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Nonstationary Flows of Viscous and Ideal Fluids in R^3 *

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The Cauchy problem for the nonstationary Navier–Stokes equation in R^3 is considered. It is shown that the solution exists on a time interval independent of the viscosity ν and tends as $\nu \to 0$ to the solution of the limiting equation, provided that the initial velocity field and the external force field are sufficiently smooth and small at infinity (in the sense that they belong to the Sobolev space over R^3 of order 3). Such a result is not altogether new but the proof, which depends on the theory of nonlinear evolution equation in Hilbert space, is simpler than the existing one due to Swann.

1. INTRODUCTION

The following results are proved in Swann [1] (see the original for more precise statements). The Cauchy problem for the Navier–Stokes equation in R^3 has a unique classical solution u_ν on a time interval $[0, T_0]$ independent of the viscosity ν, if the initial velocity field and the external force field are sufficiently smooth and decay sufficiently fast at infinity. As $\nu \to 0$, u_ν converges on $[0, T_0]$ to a limit u_0, which is a unique classical solution of the limit equation for the same data.

Swann's proof makes use of the Schauder fixed-point theorem combined with certain estimates related to linear evolution equations. In the present paper, we shall give a simplified proof using elementary results on nonlinear evolution equations of a familiar type, although our results are not exactly the same as Swann's.

We write the Navier–Stokes equation as an abstract evolution equation (see Kato–Fujita [2], Fujita–Kato [3], Kato [4])

$$\begin{aligned} du/dt + \nu A u &= Fu + b(t), \qquad t > 0, \\ u(0) &= a. \end{aligned} \tag{1.1}$$

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Here the unknown $u(t)$ (velocity field) takes values in the real Hilbert space H_σ, which is the subspace of $L^2(R^3)^3$ consisting of all solenoidal ($\operatorname{div} u = 0$) vectors; $\nu > 0$ is the (kinematic) viscosity; $A = -\Delta$ (negative Laplacian) is a nonnegative self-adjoint operator in H_σ; F is a bilinear operator

$$Fu = F(u, u), \qquad F(u, v) = -P(u \cdot \operatorname{grad})v, \tag{1.2}$$

where P is the orthogonal projection of $L^2(R^3)^3$ onto H_σ; $b(t) \in H_\sigma$ is the projection of the given external force to H_σ; and $a \in H_\sigma$ is the initial velocity.

We shall denote by $(1.1)_0$ the Eq. (1.1) for $\nu = 0$.

It was shown in [2] that (1.1) has a unique strong solution u_ν for $0 \leqslant t \leqslant T_\nu$ if $a \in D(A^{1/4})$ and $b(t) \in H_\sigma$ is Hölder continuous in t. [$D(S)$ denotes the domain of the operator S. Actually [2] dealt with the flow in a bounded domain of R^3, but the same proof applies to the present case.] Here $T_\nu > 0$ depends on ν and perhaps tends to 0 with ν. (1.1) implies a uniform estimate of the form $\| u_\nu(t)\| \leqslant \| a \| + \int_0^t \| b(s)\| \, ds$, but it is difficult to extend the solution u_ν beyond the interval $[0, T_\nu]$ since we have no similar estimate for $\| A^{1/4}u_\nu(t)\|$ (cf. [4]).

To obtain a solution that exists on an interval independent of ν, we shall assume that a and $b(t)$ belong to $D(A^{m/2}) = H_\sigma{}^m$ with a positive integer m. [$H_\sigma{}^m$ denotes the subspace, consisting of all solenoidal vectors, of the Sobolev space $H^m(R^3)^3$ of order m and exponent 2. We denote by $\| \;\|_m$ and $(\, , \,)_m$ the Sobolev m-norm and the associated inner product.] It should be noted that $H_\sigma{}^m$ reduces A, and that P may be regarded also as the orthogonal projection of $H^m(R^3)^3$ onto $H_\sigma{}^m$.

