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The $t\#$-property for integral domains

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Abstract

Generalizing work of Gilmer and Heinzer, we define a $t\#$-domain to be a domain R in which $\bigcap_{M\in\mathscr{M}_1} R_M \neq \bigcap_{M\in\mathscr{M}_2} R_M$ for any two distinct subsets $\mathscr{M}_1$ and $\mathscr{M}_2$ of the set of maximal t-ideals of R. We provide characterizations of these domains, and we show that polynomial rings over $t\#$-domains are again $t\#$-domains. Finally, we study overrings of $t\#$-domains.

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0. Introduction

Let R denote an integral domain with quotient field K. Then R is said to be a *#-domain* or to satisfy the *#-condition* if $\bigcap_{M\in\mathscr{M}_1} R_M \neq \bigcap_{M\in\mathscr{M}_2} R_M$ whenever $\mathscr{M}_1$ and $\mathscr{M}_2$ are distinct subsets of the set of maximal ideals of R. Prüfer domains satisfying the #-condition were first studied in [8,10]. Domains each of whose overrings satisfy the #-condition were also studied in [10] (in the Prüfer case); these domains have come to be called *##-domains*.

Although the papers mentioned above contain very interesting results, those results are essentially restricted to the class of Prüfer domains. This paper represents an effort to extend, by a modification of the definitions, results about the #- and ##-conditions

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to a much wider class of domains. In the Section 1, we introduce the $t\#$-condition: A domain R satisfies the $t\#$-condition if $\bigcap_{M\in\mathscr{M}_1} R_M \neq \bigcap_{M\in\mathscr{M}_2} R_M$ for any two distinct subsets $\mathscr{M}_1, \mathscr{M}_2$ of the set of maximal t-ideals of R. We discuss the extent to which the properties shown in [10] to be equivalent to the #-property carry over to our setting. For example, [10, Theorem 1 (a) $\Leftrightarrow$ (b)] states that the domain R has the #-property if and only if each maximal ideal M of R contains a finitely generated ideal which is contained in no other maximal ideal of R; we show that this result has a natural counterpart in the class of v-coherent domains (which includes all Noetherian domains). (All relevant definitions are given below.) In addition, we show that for *any* domain R, R has the $t\#$-property if and only if each maximal t-ideal M of R contains a *divisorial* ideal contained in no other maximal t-ideal of R. We also give examples to show that “divisorial” cannot be replaced by “finitely generated” in general.

In Section 2, we attempt to generalize the ##-property. In the case of Prüfer domains, the definition of the ##-property is reasonable since the overrings have nice properties (e.g., they are flat). To obtain a useful definition of the $t\#\#$-property for more general classes of rings, however, one must decide which overrings should be required to have the $t\#$-property. For example, we could say that R has the $t\#\#$-property if each t-linked overring of R has the $t\#$-property. Another possibility is to require that the overrings of R which are generalized rings of quotients of R should have the $t\#$-property. In the end we avoid making a definition at all. Instead, we explore several classes of overrings, primarily in the context of v-coherent domains, and we obtain quite satisfactory results for Prüfer v-multiplication domains.

Section 3 is devoted to a study of the $t\#$-property for polynomial rings. We show that if R has the $t\#$-property, then so does $R[\{X_\alpha\}]$ and that the converse is true if $R[\{X_\alpha\}]$ is assumed to be v-coherent. We also consider the $t\#$-property in two commonly studied localizations of $R[\{X_\alpha\}]$.

1. The *t*#-property

For a nonzero fractional ideal I of a domain R with quotient field K, we set $I^{-1} = (R :_K I) = \{x \in K \mid xI \subseteq R\}$, $I_v = (I^{-1})^{-1}$, and $I_t = \bigcup J_v$, where the union is taken over all nonzero finitely generated subideals J of I. The reader is referred to [9] for properties of these (and other) star operations. We also recall that I is said to be *divisorial* if $I = I_v$ and to be a *t-ideal* if $I = I_t$. Finally, we denote the set of maximal t-ideals of R by t-Max(R).

We begin by repeating the definition of the $t\#$-property.

Definition 1.1. A domain R has the *$t\#$-property* (or is a *$t\#$-domain*) if $\bigcap_{M\in\mathscr{M}_1} R_M \neq \bigcap_{M\in\mathscr{M}_2} R_M$ for any two distinct subsets $\mathscr{M}_1$ and $\mathscr{M}_2$ of t-Max(R).

Theorem 1.2. *The following statements are equivalent for a domain R:*

(1) *R is a $t\#$-domain.*
(2) *For each $N \in t$-Max(R), we have $\bigcap_{M\in t\text{-Max}(R)\setminus\{N\}} R_M \nsubseteq R_N$.*
(3) *For each $N \in t$-Max(R), we have $R \neq \bigcap_{M\in t\text{-Max}(R)\setminus\{N\}} R_M$.*

(4) $\bigcap_{M\in\mathcal{M}_1} R_M$ and $\bigcap_{M\in\mathcal{M}_2} R_M$ *are incomparable for each pair of disjoint subsets* $\mathcal{M}_1$ *and* $\mathcal{M}_2$ *of* t-Max(R).

(5) *For each maximal t-ideal M of R, there is a divisorial ideal of R which is contained in M and no other maximal t-ideal of R.*

(6) *For each maximal t-ideal M of R, there is an element* $u\in K\setminus R$ *such that M is the only maximal t-ideal containing* $(R :_R u)$.

Proof. By [11, Proposition 4] we have $R=\bigcap_{M\in t\text{-Max}(R)} R_M$. Using this and the definitions, the following implications are straightforward: (1) ⇔ (2), (2) ⇔ (4); (2) ⇔ (3); and (6) ⇒ (5).

To prove (2)⇔(6), observe that an element $u\in K$ satisfies $u\in\bigcap_{M\in t\text{-Max}(R)\setminus\{N\}} R_M\setminus R_N$ if and only if $(R :_R u)$ is contained in N and no other maximal t-ideal of R.

Now assume (5). Let $N\in t$-Max(R), and pick a divisorial ideal I with $I\subseteq N$ and $I\nsubseteq M$ for each $M\in t\text{-Max}(R)\setminus\{N\}$. Then $I^{-1}\subseteq\bigcap_{M\in t\text{-Max}(R)\setminus\{N\}} R_M$ (since for each M, we have $(R :_R I^{-1})=I\nsubseteq M)$, but $I^{-1}\nsubseteq R$. Hence (5) ⇒ (3), and the proof is complete. □

Corollary 1.3. *If R is a domain with the property that each maximal t-ideal is divisorial, then R is a t#-domain.*

Proof. This is clear from the equivalence of conditions (1) and (5) of Theorem 1.2. □

Recall that a *Mori domain* is a domain satisfying the ascending chain condition on (integral) divisorial ideals. Equivalently, a domain R is a Mori domain if for each ideal I of R there is a finitely generated ideal $J\subseteq I$ with $I_v=J_v$. In particular, the v- and t-operations on a Mori domain are the same. Hence Corollary 1.3 implies that Mori domains are t#-domains.

