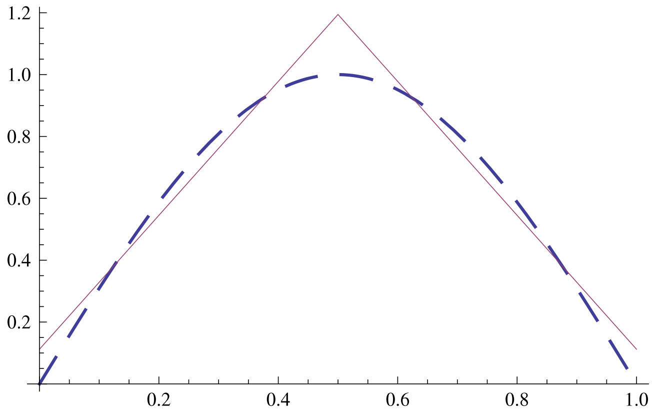

Fig. 3. Hybrid approximation of the function $f(x) = \sin(\pi x)$ for $m = 2$, $n = 2$: - - - exact curve, — hybrid curve.

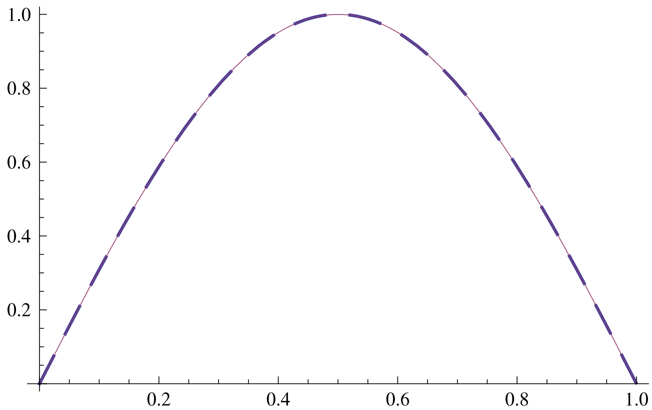

Fig. 4. Hybrid approximation of the function $f(x) = \sin(\pi x)$ for $m = 4$, $n = 4$: - - - exact curve, — hybrid curve.

and the approximate value of the integral in this case is

$$\int_0^1 f(x)\,\mathrm{d}x \approx \frac{1}{n}\sum_{i=1}^{n} f\left(\frac{2i-1}{2n}\right). \tag{43}$$

For $m = 2$,

$$\int_0^1 f(x)\,\mathrm{d}x \approx \frac{1}{2n}\sum_{i=1}^{2n} f\left(\frac{2i-1}{4n}\right). \tag{44}$$

For $m = 3$,

$$\int_0^1 f(x)\,\mathrm{d}x \approx \frac{1}{8n}\sum_{i=1}^{n}\left(3f\left(\frac{6i-5}{6n}\right) + 2f\left(\frac{6i-3}{6n}\right) + 3f\left(\frac{6i-1}{6n}\right)\right). \tag{45}$$

For $m = 4$,

$$\int_0^1 f(x)\,\mathrm{d}x \approx \frac{1}{48n}\sum_{i=1}^{n}\left(13f\left(\frac{8i-7}{8n}\right) + 11f\left(\frac{8i-5}{8n}\right) + 11f\left(\frac{8i-3}{8n}\right) + 13f\left(\frac{8i-1}{8n}\right)\right). \tag{46}$$

For $m = 5$,

$$\int_0^1 f(x)\,\mathrm{d}x \approx \frac{1}{1152n}\sum_{i=1}^{n}\left(275f\left(\frac{10i-9}{10n}\right) + 100f\left(\frac{10i-7}{10n}\right) + 402f\left(\frac{10i-5}{10n}\right) + 100f\left(\frac{10i-3}{10n}\right) + 275f\left(\frac{10i-1}{10n}\right)\right). \tag{47}$$

For $m = 6$,

$$\int_0^1 f(x)\,\mathrm{d}x \approx \frac{1}{1280n}\sum_{i=1}^{n}\left(247f\left(\frac{12i-11}{12n}\right)+139f\left(\frac{12i-9}{12n}\right)+254f\left(\frac{12i-7}{12n}\right)\right.$$
$$\left.+254f\left(\frac{12i-5}{12n}\right)+139f\left(\frac{12i-3}{12n}\right)+247f\left(\frac{12i-1}{12n}\right)\right). \tag{48}$$