The main result of this paper is given by the following theorem. Here we denote by $C[0, T; X]$ the set of all X-valued continuous functions on the closed interval $[0, T]$, by $AC[0, T; X]$ the set of all X-valued absolutely continuous functions on $[0, T]$, and so on.

THEOREM. *Let $a \in H_\sigma{}^m$ and $b \in L^1[0, T; H_\sigma{}^m]$, where $m \geqslant 3$ and $T > 0$.*

(i) *There exists $T_0 > 0$, $T_0 \leqslant T$, depending on $\| a \|_m$ and $\| b(\cdot)\|_m$ but not on ν, such that* (1.1) *has a unique solution*

$$u_\nu \in C[0, T_0; H_\sigma^m] \cap AC[0, T_0; H_\sigma^{m-1}] \cap L^1[0, T_0; H_\sigma^{m+1}]. \tag{1.3}$$

Furthermore, $\{u_\nu\}$ is bounded in $C[0, T_0; H_\sigma{}^m]$ for all $\nu > 0$.

(ii) *For each* $t \in [0, T_0]$, $u_0(t) = \lim_{\nu \to 0} u_\nu(t)$ *exists strongly in* H_σ^{m-1} *and weakly in* $H_\sigma{}^m$, *uniformly in t.* u_0 *is a unique solution of* $(1.1)_0$ *satisfying*

$$u_0 \in C[0, T_0; H_\sigma{}^m] \cap AC[0, T_0; H_\sigma^{m-1}]. \tag{1.4}$$

Remarks. 1. Since $m \geqslant 3$, it is obvious that u_ν and u_0 are classical solutions of (1.1) and $(1.1)_0$, respectively.

2. The solutions are smoother if b is. For example, we have $u_0 \in C^1[0, T_0; H_\sigma^{m-1}]$ if $b \in C[0, T; H_\sigma^{m-1}]$ in addition to the assumption of the theorem. In this case, it is interesting to note, u_0 is partly better behaved than u_ν, for du_ν/dt need not be continuous in H_σ^{m-1} at $t = 0$.

3. Similar results were obtained, by a different method, by Ebin and Marsden [5] for flows in a compact manifold (without boundary) of arbitrary dimension. For flows in R^2, the existence of solutions u_ν and u_0 and the convergence $u_\nu \to u_0$ for *all* time $t \geqslant 0$ are known; see Golovkin [6], McGrath [7], Ladyzhenskaya [8]. It is straightforward to generalize our theorem to higher dimensions, with m increased accordingly.

2. Proof of the Theorem, Part (i)

To prove the theorem, we need the following estimates for the nonlinear operator F.

$$\| F(u, v)\|_m \leqslant c \| u \|_m \| v \|_{m+1}, \quad m \geqslant 2, \quad u \in H_\sigma{}^m, \quad v \in H_\sigma^{m+1}. \tag{2.1}$$

$$|(F(u, v), v)_m | \leqslant c' \| u \|_m \| v \|_m^2, \quad m \geqslant 3, \quad u, v \text{ as above}. \tag{2.2}$$

$$|(F(u, v), v)_2 | \leqslant c' \| u \|_3 \| v \|_2^2, \qquad u, v \in H_\sigma{}^3. \tag{2.2'}$$

Here c and c' may depend on m. These inequalities will be proved in Section 4.

We first construct a local solution u_ν of (1.1). This can be done by successive approximation as in [2], or by the fixed-point theorem for a contraction map. We shall give the details in Section 5, and state the result here. The local solution u_ν belongs to the class (1.3) with T_0 replaced by a $T_\nu > 0$ depending on ν, $\| a \|_m$ and $\| b(\cdot)\|_m$.