Remark 1.4. We have stated Theorem 1.2 for the t-operation, since our primary interest is in that particular star operation. However, suppose that for a finite-type star operation $*$, we call a domain R a $*$#-*domain* if for each pair of nonempty subsets $\mathcal{M}_1$ and $\mathcal{M}_2$ of $*$-Max(R) with $\mathcal{M}_1\neq\mathcal{M}_2$, we have $\bigcap_{M\in\mathcal{M}_1} R_M\neq\bigcap_{M\in\mathcal{M}_2} R_M$. Then Theorem 1.2 remains true with t replaced everywhere (including the proof) by $*$. This is of some interest even in the case where $*$ is the trivial star operation ($I^*=I$ for each ideal I; this is often referred to as the d-operation). For the trivial star operation, Olberding has proved the equivalence of statements (1) and (5) in [20, Proposition 2.2].

Theorem 1.2 (1) ⇔ (6) generalizes [10, Theorem 1 (a) ⇔ (b)], which states that a Prüfer domain R is a #-domain if and only if each maximal ideal of R contains a finitely generated ideal which is contained in no other maximal ideal of R. This follows upon recalling that for R Prüfer (i) each ideal is a t-ideal (so that t-Max$(R)=$Max(R)) and (ii) for each $u\in K$, $(R :_R u)$ is finitely generated (in fact, two generated). In general, one cannot hope to show that each maximal t-ideal of a t#-domain contains a finitely generated ideal which is contained in no other maximal t-ideal, as the following example shows. (Example 1.7 is another such example. However, that example has (Krull) dimension two, and we think it might be of some interest to have a one-dimensional example.)

Example 1.5. Let T be an almost Dedekind domain with exactly one noninvertible maximal ideal M. (One such example is constructed in [9, Example 42.6].) By [8, Theorem 3], T is not a #-domain. For our purposes, it does no harm to assume that T/M has a proper subfield. This follows from the fact that $T(X) = T[X]_S$, where S is the multiplicatively closed subset of $T[X]$ consisting of those polynomials g having unit content (the ideal generated by the coefficients of g), is also an almost Dedekind domain with exactly one nonivertible maximal ideal, namely $MT(X)$, whose residue field $T(X)/MT(X) \approx (T/M)(X)$ has infinitely many proper subfields [9, Proposition 36.7]. Let F be such a proper subfield of T/M, and let R be defined by the following pullback diagram of canonical homomorphisms:

$$\begin{array}{ccc} R & \longrightarrow & F \\ \downarrow & & \downarrow \\ T & \longrightarrow & k = T/M. \end{array}$$

We claim that R is a t#-domain. (In fact, since R is one dimensional, it is a #-domain.) We show that R satisfies condition (5) of Theorem 1.2. For this it suffices to observe that each maximal ideal of R is divisorial. This is clear for M, and if P is a maximal ideal of R with $P \neq M$, then by [7, Theorem 2.35] P is actually invertible. Hence R is a (t)#-domain. Since T is a non-t#-Prüfer domain with offending maximal ideal M, however, there is no finitely generated ideal of T contained in M but no other maximal ideal of T; clearly, a similar statement applies to R.

If we restrict our attention to domains in which conductors are required to be finitely generated, i.e., to *finite conductor domains*, then the t#-property becomes equivalent to the property that each maximal t-ideal contain a finitely generated ideal contained in no other maximal t-ideal. In fact, we can obtain such a result by requiring a little less than finite generation of conductors. Recall that a domain R is said to be *v-coherent* if for each finitely generated ideal I of R, I^{-1} has finite type (i.e., there is a finitely generated ideal J with $I^{-1} = J_v$). This condition was first studied (under a different name) by Nour el Abidine [19]. It is easy to see that a finite conductor domain is v-coherent. We have the following result.

Theorem 1.6. *For a v-coherent domain, the conditions of Theorem* 1.2 *are each equivalent to: Each maximal t-ideal of R contains a finitely generated ideal which is contained in no other maximal t-ideal of R.*

Proof. The stated condition clearly implies condition (5) of Theorem 1.2. On the other hand, condition (6), in the presence of v-coherence, implies the stated condition. □

Now [10, Theorem 1] contains a third equivalence, namely, that R is uniquely representable as an intersection of a family $\{V_\alpha\}$ of valuation overrings such that there are no containment relations among the V_α. Since each valuation overring of a Prüfer domain is a localization, this suggests exploring the possibility that the $t\#$-property on a domain R is equivalent to the condition that R contain a unique set of incomparable t-primes $\{P_\alpha\}$ such that $R=\bigcap R_{P_\alpha}$. One implication is easy. If we assume the existence of a unique set of t-primes $\{P_\alpha\}$ such that $R=\bigcap R_{P_\alpha}$, then that set must be t-Max(R), and so R is a $t\#$-domain. In Theorem 1.8, we provide a converse in two cases. First, we give an example showing that the converse does not hold in general. Recall that a domain R is a *Prüfer v-multiplication domain* (PVMD) if R_M is a valuation domain for each maximal t-ideal M of R.

Example 1.7. In [13] Heinzer and Ohm give an example of an essential domain D which is not a PVMD. In their example k is a field, and $y,z,x_1,x_2,\ldots$ are independent indeterminants over k; $R=k(x_1,x_2,\ldots)[y,z]_{(y,z)}$; for each i, V_i is a rank one discrete valuation ring containing $k(\{x_j\}_{j\neq i})$ such that y, z, and x_i all have value 1; and $D=R\cap(\bigcap_i V_i)$. Then D is a two-dimensional domain, and in [18] it is shown that the maximal ideals of D are $M,P_1,P_2,\ldots$, where M is the contraction of the maximal ideal of $R=D_M$, and P_i is the contraction of the maximal ideal of $V_i=D_{P_i}$. Note that each P_i has height one and is therefore a t-ideal. We observe that each element of R is also in V_i for all but finitely many i; this is the case since an element of R involves only finitely many of the x_j, and x_i is a unit of V_j for all $j\neq i$. Similarly, each element of the maximal ideal of R is in the maximal ideal of V_i for all but finitely many i. It follows that if I is a finitely generated ideal of D contained in M, then I, and hence also I_t, is contained in all but finitely many of the P_i. Suppose that for such an I we have $I_t\nsubseteq M$. Write $1=x+m$ with $x\in I_t$ and $m\in M$. By the observations stated above, x and m must be simultaneously in all but finitely many of the P_i, a contradiction. Thus M is a t-ideal.[2] We show that R has the $t\#$-property by showing that M and the P_i satisfy condition (5) of Theorem 1.2. For each i, the divisorial ideal $x_iD\subseteq P_i$, while $x_i\notin M$ and $x_i\notin P_j$ for $j\neq i$. As for M, note that $y/z\in\bigcap D_{P_i}\setminus D_M$. Hence $(D:_D y/z)\subseteq M$, but $(D:_D y/z)\nsubseteq P_i$ for $i=1,2,\ldots$. Finally, denoting the height-one primes contained in M by $\{Q_\alpha\}$, we observe that $D=(\bigcap_\alpha D_{Q_\alpha})\cap(\bigcap_i D_{P_i})$ for the set of incomparable t-primes $\{Q_\alpha\}\cup\{P_i\}\neq t$-Max$(R)$.

In our next result, we use the fact that a PVMD is v-coherent [19].