3.2.2. Numerical technique for double integrals

We consider the double integral

$$\int_0^1\int_0^1 f(x,y)\,\mathrm{d}x\,\mathrm{d}y. \tag{49}$$

Using hybrid functions we can approximate the function $f(x, y)$ as

$$f(x,y) \approx \sum_{i=1}^{n}\sum_{j=0}^{m-1}\sum_{k=1}^{n}\sum_{l=0}^{m-1} c_{ijkl}\psi_{ij}(x)\psi_{kl}(y). \tag{50}$$

Substituting these in (49), we obtain an approximate value of the integral as

$$\int_0^1\int_0^1 f(x,y)\,\mathrm{d}x\,\mathrm{d}y \approx \frac{1}{n^2}\sum_{i=1}^{n}\sum_{k=1}^{n} c_{i0k0}. \tag{51}$$

In order to calculate the hybrid coefficients c_{i0k0}, we consider the points

$$x_p = \frac{2k-1}{2mn}, \quad p = 1, 2, \ldots, mn, \tag{52}$$

$$y_q = \frac{2k-1}{2mn}, \quad q = 1, 2, \ldots, mn. \tag{53}$$

Substituting these points in (50), we obtain a system of equations:

$$f(x,y) = \sum_{i=1}^{n}\sum_{j=0}^{m-1}\sum_{k=1}^{n}\sum_{l=0}^{m-1} c_{ijkl}\psi_{ij}(x_p)\psi_{kl}(y_q), \quad p = 1, 2, \ldots, mn,\ q = 1, 2, \ldots, mn. \tag{54}$$

The coefficients c_{i0k0} can be easily calculated from this system. We need to calculate n^2 coefficients in this case. Integral approximations are given below for the first few values of m.

For $m = 1$,

$$\int_0^1\int_0^1 f(x,y)\,\mathrm{d}x\,\mathrm{d}y \approx \frac{1}{n^2}\sum_{i=1}^{n}\sum_{k=1}^{n} f\left(\frac{2i-1}{2n}, \frac{2k-1}{2n}\right). \tag{55}$$

For $m = 2$,

$$\int_0^1\int_0^1 f(x,y)\,\mathrm{d}x\,\mathrm{d}y \approx \frac{1}{4n^2}\sum_{i=1}^{2n}\sum_{k=1}^{2n} f\left(\frac{2i-1}{4n}, \frac{2k-1}{4n}\right). \tag{56}$$

For $m = 3$,

$$\int_0^1\int_0^1 f(x,y)\,\mathrm{d}x\,\mathrm{d}y \approx \frac{1}{64n^2}\sum_{i=1}^{n}\sum_{k=1}^{n}\left(9f\left(\frac{6i-5}{6n}, \frac{6k-5}{6n}\right)+6f\left(\frac{6i-5}{6n}, \frac{6k-3}{6n}\right)+9f\left(\frac{6i-5}{6n}, \frac{6k-1}{6n}\right)\right.$$
$$+6f\left(\frac{6i-3}{6n}, \frac{6k-5}{6n}\right)+4f\left(\frac{6i-3}{6n}, \frac{6k-3}{6n}\right)+6f\left(\frac{6i-3}{6n}, \frac{6k-1}{6n}\right)$$
$$\left.+9f\left(\frac{6i-1}{6n}, \frac{6k-5}{6n}\right)+6f\left(\frac{6i-1}{6n}, \frac{6k-3}{6n}\right)+9f\left(\frac{6i-1}{6n}, \frac{6k-1}{6n}\right)\right). \tag{57}$$

For $m = 4$,

$$\int_0^1 \int_0^1 f(x,y)\,\mathrm{d}x\,\mathrm{d}y \approx \frac{1}{2304n^2} \sum_{i=1}^{n} \sum_{k=1}^{n} \Bigg(169f\left(\frac{8i-7}{8n}, \frac{8k-7}{8n}\right) + 143f\left(\frac{8i-7}{8n}, \frac{8k-5}{8n}\right)$$
$$+ 143f\left(\frac{8i-7}{8n}, \frac{8k-3}{8n}\right) + 169f\left(\frac{8i-7}{8n}, \frac{8k-1}{8n}\right) + 143f\left(\frac{8i-5}{8n}, \frac{8k-7}{8n}\right)$$
$$+ 121f\left(\frac{8i-5}{8n}, \frac{8k-5}{8n}\right) + 121f\left(\frac{8i-5}{8n}, \frac{8k-3}{8n}\right) + 143f\left(\frac{8i-5}{8n}, \frac{8k-1}{8n}\right)$$
$$+ 143f\left(\frac{8i-3}{8n}, \frac{8k-7}{8n}\right) + 121f\left(\frac{8i-3}{8n}, \frac{8k-5}{8n}\right)$$
$$+ 121f\left(\frac{8i-3}{8n}, \frac{8k-3}{8n}\right) + 143f\left(\frac{8i-3}{8n}, \frac{8k-1}{8n}\right) + 169f\left(\frac{8i-1}{8n}, \frac{8k-7}{8n}\right)$$
$$+ 143f\left(\frac{8i-1}{8n}, \frac{8k-5}{8n}\right) + 143f\left(\frac{8i-1}{8n}, \frac{8k-3}{8n}\right) + 169f\left(\frac{8i-1}{8n}, \frac{8k-1}{8n}\right)\Bigg). \quad (58)$$