To show that u_ν can be extended to a solution on $[0, T_0]$ with $T_0 > 0$ independent of ν, we deduce an estimate for $\| u_\nu(t)\|_m$ independent of ν. Taking the m-inner product of (1.1) with $u_\nu(t)$ and using (2.2), we obtain

$$\tfrac{1}{2}(d/dt)\, \| u_\nu \|_m^2 + \nu(Au_\nu\,, u_\nu)_m \leqslant c'\, \| u_\nu \|_m^3 + \| b(t)\|_m \, \| u_\nu \|_m\,, \tag{2.3}$$

where u_ν stands for $u_\nu(t)$. Since the second term on the left of (2.3) is nonnegative, we obtain

$$(d/dt)\, \| u_\nu \|_m \leqslant c'\, \| u_\nu \|_m^2 + \| b(t)\|_m\,, \qquad \| u_\nu(0)\|_m = \| a \|_m\,.$$

It follows that

$$\| u_\nu(t)\|_m \leqslant \phi(t), \tag{2.4}$$

where ϕ is the solution of the scalar initial-value problem:

$$d\phi(t)/dt = c'\phi(t)^2 + \| b(t)\|_m\,, \qquad \phi(0) = \| a \|_m\,. \tag{2.5}$$

Since $\| b(t)\|_m$ is integrable in t by hypothesis, ϕ exists as a continuous function on a certain interval $[0, T_0]$, $T_0 > 0$; obviously T_0 and ϕ are independent of ν.

(2.4) is true for $t \in [0, T_\nu]$. If $T_\nu < T_0$, we can solve (1.1) for $t \geqslant T_\nu$ with the initial value $u_\nu(T_\nu) \in H_\sigma{}^m$, to continue the solution u_ν to an interval $[0, T_\nu + T_\nu']$, in which (2.4) is true. Since $T_\nu' > 0$ can be determined depending only on ν, $\| u(T_\nu)\|_m$, and $\| b(\cdot)\|_m$, it is easy to see that, after a finite number of such extensions, u_ν can be continued as a solution on $[0, T_0]$ with the estimate (2.4) throughout. This completes the proof of part (i) of the theorem. [The uniqueness follows from the uniqueness of the local solution.]

For later use we note the inequality

$$\nu \int_0^t (Au_\nu(s), u_\nu(s))_m \, ds \leqslant \psi(t), \qquad t \in [0, T_0], \tag{2.6}$$

where ψ is a continuous scalar function depending on $\| a \|_m$ and $\| b(\cdot)\|_m$ but not on ν. (2.6) is obtained by integrating (2.3) and using (2.4).

3. Proof of the Theorem, Part (ii)

For simplicity write $u_1 = u_{\nu_1}$, $u_2 = u_{\nu_2}$, where $\nu_1 < \nu_2$. Taking the difference of the corresponding equations, we obtain

$$\begin{aligned}(d/dt)(u_1 - u_2) + \nu_1 A(u_1 - u_2) + (\nu_1 - \nu_2)\, Au_2 \\ = F(u_1 - u_2\,, u_1) + F(u_2\,, u_1 - u_2).\end{aligned} \tag{3.1}$$

Write $w = u_1 - u_2$ and take the $(m - 1)$-inner product of (3.1) with $w(t)$. Noting that $A \geqslant 0$ and using (2.1) and (2.2) or (2.2'), we obtain (note that $m \geqslant 3$)

$$\tfrac{1}{2}(d/dt)\| w \|_{m-1}^2 \leqslant (\nu_2 - \nu_1)(Au_2, w)_{m-1} + (c \| u_1 \|_m + c' \| u_2 \|_m) \| w \|_{m-1}^2 .$$

But $\| u_1 \|_m$, $\| u_2 \|_m$ are uniformly bounded by (2.4). Hence

$$(d/dt)\| w \|_{m-1} \leqslant \nu_2 \| Au_2 \|_{m-1} + K \| w \|_{m-1},$$

where K is a constant independent of ν, and so (note $w(0) = 0$)

$$\begin{aligned} \| w(t)\|_{m-1} &\leqslant \nu_2 e^{Kt} \int_0^t \| Au_2(s)\|_{m-1}\, ds \\ &\leqslant (\nu_2 t)^{1/2} e^{Kt} \left(\nu_2 \int_0^t \| Au_2(s)\|_{m-1}^2\, ds\right)^{1/2}. \end{aligned} \tag{3.2}$$

Since $\| Au_2 \|_{m-1}^2 = \|(A + 1)^{-1/2} Au_2 \|_m^2 \leqslant \| A^{1/2} u_2 \|_m^2 = (Au_2, u_2)_m$, the right member of (3.2) tends to zero as $\nu_2 \to 0$ by (2.6).