Theorem 1.8. *Let R be either a PVMD or a Mori domain. Then the conditions of Theorem* 1.2 *are each equivalent to*: *There is a unique set $\{P_\alpha\}$ of incomparable t-primes such that $R=\bigcap R_{P_\alpha}$. In particular*, *a Mori domain has this property*.

Proof. One implication was discussed above. Assume that R is a $t\#$-domain, and suppose that $R=\bigcap R_{P_\alpha}$ for some set $\{P_\alpha\}$ of incomparable t-primes. Observe that the

[2] We observe that since $D_M=R$ is not a valuation domain, this yields an easy way to see that D is not a PVMD.

hypotheses guarantee that R is v-coherent. It suffices to show that each P_α is a maximal t-ideal. By way of contradiction, suppose that $P_\beta \subsetneq M$, where M is a maximal t-ideal. If R is a PVMD, then (since the P_α are incomparable and R_M is a valuation domain), $P_\alpha \nsubseteq M$ for each $\alpha \neq \beta$. By Theorem 1.6, M contains a finitely generated ideal I which is contained in M and no other maximal t-ideal. In particular, $I \nsubseteq P_\alpha$ for $\alpha \neq \beta$. Pick $a \in M \setminus P_\beta$. Then (I, a) is a finitely generated ideal contained in no P_α whatsoever. It follows that $(I, a)^{-1} \subseteq \bigcap R_{P_\alpha} = R$, whence $(I, a)_v = R$. However, since M is a t-ideal, we have $(I, a)_v \subseteq M$, a contradiction in this case. If R is Mori, then M itself is divisorial, and, since M is contained in no P_α, we obtain the contradiction that $M^{-1} \subseteq \bigcap R_{P_\alpha}$. □

Remark 1.9. We have not been able to determine whether weakening the hypothesis of Theorem 1.8 to v-coherent is sufficient. It does suffice if we make the following subtle change to the condition: there is a unique set $\{P_\alpha\}$ of t-primes such that both $R = \bigcap R_{P_\alpha}$ and the intersection is irredundant (no R_{P_α} can be deleted). To see this, suppose that R is a t#-domain, and let $\{P_\alpha\}$ be as indicated. Pick a P_β; we wish to show that it is a maximal t-ideal. The irredundancy hypothesis allows us to choose $u \in R_{P_\beta} \setminus \bigcap_{\alpha \neq \beta} R_{P_\alpha}$. We have $(R :_R u) \nsubseteq P_\beta$ and $(R :_R u) \subseteq P_\alpha$ for each $\alpha \neq \beta$. Since R is v-coherent, there is a finitely generated ideal I with $(R :_R u) = I_v$. Pick a maximal t-ideal $M \supseteq P_\beta$. If there is an element $a \in M \setminus P_\beta$, then, as in the proof of Theorem 1.8, the ideal (I, a) will furnish a contradiction.

2. Overrings of t#-domains

In [10] Gilmer and Heinzer also studied Prüfer domains with the property that each overring is a #-domain; these domains have come to be called *##-domains*. Our goal in this section is to obtain t-analogues of results on ##-domains.

Most of the characterizations of Prüfer ##-domains in [10] can be extended to PVMDs with the property that each t-linked overring is t#. However, if we want to consider a larger class of domains, e.g., v-coherent domains, the question arises as to which overrings should be considered. Put another way, it is not clear exactly how one should define the t##-property (and we shall not do so).

In what follows, it will be convenient to employ the language of localizing systems. We recall the requisite definitions. A nonempty set $\mathscr{F}$ of nonzero ideals of R is said to be a *multiplicative system of ideals* if $IJ \in \mathscr{F}$ for each $I, J \in \mathscr{F}$. The ring $R_{\mathscr{F}} = \{x \in K \mid xI \subseteq R \text{ for some } I \in \mathscr{F}\}$ is called a *generalized ring of quotients* of R. For each ideal J of R we set $J_{\mathscr{F}} = \{x \in K \mid xI \subseteq J \text{ for some } I \in \mathscr{F}\}$; $J_{\mathscr{F}}$ is an ideal of $R_{\mathscr{F}}$ containing $JR_{\mathscr{F}}$.

A particular type of multiplicative system is a *localizing system*: this is a set $\mathscr{F}$ of ideals of R such that (1) if $I \in \mathscr{F}$ and J is an ideal of R with $I \subseteq J$, then $J \in \mathscr{F}$ and (2) if $I \in \mathscr{F}$ and J is an ideal of R such that $(J :_R a) \in \mathscr{F}$ for every $a \in I$, then $J \in \mathscr{F}$. If Λ is a subset of $\operatorname{Spec} R$, then $\mathscr{F}(\Lambda) = \{I \mid \text{ is an ideal of } R \text{ such that } I \nsubseteq P \text{ for each } P \in \Lambda\}$ is a localizing system; moreover, $R_{\mathscr{F}(\Lambda)} = \bigcap_{P \in \Lambda} R_P$. A localizing system $\mathscr{F}$ is said to be *spectral* if $\mathscr{F} = \mathscr{F}(\Lambda)$ for some set of primes Λ. Finally, an *irredundant spectral localizing system* is a localizing system of ideals $\mathscr{F}(\Lambda)$, where Λ is a set of pairwise incomparable primes.

These notions have t-analogues. A set of t-ideals is a *t-multiplicative system* if it is closed under t-multiplication; a t-multiplicative system Φ is a *t-localizing system* if it satisfies the closure operations (1) and (2) above.

The localizing system $\mathscr{F}$ is said to be of *finite type* if for each $I \in \mathscr{F}$ there is a finitely generated ideal $J \in \mathscr{F}$ with $J \subseteq I$. Also, $\mathscr{F}$ is said to be *v-finite* if each t-ideal of $\mathscr{F}$ contains a v-finite ideal which is also in $\mathscr{F}$.

Denoting the set of t-ideals of R by $t(R)$, it is easy to see that if $\mathscr{F}$ is a localizing system, then $\Phi = \mathscr{F} \cap t(R)$ is a t-localizing system, $R_{\mathscr{F}} = R_\Phi$, and $\mathscr{F}$ is v-finite if and only if Φ is v-finite. Conversely, if Φ is a t-localizing system of t-ideals, then $\bar{\Phi} = \{I \mid I_t \in \Phi\}$ is a localizing system of ideals with $\Phi = \bar{\Phi} \cap t(R)$.

Let Λ be a set of pairwise incomparable t-primes. With $\mathscr{F}(\Lambda)$ as above and $\Phi(\Lambda) = \mathscr{F}(\Lambda) \cap t(R) (= \{I \mid I$ is a t-ideal and $I \nsubseteq P$ for all $P \in \Lambda\})$, we have that $I \in \mathscr{F}(\Lambda)$ if and only if $I_t \in \Phi(\Lambda)$. Hence $\overline{\Phi(\Lambda)} = \mathscr{F}(\Lambda)$.

An overring T of R is a *t-subintersection of* R if it has the form $\bigcap R_P$, where the intersection is taken over some set of t-primes P of R, i.e., if $T = R_{\Phi(\Lambda)}$ for some spectral t-localizing system $\Phi(\Lambda)$ of R, where Λ is a set of t-primes. We say that T is *t-flat over* R if $T_M = R_{M \cap R}$ for each maximal t-ideal M of T [17]. Finally, recall that T is t-linked over R if for each finitely generated ideal I of R with $(R : I) = R$ we have $(T : IT) = T$ [1].