4. Error analysis

4.1. Haar wavelets

Assume that $f(x)$ is a differentiable function with

$$|f'(x)| \le K, \quad \forall t \in (a,b), \quad (59)$$

where K is a positive constant. The Haar wavelet approximation for the function $f(x)$ is given by

$$f_M(x) = \sum_{i=1}^{2M} a_i h_i(x). \quad (60)$$

Babolian and Shahsavaran [16] have shown that the square of the error norm for wavelet approximation is given by

$$\|f(x) - f_M(x)\|^2 = \frac{K^3}{3} \cdot \frac{1}{(2M)^2}. \quad (61)$$

Therefore,

$$\|f(x) - f_M(x)\| = O\left(\frac{1}{M}\right). \quad (62)$$

From the above equation, it is obvious that the error bound is inversely proportional to the level of resolution of the Haar wavelet. This ensures the convergence of the Haar wavelet approximation when M is increased.

4.2. Hybrid functions

Proposition. *Let $x(t) \in H^k(-1,1)$ (Sobolev space), $x_J(t) = \sum_{i=0}^{J} a_i L_i(t)$ be the best approximation polynomial of $x(t)$ in L^2- norm; then*

$$\|x(t) - x_J(t)\|_{L^2[-1,1]} \le C_0 J^{-k} \|x(t)\|_{H^k(-1,1)},$$

where C_0 is a positive constant, which depends on the selected norm and is independent of $x(t), J$; see [17,18].

5. Numerical examples

The following examples are given to show the accuracy and efficiency of Haar wavelets. In Examples 4 and 5 we apply the method to double integrals while in Example 6 we apply it to a triple integral. In Examples 3 and 5, improper integrals are approximated using this method.

Example 1.

$$\int_0^1 \sin(x^2)\,\mathrm{d}x.$$

Relative errors are shown in Table 1.

Table 1
Relative errors of Example 1.

Haar		Hybrid	
$J = 4$	1.4177E−04	$m = 3, n = 5$	1.0110E−05
$J = 5$	3.5432E−05	$m = 4, n = 8$	7.9460E−07
$J = 6$	8.8574E−06	$m = 5, n = 12$	3.4471E−11
$J = 7$	2.2143E−06	$m = 6, n = 20$	9.6947E−13

Table 2
Relative errors of Example 2.

Haar		Hybrid	
$J = 4$	3.5293E−05	$m = 3, n = 5$	3.6467E−07
$J = 5$	8.8229E−06	$m = 4, n = 8$	2.8648E−08
$J = 6$	2.2057E−06	$m = 5, n = 12$	1.3262E−12
$J = 7$	5.5143E−07	$m = 6, n = 20$	3.7148E−14

Table 3
Relative errors of Example 3.

Haar		Hybrid	
$J = 4$	4.0642E−05	$m = 3, n = 5$	2.9376E−04
$J = 5$	1.0173E−05	$m = 4, n = 8$	6.6406E−05
$J = 6$	2.5431E−06	$m = 5, n = 12$	1.8220E−06
$J = 7$	6.3578E−07	$m = 6, n = 20$	3.7947E−08

Table 4
Relative errors of Example 4.

Haar		Hybrid	
$J = 4$	5.0215E−04	$m = 3, n = 5$	4.3261E−07
$J = 5$	1.2551E−04	$m = 4, n = 8$	3.4122E−08
$J = 6$	3.1375E−05	$m = 5, n = 12$	2.4702E−13
$J = 7$	7.8437E−06	–	–

Example 2.

$$\int_0^5 \sqrt{x^2 - 5x + 31}\,dx.$$

Relative errors are shown in Table 2.

Example 3 (*Improper Integral*).

$$\int_0^1 \frac{e^{-1/x}}{x^2}\,dx.$$

Relative errors are shown in Table 3.

Example 4.

$$\int_0^{\frac{\pi}{2}} \int_0^{\pi} \sin(x+y)\,dx\,dy.$$

Relative errors are shown in Table 4.

Example 5 (*Improper Integral*).

$$\int_0^1 \int_0^1 \frac{1}{\sqrt{x^2+y^2}}\,dx\,dy.$$

Relative errors are shown in Table 5.