Thus $u_0(t) = \lim_{\nu \to 0} u_\nu(t) \in H_\sigma^{m-1}$ exists in $(m - 1)$-norm, uniformly for $t \in [0, T_0]$. Clearly, $u_0(\cdot)$ is continuous in $(m - 1)$-norm. Since, however, $\| u_\nu(t)\|_m \leqslant \phi(t)$ by (2.4), it follows easily that $u_0(t) \in H_\sigma{}^m$, $u(t) \rightharpoonup u_0(t)$ (weak convergence) in $H_\sigma{}^m$ uniformly in t, $\| u_0(t)\|_m \leqslant \phi(t)$, and that $u_0(\cdot)$ is weakly continuous in $H_\sigma{}^m$.

Furthermore, we have

$$Fu_\nu(t) \rightharpoonup Fu_0(t) \text{ in } H_\sigma^{m-1}, \text{ uniformly for } t \in [0, T_0] \tag{3.3}$$

and that $Fu_0(\cdot)$ is weakly continuous in H_σ^{m-1}. To see this, we note that $Fu_\nu(\cdot)$ is strongly continuous and uniformly bounded in H_σ^{m-1} by (1.3) and (2.1). Thus (3.3) follows from

$$(Fu_\nu(t), f)_{m-1} \to (Fu_0(t), f)_{m-1}, \text{ uniformly for } t \in [0, T_0], \tag{3.4}$$

for any sufficiently smooth function $f \in H_\sigma^{m-1}$. (3.4) in turn follows from the convergence $u_\nu(t) \to u_0(t)$ in $(m - 1)$-norm and the identity

$$(Fu, f)_{m-1} = -((u \cdot \operatorname{grad})u, f)_{m-1} = (uu, \operatorname{grad} f)_{m-1}$$

obtained by an integration by parts; note that $H^{m-1}(R^3)$ is a Banach algebra.

We shall now show that u_0 is a solution of $(1.1)_0$. To this end, we integrate (1.1) on $[t', t''] \subset (0, T_0]$, obtaining

$$u_\nu(t'') - u_\nu(t') = \int_{t'}^{t''} (-\nu Au_\nu + Fu_\nu + b)\, dt.$$

Take the $(m-1)$-inner product of this equality with a smooth function $f \in H_\sigma^{m-1}$ and go to the limit $\nu \to 0$. Since (3.4) is true and $\nu(Au_\nu, f)_{m-1} = \nu(u_\nu, Af)_{m-1} \to 0$ uniformly in t, we obtain

$$(u_0(t'') - u_0(t'), f)_{m-1} = \int_{t'}^{t''} (Fu_0 + b, f)_{m-1}\, dt.$$

Hence,

$$u_0(t'') - u_0(t') = \int_{t'}^{t''} (Fu_0 + b)\, dt; \tag{3.5}$$

note that $Fu_0(t) \in H_\sigma^{m-1}$ is weakly continuous in t and $b(t) \in H_\sigma{}^m \subset H_\sigma^{m-1}$ is integrable in t. Furthermore, going to the limit shows that (3.5) is true even if $t' = 0$. Thus

$$u_0 \in AC[0, T_0; H_\sigma^{m-1}] \cap L^\infty[0, T_0; H_\sigma{}^m], \tag{3.6}$$

and u_0 is a solution of $(1.1)_0$.

It is convenient at this point to prove the uniqueness of the solution of $(1.1)_0$ within the class (3.6). Suppose we have two solutions u, v. Then $w = u - v$ satisfies $dw/dt = F(w, v) + F(u, w)$. Hence,

$$\tfrac{1}{2}(d/dt) \| w \|_{m-1}^2 \leqslant (c \| v \|_m + c' \| u \|_m) \| w \|_{m-1}^2$$

by (2.1) and (2.2) or (2.2′). Since $c \| v \|_m + c' \| u \|_m \leqslant \text{const}$ by assumption and $w(0) = 0$, we must have $\| w \|_{m-1} = 0$ and $w = 0$. This proves the uniqueness.