The following implications are easily verified: T is t-flat over $R \Rightarrow T$ is a t-subintersection of $R \Rightarrow T$ is a generalized ring of quotients of $R \Rightarrow T$ is t-linked over R.

All these conditions are equivalent for PVMDs [17, Proposition 2.10], but we believe that in general none of the arrows can be reversed if R is merely assumed to be v-coherent. Also, if R is a PVMD, then every t-linked overring of R is a PVMD [16, Theorem 3.8 and Corollary 3.9], but if R is just v-coherent, we know only that generalized rings of quotients of R are v-coherent [6, Proposition 3.1].

We shall begin by considering t-flat overrings of v-coherent domains. Recall that, for any domain R, an overring T of R is t-flat over R if and only if T is a generalized ring of quotients with respect to a v-finite t-localizing system of ideals [2, Theorem 2.6].

On the other hand, we know that if R is Prüfer then every overring is flat, and we also know that R is a ##-domain if each irredundant spectral localizing system is finitely generated [5]. We shall show that for v-coherent domains the property that each irredundant spectral t-localizing system is v-finite is equivalent to the property that each t-subintersection of R is t-flat and t#.

Lemma 2.1. *Let R be a v-coherent domain and Φ a t-localizing system of t-ideals. Then the following statements are equivalent.*

(1) *Φ is v-finite.*
(2) *The set Λ of maximal elements of $t\text{-Spec}(R) \setminus \Phi$ is not empty, and $M \in t\text{-Max}(R_\Phi)$ if and only if $M = P_\Phi$ for some $P \in \Lambda$.*

Under these conditions, $\Phi=\Phi(\Lambda)$. *In particular, if* Λ *is a set of pairwise incomparable* t*-primes of* R, *then* $\Phi=\Phi(\Lambda)$ *is* v*-finite if and only if* $t\text{-Max}(R_\Phi)=\{P_\Phi \mid P\in\Lambda\}$.

Proof. Set $\mathscr{F}=\bar{\Phi}=\{I \mid I_t\in\Phi\}$ and use (i) $\Leftrightarrow$ (vi) of [6, Theorem 3.3]. □

Proposition 2.2. *Let* R *be a* v*-coherent domain. If* Λ *and* Λ' *are two sets of pairwise incomparable* t*-primes such that* $\Phi(\Lambda)$ *and* $\Phi(\Lambda')$ *are* v*-finite and* $R_{\Phi(\Lambda)}=R_{\Phi(\Lambda')}$, *then* $\Lambda=\Lambda'$.

Proof. By Lemma 2.1, we have $t\text{-Max}(T)=\{P_\Phi \mid P\in\Lambda\}=\{Q_{\Phi'} \mid Q\in\Lambda'\}$ and, upon contracting to R, we obtain $\Lambda=\Lambda'$. □

Recalling that an overring T of a domain R is t-flat over R if and only if $T=R_\Phi$ for some v-finite t-localizing system Φ, the preceding two results immediately imply:

Corollary 2.3. *Let* R *be a* v*-coherent domain and let* T *be a* t*-flat overring of* R. *Then there exists a uniquely determined set* Λ *of pairwise incomparable* t*-primes for which* $T=R_{\Phi(\Lambda)}$ *and* $\Phi(\Lambda)$ *is* v*-finite. The set* Λ *is given by* $\Lambda=\{M\cap R \mid M\in t\text{-Max}(T)\}$.

Proposition 2.4. *Let* R *be a* v*-coherent domain. Then the following statements are equivalent.*

(1) *For each set* Λ *of pairwise incomparable* t*-primes of* R, $\Phi(\Lambda)$ *is* v*-finite.*
(2) *If* Λ *and* Λ' *are two sets of pairwise incomparable* t*-primes of* R *such that* $R_{\Phi(\Lambda)}=R_{\Phi(\Lambda')}$, *then* $\Lambda=\Lambda'$.
(3) *If* T *is a* t*-subintersection of* R *and is represented as* $T=\bigcap_{P\in\Lambda}R_P$ *for some set* Λ *of pairwise incomparable* t*-primes, then that representation is irredundant.*
(4) *For each* t*-prime* P *and each set* Λ *of pairwise incomparable* t*-primes of* R *not containing* P, *there exists an element* $u\in K$ *such that* $(R:_R u)\subseteq P$ *and* $(R:_R u)\nsubseteq Q$, *for each* $Q\in\Lambda$.
(5) *For each* t*-prime* P *and each set* Λ *of pairwise incomparable* t*-primes of* R *not containing* P, *there exists a finitely generated ideal* J *of* R *such that* $J\subseteq P$ *and* $J\nsubseteq Q$ *for each* $Q\in\Lambda$.
(6) *For each* t*-prime* P *and each set* Λ *of pairwise incomparable* t*-primes of* R *not containing* P, $R_P\nsupseteq R_{\Phi(\Lambda)}$.
(7) *For each set* Λ *of pairwise incomparable* t*-primes of* R, $R_{\Phi(\Lambda)}$ *is* t*-flat over* R *and has the* $t\#$*-property.*

Proof. (1) $\Rightarrow$ (2) by Proposition 2.2.

(2) $\Rightarrow$ (3) is clear.

(3) $\Rightarrow$ (1): Given a set Λ of incomparable primes, consider the t-subintersection $T=R_{\Phi(\Lambda)}$ of R. Since T is v-coherent [6, Proposition 3.1] and the intersection is irredundant, we obtain $t\text{-Max}(T)=\{P_{\Phi(\Lambda)} \mid P\in\Lambda\}$ as in Remark 1.9. It follows that $\Phi(\Lambda)$ is v-finite (Lemma 2.1).

(1) $\Rightarrow$ (7): Let Λ be a set of pairwise incomparable t-primes of R and $T = R_{\Phi(\Lambda)}$. Since $\Phi(\Lambda)$ is v-finite, then T is t-flat over R, and $\Lambda = \{M \cap R \mid M \in t\text{-Max}(R)\}$ is uniquely determined by Corollary 2.3. Hence we cannot delete any $P \in \Lambda$, and so the intersection is irredundant. In addition, by t-flatness, $T_M = R_{M\cap R}$; hence T is a t#-domain.

(7) $\Rightarrow$ (3): If the t-subintersection $T = R_{\Phi(\Lambda)}$ of R is t-flat, then $\Lambda = \{M\cap R \mid M \in t\text{-Max}(R)\}$ by Corollary 2.3. If T is also a t#-domain, then $T = \bigcap T_M = \bigcap R_{M\cap R}$ is an irredundant t-subintersection.

(2) $\Rightarrow$ (4): Given Λ and P as specified, set $\Lambda' = (\Lambda \setminus \{Q \in \Lambda \mid Q \subseteq P\}) \cup \{P\}$. Then $\Lambda \neq \Lambda'$, so that $R_{\Phi(\Lambda)} \neq R_{\Phi(\Lambda')}$ by (2). Since we clearly have $R_{\Phi(\Lambda')} \subseteq R_{\Phi(\Lambda)}$, there is an element $u \in R_{\Phi(\Lambda)} \setminus R_{\Phi(\Lambda')}$, and for this u we have $(R :_R u) \subseteq P$ and $(R :_R u) \nsubseteq Q$ for each $Q \in \Lambda$.