Table 5
Relative errors of Example 5.

Haar		Hybrid	
$J = 4$	7.1275E−03	$m = 3, n = 5$	8.9863E−03
$J = 5$	3.5719E−03	$m = 4, n = 8$	4.6532E−03
$J = 6$	1.7880E−03	$m = 5, n = 12$	1.8550E−03
$J = 7$	8.9450E−04	–	–

Table 6
Relative errors of Example 6.

Haar	
$J = 4$	1.2863E−05
$J = 5$	3.2159E−06
$J = 6$	8.0398E−07

Example 6.

$$\int_1^2 \int_1^2 \int_1^2 \frac{1}{x+y+z} \, dx \, dy \, dz.$$

Relative errors are shown in Table 6.

Note that in the case of improper integrals (Examples 3 and 5), the hybrid function approach is slightly better than the Haar wavelet approach due to the effects of singularity, while in other examples we got significantly better results using hybrid functions. This suggests that in the case of improper integrals one should prefer the Haar wavelet approach.

6. Conclusion

A comparative analysis of Haar wavelets and hybrid functions is performed to find numerical approximations of different types of integral. The simple applicability of Haar wavelets and the fast convergence of hybrid functions provide a solid foundation for using these functions in the context of numerical approximation of integral equations, partial differential equations and ordinary differential equations.

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References

[1] C.F. Chen, C.H. Hsiao, Haar wavelet method for solving lumped and distributed-parameter systems, IEE Proc. Control Theory Appl. 144 (1997) 87–94.
[2] C.H. Hsiao, Hybrid function method for solving Fredholm and Volterra integral equations of the second kind, J. Comput. Appl. Math. 230 (2009) 59–68.
[3] C.F. Chen, C.H. Hsiao, Wavelet approach to optimising dynamic systems, IEE Proc. Control Theory Appl. 146 (1997) 213–219.
[4] Chun Hui Hsiao, Wen-June Wang, Haar wavelet approach to nonlinear stiff systems, Math. Comput. Simulation 57 (2001) 347–353.
[5] C.H. Hsiao, Haar wavelet approach to linear stiff systems, Math. Comput. Simulation 64 (2004) 561–567.
[6] Ü. Lepik, Numerical solution of differential equations using Haar wavelets, Math. Comput. Simulation 168 (2005) 127–143.
[7] Ü. Lepik, E. Tamme, Application of the Haar wavelets for solution of linear integral equations, in: Dynamical Systems and Applications, Antalaya, Proceedings, 2004, pp. 494–507.
[8] Ü. Lepik, E. Tamme, Solution of nonlinear Fredholm integral equations via the Haar wavelet method, Proc. Estonian Acad. Sci. Phys. Math. 56 (2007) 17–27.
[9] K. Maleknejad, F. Mirzaee, Using rationalized Haar wavelet for solving linear integral equations, Appl. Math. Comput. 160 (2005) 579–587.
[10] Ü. Lepik, Haar wavelet method for nonlinear integro–differential equations, Appl. Math. Comput. 176 (2006) 324–333.
[11] Ü. Lepik, Numerical solution of evolution equations by the Haar wavelet method, Appl. Math. Comput. 185 (2007) 695–704.
[12] Xing Tao Wang, Numerical solution of optimal control for scaled systems by hybrid functions, Int. J. Innovative Comput. Inf. Control 4 (2008) 849–855.
[13] H.R. Marzban, M. Razzaghi, Numerical solution of the controlled duffing oscillator by hybrid functions, Appl. Math. Comput. 140 (2003) 179–190.
[14] H.R. Marzban, H.R. Tabrizidooz, M. Razzaghi, Hybrid functions for nonlinear initial-value problems with applications to Lane–Emden type equations, Phys. Lett. A 372 (2008) 5883–5886.
[15] H.R. Marzban, M. Razzaghi, Optimal control of linear delay systems via hybrid of block-pulse and Legendre polynomials, J. Franklin Inst. 341 (2004) 279–293.
[16] E. Babolian, A. Shahsavaran, Numerical solution of nonlinear Fredholm integral equations of the second kind using Haar wavelets, J. Comput. Appl. Math. 225 (2009) 87–95.
[17] K. Maleknejad, M. Tavassoli Kajani, Solving second kind integral equations by Galerkin methods with hybrid Legendre and block-pulse functions, Appl. Math. Comput. 145 (2003) 623–629.
[18] C. Canuto, M.Y. Hussaini, A. Quarteroni, T.A. Zang, Spectral Methods on Fluid Dynamics, Springer Verlag, 1988.