To complete the proof of (ii), it remains to show that $u_0 \in C[0, T_0; H_\sigma{}^m]$. Since $\| u_0(t) \|_m \leqslant \phi(t)$ as shown above and since $\phi(0) = \| a \|_m$, we have $\limsup_{t\to 0} \| u_0(t) \|_m \leqslant \| a \|_m = \| u_0(0) \|_m$. Since u_0 is weakly continuous in $H_\sigma{}^m$, it follows that u_0 is strongly continuous in $H_\sigma{}^m$ at $t = 0$. To prove the same result at any $t_0 \in [0, T_0]$, let v be the solution of the Cauchy problem for $(1.1)_0$ for $t \geqslant t_0$ with the initial value $u_0(t_0)$. By the result just proved, v is continuous in $H_\sigma{}^m$ at $t = t_0$. But u_0 coincides with v for $t \geqslant t_0$ by the uniqueness proved above. Hence u_0 is right-continuous at t_0. Furthermore, since $(1.1)_0$ is reversible in time t (in an obvious sense), u_0 must be left-continuous as well. Thus $u_0 \in C[0, T_0; H_\sigma{}^m]$ as we wished to show.

To prove the assertion of Remark 2, it suffices to note that $Fu_0 \in C[0, T_0; H_\sigma^{m-1}]$ by (2.1) and $u_0 \in C[0, T_0; H_\sigma{}^m]$. Hence du_0/dt belongs to $C[0, T_0; H_\sigma^{m-1}]$ if b does.

4. Proof of (2.1), (2.2), and (2.2′)

In this section we denote by c a constant which may differ at each occurrence.

The proof of (2.1) is easy. Since P has norm one in any $H^m(R^3)^3$ and since $H^m(R^3)$ is a Banach algebra for $m \geqslant 2$, we have

$$\| F(u, v) \|_m \leqslant \| (u \cdot \operatorname{grad}) v \|_m \leqslant c \| u \|_m \| \operatorname{grad} v \|_m \leqslant c \| u \|_m \| v \|_{m+1} .$$

To prove (2.2) and (2.2′), we may assume $u, v \in C_0^\infty(R^3)^3$ in view of (2.1). Then

$$\begin{aligned} (F(u, v), v)_m &= -((u \cdot \operatorname{grad}) v, v)_m \\ &= - \sum_{|\alpha| \leqslant m} (D^\alpha (u \cdot \operatorname{grad}) v, D^\alpha v)_0 , \end{aligned} \tag{4.1}$$

where the D^α are multiindexed derivatives in x and $(\ , \ )_0$ is the L^2-inner product. Application of the Leibniz rule gives

$$D^\alpha (u \cdot \operatorname{grad}) v = (u \cdot \operatorname{grad}) D^\alpha v + \sum_{0 < \beta \leqslant \alpha} c_{\alpha, \beta} (D^\beta u \cdot \operatorname{grad}) D^{\alpha - \beta} v. \tag{4.2}$$

The contribution to (4.1) of the first term of (4.2) is zero, as is seen by an integration by parts using $\operatorname{div} u = 0$. The contribution of each of the remaining terms satisfies (2.2) or (2.2′) separately. This is easily seen by using the Schwarz inequality and the following lemma.