(4) $\Rightarrow$ (5): Since R is v-coherent, then the ideal $(R :_R u)$ contains a finitely generated subideal J with $J_v = (R :_R u)$; this J does what is required.

(5) $\Rightarrow$ (6): Given J as indicated, one shows easily that $(R : J) \subseteq R_{\Phi(\Lambda)}$ but $(R : J) \nsubseteq R_P$, whence $R_P \nsupseteq R_{\Phi(\Lambda)}$.

(6) $\Rightarrow$ (2): Suppose that Λ and Λ' are two sets of pairwise incomparable primes for which $R_{\Phi(\Lambda)} = R_{\Phi(\Lambda')}$ but $\Lambda \neq \Lambda'$. We may then assume that there is a prime $P \in \Lambda \setminus \Lambda'$. If $P \nsubseteq Q$ for all $Q \in \Lambda'$, then (6) yields $R_P \nsupseteq R_{\Phi(\Lambda')} = R_{\Phi(\Lambda)}$, a contradiction. We then denote by Λ'' the maximal elements in the set $(\Lambda \cup \{Q \in \Lambda' \mid P \subseteq Q\}) \setminus \{P\}$. (Choosing the maximal elements is possible since both Λ and Λ' contain pairwise incomparable elements.) Pick $Q_0 \in \Lambda'$ with $P \subseteq Q_0$. Then $Q_0 \in \Lambda''$, and we have $R_P \supseteq R_Q \supseteq R_{\Phi(\Lambda'')}$, which contradicts (6). □

Remark 2.5. The equivalent conditions of Proposition 2.4 hold automatically for a Mori domain, since in such a domain each t-ideal is v-finite.

Proposition 2.6. *Let R be a v-coherent domain such that each t-subintersection of R is t-flat over R. Then the following statements are equivalent*:

1. *Each t-flat overring of R is a t#-domain.*
2. *Each t-subintersection of R is a t#-domain.*
3. *For each set Λ of pairwise incomparable t-primes of R, $\Phi(\Lambda)$ is v-finite.*
4. *If T is a t-subintersection of R, there exists a unique set of pairwise incomparable t-primes Λ of R such that $T = R_{\Phi(\Lambda)}$; moreover, $\Lambda = \{M \cap R \mid M \in t\text{-Max}(T)\}$.*
5. *If T is a t-flat overring of R and $T = \bigcap_{Q\in\Lambda} T_Q$ for some set Λ of pairwise incomparable t-primes of T, then $\Lambda = t\text{-Max}(T)$.*

Proof. (1) $\Leftrightarrow$ (2) and (5) $\Rightarrow$ (1) are clear.

(2) $\Leftrightarrow$ (3) by Proposition 2.4.

(3) $\Rightarrow$ (4) by Corollary 2.3.

(4) $\Rightarrow$ (5): Assume that T is a t-flat overring of R and that we have $T = \bigcap_{Q\in\Lambda} T_Q$, where Λ is a set of pairwise incomparable t-primes of T. By t-flatness, $T_Q = R_{Q\cap R}$ for each $Q \in \Lambda$. Hence $T = R_{\Phi(\Gamma)}$, where $\Gamma = \{Q \cap R \mid Q \in \Lambda\}$. We then have $\Lambda = t\text{-Max}(T)$ by (4) (and t-flatness). □

If R is a Mori domain, then, as mentioned in Remark 2.5, the equivalent conditions of Proposition 2.4 hold. It then follows from [2, Theorem 2.6] that each t-subintersection of R is t-flat; hence the equivalent conditions of Proposition 2.6 hold also.

For a PVMD, t-subintersections are automatically t-flat; in fact, t-linked overrings are t-flat by [17, Proposition 2.10]. Thus the hypotheses of Proposition 2.6 hold for PVMDs. Our next proposition adds several more equivalences for PVMDs. We need the following lemma.

Lemma 2.7. *Let R be a PVMD and let P be a t-prime of R which is not t-invertible. Then* $(P:P)=(R:P)=R_P\cap S$, *where* $S=\bigcap_{M\in t\text{-Max}(R), M\not\supseteq P} R_M$.

Proof. By [15, Proposition 2.3, Lemma 1.2], $(R:P)=(P:P)$. The result now follows from [14, Theorem 4.5]. □

Proposition 2.8. *For a PVMD R, the conditions of Proposition* 2.6 *are also equivalent to each of the following*:

(6) *If $\Lambda\subseteq t$-Max(R), then $\Phi(\Lambda)$ is v-finite.*
(7) *Each t-prime ideal P of R contains a finitely generated ideal which is not contained in any maximal t-ideal of R not containing P.*
(8) *For each t-prime P of R, there exists an element $u\in K$ such that $(R:_R u)\subseteq P$ and $(R:_R u)\not\subseteq M$, for each maximal t-ideal M not containing P.*
(9) *For each t-prime ideal P of R, $R_P\not\supseteq\bigcap R_M$, where M ranges over the set of maximal t-ideals not containing P.*
(10) *Each t-linked overring of R is a t#-domain.*
(11) *$(P:P)$ is a t#-domain for each t-prime P of R.*

Proof. (3) ⇒ (6) is clear.

(6) ⇒ (7): If Λ is the set of maximal t-ideals not containing P, then $P\in\Phi(\Lambda)$ and $\Phi(\Lambda)$ is v-finite.

(7) ⇒ (1): Let T be a t-subintersection of R. Then T is t-flat over R, and we have $T=\bigcap_{M\in t\text{-Max}(T)} R_{M\cap R}$. Fix $N\in t$-Max(T) and let J be a finitely generated ideal of R contained in $P=N\cap R$ and not contained in the maximal t-ideals of R not containing P. Since in a PVMD two incomparable t-primes are t-comaximal, then J is not contained in $M\cap R$ for each maximal t-ideal $M\neq N$ of T. It follows that JT is a finitely generated ideal contained in N and not contained in M for $M\neq N$. We conclude by applying Theorem 1.6.

(3) ⇒ (8) by Proposition 2.4.

(8) ⇒ (7) by v-coherence.

(8) ⇔ (9) because, for each prime P and $u\in K$, $(R{:_R}u)\subseteq P$ iff $u\notin R_P$.

(1) ⇔ (10) because each t-linked overring of a PVMD is t-flat [17, Proposition 2.10].

(11) ⇒ (9): Let $T=(P:P)$. If P is t-invertible then $R=T$. Otherwise, $T=(R:P)=R_P\cap(\cap R_{M_\alpha})$, where M_α ranges over the set of maximal t-ideals of R not containing P (Lemma 2.7). In either case, setting $\Lambda=\{P\}\cup\{M_\alpha\}$, we have that $T=R_{\Phi(\Lambda)}$. Since R is

v-coherent, the set of ideals $\{Q_{\Phi(\Lambda)} = QR_Q \cap T; Q \in \Lambda\}$ is a set of incomparable t-primes of T [6, Proposition 3.2]. For each $Q \in \Lambda$, we have $R_Q = T_{Q_{\Phi(\Lambda)}}$ and by hypothesis T is a $t\#$-domain. Hence by Theorem 1.6 $R_{\Phi(\Lambda)}$ is an irredundant intersection. It follows that $R_P \not\supseteq \cap R_{M_\alpha}$.

(10) $\Rightarrow$ (11): According to [1, Proposition 2.2(5)], $(A_v : A_v)$ is t-linked over R for each ideal A of R. In fact, it is easy to see that replacing "v" by "t" in the proof of that result shows that $(A_t : A_t)$ is t-linked. In particular, if P is a t-prime of R, then $(P : P)$ is t-linked. □

Comparing conditions (3) and (6) of Propositions 2.6 and 2.8, we observe that for PVMDs one has to consider only subsets of t-Max(R) rather than all sets of incomparable t-primes.

The equivalence of conditions (7) and (8) above is also proved in [3, Lemma 3.6]. The equivalence of conditions (10) and (11) for Prüfer domains is [20, Proposition 2.5].

When R is Prüfer, Proposition 2.8 recovers [5, Theorem 2.4]. In [5, Theorem 2.5] it is also proved that for Prüfer domains the ##-condition is equivalent to the $\#_P$-condition introduced by Popescu in [21]. We recall that R is a $\#_P$-domain if, given two sets of prime ideals $\Lambda_1 \neq \Lambda_2$ with the property that $P + Q = R$ for each pair of distinct ideals $P \in \Lambda_1$ and $Q \in \Lambda_2$, we have $R_{\Phi(\Lambda_1)} \neq R_{\Phi(\Lambda_2)}$.

We can define the $t\#_P$-property analogously: R is a *$t\#_P$-domain* if, given two sets of prime t-ideals $\Lambda_1 \neq \Lambda_2$ with the property that $(P + Q)_t = R$ for each pair of distinct ideals $P \in \Lambda_1$ and $Q \in \Lambda_2$, we have $R_{\Phi(\Lambda_1)} \neq R_{\Phi(\Lambda_2)}$.

We will show that, with this definition, [5, Theorem 2.5] can be extended to PVMDs. Recall that, if R is a PVMD, then for any two incomparable prime t-ideals P and Q we have $(P + Q)_t = R$ (since R_M is a valuation domain for each maximal t-ideal M of R).

Proposition 2.9. *Let R be a v-coherent domain, and assume that the equivalent conditions of Proposition* 2.4 *are satisfied. Then R is a $t\#_P$-domain.*

Proof. Let $\Lambda_1 \neq \Lambda_2$ be two sets of prime t-ideals of R with the property that $(P+Q)_t = R$ for each pair of distinct ideals $P \in \Lambda_1$ and $Q \in \Lambda_2$, and let $P \in \Lambda_1 \setminus \Lambda_2$. Since $(P+Q)_t = R$ for $Q \in \Lambda_2$, we have $(P+M)_t = R$ for each M in the set $\Gamma = \{N \in t\text{-Max}(R) \mid Q \subseteq N \text{ for some } Q \in \Lambda_2\}$. Since Γ is a set of incomparable t-primes not containing P, we may apply Proposition 2.4 (4) to obtain an element $u \in K$ such that $(R :_R u) \subseteq P$ but $(R :_R u) \not\subseteq M$ for each $M \in \Gamma$. It is then easy to see that $u \in R_{\Phi(\Lambda_2)} \setminus R_{\Phi(\Lambda_1)}$. □

Our next result shows that for PVMDs the $t\#_P$-condition is equivalent to the conditions of Propositions 2.6 and 2.8.

Proposition 2.10. *Let R be a PVMD. Then R is a $t\#_P$-domain if and only if each t-linked overring of R is a $t\#$-domain.*

Proof. In a PVMD any two incomparable t-primes are t-comaximal. Hence if R is a $t\#_P$-domain, then R must satisfy condition (3) of Proposition 2.4. The fact that conditions (4) of Proposition 2.6 and (10) of Proposition 2.8 are equivalent then shows that each t-linked overring of R is a $t\#$-domain. The converse follows from Proposition 2.9. □

The next result generalizes [5, Theorem 2.6].

Proposition 2.11. *The following statements are equivalent for a v-coherent domain R:*

(1) *For each set Λ of t-primes of R, $\Phi(\Lambda)$ is v-finite.*
(2) *R satisfies the ascending chain conditions on t-primes, and R satisfies the equivalent conditions of Proposition* 2.4.

Proof. (1) ⇒ (2): Let Λ be a nonempty set of t-primes of R. Since $\Phi(\Lambda)$ is v-finite, Lemma 2.1 implies that Λ has maximal elements. Hence R satisfies the acc on t-primes. Condition (1) of Proposition 2.4 holds by hypothesis.

(2) ⇒ (1): Let Λ be a nonempty set of t-primes. Then acc on t-primes implies that each element of Λ is contained in a maximal element. Hence if Λ_0 is the set of maximal elements of Λ, then $\Phi(\Lambda) = \Phi(\Lambda_0)$ is v-finite by Proposition 2.4. □

The preceding result can be improved for PVMD's in a way which generalizes [10, Theorem 4]. We first recall some results from [2] and prove a variation on [10, Lemma 4].

Lemma 2.12. *Let R be any domain. Then R satisfies the ascending chain condition on radical t-ideals if and only if each prime t-ideal is the radical of a v-finite t-ideal.*

If R does satisfy the acc on radical t-ideals, then every t-ideal has only finitely many minimal (t-)primes.

Proof. [2, Lemmas 3.7 and 3.8]. □

Lemma 2.13 (cf. [10, Lemma 4]). *Let $I = (a_1, \dots, a_n)$ be a finitely generated ideal of a PVMD R. Then each minimal prime ideal of I_v is minimal over some (a_i). Moreover, if I_v has only finitely many minimal primes, then each minimal prime of I_v is the radical of a v-finite divisorial ideal.*

Proof. Let P be minimal over I_v. Then P is a t-prime, and, since primes contained in P are also t-primes, P is also minimal over I. The proof of the first statement now proceeds as in the proof of the corresponding part of [10, Lemma 4]. Now assume that I_v has only finitely many minimal primes $P_1, \dots, P_k$, $k \geqslant 2$. Since there are no containment relations among the P_i (and since the t-spectrum of a PVMD is treed), we have $(P_1 + P_2 \cdots P_k)_t = R$. Hence there are finitely generated ideals $A \subseteq P_1$ and $B \subseteq P_2 \cdots P_k$ with $(A+B)_v = R$. We claim that P_1 is the radical of $(I+A)_v$. To see this, suppose that Q is a prime which is minimal over $(I + A)_v$. Then Q is a t-prime and must contain a prime minimal over I_v; that is, Q must contain one of the P_i. However, Q cannot contain P_i for $i \geqslant 2$, since then Q would contain B (and $(A+B)_v = R$). Hence Q contains, and is therefore equal to, P_1. □

Proposition 2.14. *Let R be PVMD. Then the statements in Proposition* 2.11 *are equivalent to each of the following*:

(3) *R satisfies the ascending chain condition on radical t-ideals.*
(4) *R satisfies the ascending chain condition on t-primes, and, for each finitely generated ideal I, the set of minimal primes of I_v is a finite set.*
(5) *Each t-prime of R is branched and each t-linked overring of R is a t#-domain.*
(6) *R satisfies the ascending chain condition on t-primes and each t-linked overring of R is a t#-domain.*

Proof. (2) ⇒ (4): By (3) ⇔ (5) of Proposition 2.4, for each t-prime P of R, we have that $R_P \not\supseteq \bigcap R_M$, where the intersection is taken over those maximal t-ideals of R which do not contain P. Hence each principal ideal has only finitely many minimal (t-)primes by [3, Lemma 3.9]. Thus if $I=(a_1,\dots,a_n)$ is finitely generated, then I_v can have only finitely minimal primes, since Lemma 2.13 implies that each such minimal prime must be minimal over one of the a_i.