Lemma. *Let $0 < \beta \leqslant \alpha$. Then*

$$\| (D^\beta u \cdot \operatorname{grad}) D^{\alpha - \beta} v \|_0 \leqslant \begin{cases} c \| u \|_3 \| v \|_{|\alpha|} , & | \beta | = 1, 2, \\ c \| u \|_{|\beta|} \| v \|_{|\alpha| - |\beta| + 3} , & | \beta | \geqslant 3. \end{cases} \tag{4.3}$$

Proof. We use the well-known estimates

$$\| fg \|_0 \leqslant c \| f \|_2 \| g \|_0 , \qquad \| fg \|_0 \leqslant c \| f \|_1 \| g \|_1 . \tag{4.4}$$

If $| \beta | = 1$, then we have $\| D^\beta u \|_2 \leqslant \| u \|_3$ and $\| \operatorname{grad} D^{\alpha - \beta} v \|_0 \leqslant \| v \|_{|\alpha|}$. If $| \beta | = 2$, then $\| D^\beta u \|_1 \leqslant \| u \|_3$ and $\| \operatorname{grad} D^{\alpha - \beta} v \|_1 \leqslant \| v \|_{|\alpha|}$. If $| \beta | \geqslant 3$, then $\| D^\beta u \|_0 \leqslant \| u \|_{|\beta|}$ and $\| \operatorname{grad} D^{\alpha - \beta} v \|_2 \leqslant \| v \|_{|\alpha| - |\beta| + 3}$. In each case we have (4.3) by (4.4).

5. Construction of the Local Solution of (1.1)

Since ν is fixed throughout this section, we may assume $\nu = 1$ without loss of generality.

We first construct a local solution of the integral equation

$$u(t) \equiv Gu(t) = e^{-tA}a + \int_0^t e^{-(t-s)A}[Fu(s) + b(s)]\, ds. \tag{5.1}$$

To this end we shall use the method of contraction map.

For simplicity we write $X_j = C[0, T'; H_\sigma{}^j]$, $Y_j = L^1[0, T'; H_\sigma{}^j]$, $j = 1, 2, \ldots$, and set $Z = X_m \cap Y_{m+1}$. Here $T' > 0$ is to be determined later. For the norm in Z we choose

$$\| v \|_Z = \max(K^{-1} \| v \|_{X_m}, L^{-1} \| v \|_{Y_{m+1}}), \tag{5.2}$$

where $K, L > 0$ are also to be determined later.

First we note that $u, v \in Z$ implies $F(u, v) \in Y_m$, with

$$\| F(u, v) \|_{Y_m} \leqslant c \| u \|_{X_m} \| v \|_{Y_{m+1}}; \tag{5.3}$$

this follows directly from (2.1).

Next we compute $Gu - Gv$ for $u, v \in Z$. We have

$$Gu(t) - Gv(t) = \int_0^t e^{-(t-s)A}[F(u - v, u) + F(v, u - v)]\, ds. \tag{5.4}$$

Since e^{-tA} has norm one as an operator in $H_\sigma{}^m$, we have

$$\begin{aligned} \| Gu - Gv \|_{X_m} &\leqslant \| F(u - v, u) \|_{Y_m} + \| F(v, u - v) \|_{Y_m} \\ &\leqslant c \| u - v \|_{X_m} \| u \|_{Y_{m+1}} + c \| v \|_{X_m} \| u - v \|_{Y_{m+1}} \\ &\leqslant cKL(\| u \|_Z + \| v \|_Z) \| u - v \|_Z, \end{aligned} \tag{5.5}$$

where we used (5.3) and (5.2).

On the other hand, e^{-tA} has norm $t^{-1/2}e^t$ as an operator from $H_\sigma{}^m$ to H_σ^{m+1}. It follows from (5.4) that $\| Gu(\cdot) - Gv(\cdot) \|_{m+1}$ is majorized by the convolution of $t^{-1/2}e^t$ and $\| F(u - v, u) \|_m + \| F(v, u - v) \|_m$. Hence

$$\| Gu - Gv \|_{Y_{m+1}} \leqslant 2e^{T'} T'^{1/2} cKL(\| u \|_Z + \| v \|_Z) \| u - v \|_Z. \tag{5.6}$$

We now assume that K and L are related by

$$L = \gamma K \qquad \text{where} \qquad \gamma = 2e^{T'} T'^{1/2}. \tag{5.7}$$

Recalling (5.2) and comparing (5.5) and (5.6), we thus obtain

$$\| Gu - Gv \|_Z \leqslant cL(\| u \|_Z + \| v \|_Z) \| u - v \|_Z . \tag{5.8}$$