(4) ⇒ (3): Let P be a t-prime of R. By Lemma 2.12, it suffices to show that P is the radical of a v-finite t-ideal. By the ascending chain condition on t-primes, the set of t-primes properly contained in P has a maximal element Q. Thus, for $x \in P \setminus Q$, P is minimal over the principal ideal xR. By assumption, xR has only finitely many minimal primes. Hence Lemma 2.13 yields that P is the radical of a v-finite t-ideal, as desired.

(3) ⇒ (2): Clearly, R satisfies the ascending chain condition on t-primes. Let P be a t-prime of R. By Lemma 2.12 P is the radical of J_v for some finitely generated ideal J of R. Since any t-prime containing J also contains P, it is clear that condition (5) of Proposition 2.4 holds.

(2) ⇔ (6) because each t-linked overring of a PVMD is a t-flat t-subintersection ([16, Theorem 3.8] and [17, Proposition 2.10]).

(5) ⇔ (6): Since each localization of a PVMD at a t-prime is a valuation domain, each t-prime of R is branched if and only if R satisfies the ascending chain condition on t-primes. □

The PVMDs with the property that each t-localizing system of ideals is v-finite have been studied in [2,3]. They are called Generalized Krull domains. By [2, Theorem 3.9], R is a Generalized Krull domain if and only if each principal ideal has only finitely many minimal primes and $P \neq (P^2)_t$ for each t-prime P. On the other hand, the first condition is satisfied under the equivalent conditions of Proposition 2.8 [3, Lemma 3.9]. Hence we obtain the following result.

Corollary 2.15. *A PVMD R is a Generalized Krull domain if and only if each t-linked overring of R is a t#-domain and $P \neq (P^2)_t$ for each t-prime P.*

3. Polynomial rings

In this section, we denote by $\{X_\alpha\}$ a set of independent indeterminates over R. Let us call a prime ideal Q of $R[\{X_\alpha\}]$ an *upper to zero* if $Q \cap R = 0$. For f in the quotient field of $R[\{X_\alpha\}]$, the *content of f*, written $c(f)$ is the fractional R-ideal generated by the coefficients of f; we also write $c(I)$ for the fractional ideal generated by the coefficients of all the polynomials in the fractional $R[\{X_\alpha\}]$-ideal I.

Lemma 3.1. *Let Q be an upper to zero in $R[\{X_\alpha\}]$. Then the following statements are equivalent.*

(1) $Q = fK[\{X_\alpha\}] \cap R[\{X_\alpha\}]$ *for some irreducible polynomial* $f \in K[\{X_\alpha\}]$. (*Note that we may take* $f \in R[\{X_\alpha\}]$.)
(2) $\operatorname{ht} Q = 1$.
(3) $R[\{X_\alpha\}]_Q$ *is a DVR*.

Proof. A localization argument establishes (1) $\Leftrightarrow$ (2), and (3) $\Rightarrow$ (2) is trivial. Assume (2). If $\{X_\alpha\}$ is finite, then (3) follows from a standard induction argument. If $\{X_\alpha\}$ is infinite, then we may pick $X_1, \ldots, X_n \in \{X_\alpha\}$ with $Q \cap R[X_1, \ldots, X_n] \neq 0$. Then $V = R[X_1, \ldots, X_n]_{Q \cap R[X_1, \ldots, X_n]}$ is a DVR with maximal ideal $M = (Q \cap R[X_1, \ldots, X_n])V$, and, since $\operatorname{ht} Q = 1$, we must have Q extended from $Q \cap R[X_1, \ldots, X_n]$. It is then easy to see that $R[\{X_\alpha\}]_Q = V[\{X_\alpha\}]_{M[\{X_\alpha\}]}$ is a DVR. □

Lemma 3.2. *Let Q be an upper to zero in $R[\{X_\alpha\}]$ which is also a maximal t-ideal. Then* $\operatorname{ht} Q = 1$.

Proof. First suppose $\{X_\alpha\} = \{X_1, \ldots, X_n\}$. The result clearly holds if $n = 1$ (even if Q is not a maximal t-ideal!). Suppose $n > 1$, and let $q = Q \cap R[X_1, \ldots, X_{n-1}]$. If $q = 0$, then $\operatorname{ht} Q = 1$ by the case $n = 1$. If $q \neq 0$, then by [15, Theorem 1.4], q is a maximal t-ideal of $R[X_1, \ldots, X_{n-1}]$, and $Q = q[X_n]$. By induction $\operatorname{ht} q = 1$, and $V = R[X_1, \ldots, X_{n-1}]_q$ is a DVR by Lemma 3.1. Hence $R[X_1, \ldots, X_n]_Q = V[X_n]_Q$ is also a DVR, and $\operatorname{ht} Q = 1$.

For the general case, we may pick $X_1, \ldots, X_n \in \{X_\alpha\}$ with $q = Q \cap R[X_1, \ldots, X_n] \neq 0$. By [4, Proposition 2.2], q is a maximal t-ideal of $R[X_1, \ldots, X_n]$, and Q is extended from q. The argument now proceeds as in the induction step above. □

The following extends [15, Theorem 1.4, Corollary 1.5] to the case of infinitely many indeterminates.

Theorem 3.3. *Let Q be an upper to zero in $R[\{X_\alpha\}]$. Then the following statements are equivalent.*

(1) *Q is a maximal t-ideal.*
(2) *Q is t-invertible.*
(3) $c(Q)_t = R$, *and* $\operatorname{ht} Q = 1$. (*In this case, a standard argument shows that Q contains an element g with $c(g)_v = R$.*)

In case these equivalent statements hold, then $Q = fK[\{X_\alpha\}] \cap R[\{X_\alpha\}]$ *for some* $f \in R[\{X_\alpha\}]$ *such that f is irreducible in* $K[\{X_\alpha\}]$; *moreover, we have* $Q = (f, g)_v$.

Proof. (3) ⇒ (1): Since $\operatorname{ht} Q = 1$, Q is a t-ideal. Hence Q is contained in a maximal t-ideal, say N. Since $c(Q)_t = R$, we cannot have N extended from $N \cap R$, whence N is an upper to zero by [4, Proposition 2.2]. By Lemma 3.2, $\operatorname{ht} N = 1$, whence $Q = N$, and Q is a maximal t-ideal.

The proofs of (1) ⇒ (3) and (2) ⇒ (1) are as in [15, Theorem 1.4].

(1) ⇒ (2): This also goes through essentially as in the proof of [15, Theorem 1.4]. That proof contains a reference to [12, Proposition 1.8], which is stated for the case of one indeterminate. However, the proof of this latter result extends to the case of an arbitrary set of indeterminates. (The content formula [9, Corollary 28.3] is needed.)