A similar (and simpler) computation gives

$$\| G0 \|_Z \leqslant (\| a \|_m + B) \gamma L^{-1}, \qquad B = \int_0^T \| b(t) \|_m \, dt. \tag{5.9}$$

These results show that G maps Z into itself, with

$$\| Gu \|_Z \leqslant (\| a \|_m + B) \gamma L^{-1} + cL \| u \|_Z^2 . \tag{5.10}$$

We want G to map the unit ball S of Z into itself. This is achieved by choosing L so that $(\| a \|_m + B) \gamma L^{-1} + cL = 1$. Such an $L > 0$ exists if

$$4c\gamma(\| a \|_m + B) < 1; \tag{5.11}$$

indeed we can then take

$$L = (2c)^{-1}\{1 - [1 - 4\gamma c(\| a \|_m + B)]^{1/2}\} < (2c)^{-1}. \tag{5.12}$$

Since γ is given by (5.7), condition (5.11) is satisfied by choosing T' sufficiently small.

With such choices of T', L, and K, G maps S into S. At the same time, we see from (5.8) and (5.12) that G is a strict contraction on S. Therefore G has a unique fixed point u in S, which is a solution of the integral equation (5.1).

For this u we have $Fu \in Y_m$ by (5.3). Since $b \in Y_m$, we have $Fu + b \in Y_m$. It follows from the well-known property of e^{-tA} (see, e.g., [2]) that, if we denote by $v(t)$ the integral on the right of (5.1),

$$Av(t) = \int_0^t [A^{1/2} e^{-(t-s)A}][A^{1/2}(Fu + b)] \, ds$$

is in Y_{m-1}; note that $A^{1/2}(Fu + b) \in Y_{m-1}$ by $Fu + b \in Y_m$ and $\| A^{1/2} e^{-tA} \| \leqslant t^{-1/2}$. Thus, $dv/dt = -Av + Fu + b \in Y_{m-1}$ and so $v \in AC[0, T'; H_\sigma^{m-1}]$. On the other hand $e^{-tA}a \in AC[0, T'; H_\sigma^{m-1}]$ too, with

$$\|(d/dt) \, e^{-tA} a \|_{m-1} = \| A e^{-tA} a \|_{m-1} \leqslant t^{-1/2} \| A^{1/2} a \|_{m-1} \leqslant t^{-1/2} \| a \|_m .$$

Thus, u is a solution of (1.1) satisfying (1.3) locally (i.e., with T_0 replaced by T').

The uniqueness of the local solution can be proved by a standard method (cf. [2]).

References

1. H. Swann, The convergence with vanishing viscosity of nonstationary Navier–Stokes flow to ideal flow in R_3, *Trans. Amer. Math. Soc.* **157** (1971), 373–397.
2. T. Kato and H. Fujita, On the nonstationary Navier–Stokes system, *Rend. Sem. Mat. Univ. Padova* **32** (1962), 243–260.
3. H. Fujita and T. Kato, On the Navier–Stokes initial value problem, I, *Arch. Rational Mech. Anal.* **16** (1964), 269–315.
4. T. Kato, Nonlinear evolution equations in Banach spaces, *in* "Proceedings of the Symposium on Applied Mathematics," pp. 50–67, Vol. 17, American Mathematical Society, 1965.
5. D. G. Ebin and J. Marsden, Groups of diffeomorphisms and the motion of an incompressible fluid, *Ann. of Math.* **92** (1970), 102–163.
6. K. K. Golovkin, Vanishing viscosity in the Cauchy problem for equations of hydrodynamics (Russian), *Trudy Mat. Inst. Steklov.* **92** (1966), 31–49.
7. F. J. McGrath, Nonstationary plane flow of viscous and ideal fluids, *Arch. Rational Mech. Anal.* **27** (1968), 329–348.
8. O. A. Ladyzhenskaya, "The Mathematical Theory of Viscous Incompressible Flow," 2nd English ed., Gordon and Breach, New York, 1969.