To prove the last statement, note that $Q = fK[\{X_\alpha\}] \cap R[\{X_\alpha\}]$ by Lemmas 3.1 and 3.2. The fact $Q = (f, g)_v$ may be proved as in [15, Corollary 1.5]. □

Theorem 3.4. *If R is a t#-domain, then so is $R[\{X_\alpha\}]$.*

Proof. We wish to show that condition (5) of Theorem 1.2 is satisfied. Thus let N be a maximal t-ideal of $R[\{X_\alpha\}]$. By [4, Proposition 2.2], either $N \cap R = 0$ or $N = (N \cap R)R[\{X_\alpha\}]$. In the former case N is divisorial (being a t-invertible t-ideal), and N is certainly not contained in any other maximal t-ideal of $R[\{X_\alpha\}]$. In the latter case, $N \cap R$ contains a divisorial ideal I which is contained in no other maximal t-ideal of R, and it follows that $IR[\{X_\alpha\}]$ is a divisorial ideal of $R[\{X_\alpha\}]$ which is contained in N and no other maximal t-ideal of $R[\{X_\alpha\}]$. □

We have been unable to prove the converse of Theorem 3.4. (Indeed, we doubt that the converse is true.) However, we can prove that several standard localizations of $R[\{X_\alpha\}]$ are simultaneously $t\#$. We denote by $R(\{X_\alpha\})$ the ring of fractions of $R[\{X_\alpha\}]$ with respect to the multiplicatively closed subset of $R[\{X_\alpha\}]$ consisting of the polynomials of unit content. Finally, if $S = \{f \in R[\{X_\alpha\}] \mid c(f)_v = R\}$, we denote by $R\langle\{X_\alpha\}\rangle$ the ring $R[\{X_\alpha\}]_S$. We then have the following description of the maximal t-ideals in these rings.

Lemma 3.5. *Denote by $\mathscr{U}_1$ the set of uppers to zero which are also maximal t-ideals in $R[\{X_\alpha\}]$ and by $\mathscr{U}_2$ the set of those elements $P \in \mathscr{U}_1$ which satisfy $c(P) \neq R$. Then*
(1) $t\text{-Max}(R[\{X_\alpha\}]) = \{MR[\{X_\alpha\}] \mid M \in t\text{-Max}(R)\} \bigcup \mathscr{U}_1$;
(2) $t\text{-Max}(R(\{X_\alpha\})) = \{MR(\{X_\alpha\}) \mid M \in t\text{-Max}(R)\} \bigcup \{PR(\{X_\alpha\}) \mid P \in \mathscr{U}_2\}$;
(3) $t\text{-Max}(R\langle\{X_\alpha\}\rangle) = \{MR\langle\{X_\alpha\}\rangle \mid M \in t\text{-Max}(R)\}$.

Proof. (1) Each maximal t-ideal of $R[\{X_\alpha\}]$ must have the form indicated by [4, Proposition 2.2]. The reverse inclusion follows from [4, Lemma 2.1].

(2) By [16, Corollary 2.3], $MR(\{X_\alpha\})$ is a t-ideal of $R(\{X_\alpha\})$ for each $M \in t\text{-Max}(R)$. Suppose that for some $N \in t\text{-Max}(R(\{X_\alpha\}))$ we have $N \supseteq MR(\{X_\alpha\})$. Then since $R(\{X_\alpha\})$ is a ring of fractions of $R[\{X_\alpha\}]$, N is extended from a maximal t-ideal of $R[\{X_\alpha\}]$, which in turn must be extended from a maximal t-ideal of R. It follows that $N = MR(\{X_\alpha\})$. Hence $MR(\{X_\alpha\}) \in t\text{-Max}(R(\{X_\alpha\}))$. If $P \in \mathscr{U}_2$, then, since $c(P) \neq R$, $PR(\{X_\alpha\}) \neq R(\{X_\alpha\})$. Moreover, since $\operatorname{ht} P = 1$ by Theorem 3.3, $\operatorname{ht} PR(\{X_\alpha\}) = 1$ also, and $PR(\{X_\alpha\})$ is a t-prime of $R(\{X_\alpha\})$. Any maximal t-ideal of $R(\{X_\alpha\})$

containing $PR(\{X_\alpha\})$ must be extended from a t-prime of $R[\{X_\alpha\}]$ containing P. Therefore, since $P \in t\text{-Max}(R[\{X_\alpha\}])$, $PR(\{X_\alpha\}) \in t\text{-Max}(R(\{X_\alpha\}))$. That each maximal t-ideal of $R(\{X_\alpha\})$ must be of the form indicated follows from (1) (and the fact that $R(\{X_\alpha\})$ is a ring of fractions of $R[\{X_\alpha\}]$).

(3) This follows from the facts that $R\langle\{X_\alpha\}\rangle$ is a localization of $R(\{X_\alpha\})$ and that each $P \in \mathscr{U}_1$ satisfies $c(P)_t = R[\{X_\alpha\}]$ by Theorem 3.3 so that $PR\langle\{X_\alpha\}\rangle = R\langle\{X_\alpha\}\rangle$. □

In the proof of the following result, we often invoke Lemma 3.5 without explicit reference.

Theorem 3.6. *The following statements are equivalent for a domain* R:

(1) $R[\{X_\alpha\}]$ *is a* $t\#$*-domain.*
(2) $R(\{X_\alpha\})$ *is a* $t\#$*-domain.*
(3) $R\langle\{X_\alpha\}\rangle$ *is a* $t\#$*-domain.*

If, in addition, $R[\{X_\alpha\}]$ *is v-coherent, then these conditions are equivalent to*: R *is a* $t\#$*-domain.*

Proof. (1) ⇒ (2): If $M \in t\text{-Max}(R)$, then by Theorem 1.2, M contains a divisorial ideal I which is contained in no other maximal t-ideal of R. It follows that $IR(\{X_\alpha\})$ is a divisorial ideal of $R(\{X_\alpha\})$ [16, Corollary 2.3], and it is clear that $IR(\{X_\alpha\})$ is contained in $MR(\{X_\alpha\})$ but in no other maximal t-ideal of $R(\{X_\alpha\})$. Hence each maximal t-ideal of $R(\{X_\alpha\})$ of the form $MR(\{X_\alpha\})$ contains a divisorial ideal contained in no other maximal t-ideal of $R(\{X_\alpha\})$. On the other hand, if $PR(\{X_\alpha\})$ is a maximal t-ideal of $R(\{X_\alpha\})$ with $P \in \mathscr{U}_2$, then P is divisorial, from which it follows that $PR(\{X_\alpha\})$ is also divisorial (and is clearly not contained in any other maximal t-ideal of $R(\{X_\alpha\})$). By Theorem 1.2, $R(\{X_\alpha\})$ is a $t\#$-domain.

(2) ⇒ (3): Similar (but easier).

(3) ⇒ (1): Let M be a maximal t-ideal of R. By hypothesis and Theorem 1.2 ((1) ⇔ (6)), there is an element $u \in K(\{X_\alpha\})$ such that $(R\langle\{X_\alpha\}\rangle :_{R\langle\{X_\alpha\}\rangle} u)$ is contained in $MR\langle\{X_\alpha\}\rangle$ and no other maximal t-ideal of $R\langle\{X_\alpha\}\rangle$. Let $I = (R[\{X_\alpha\}] :_{R[\{X_\alpha\}]} U)$. Then I is divisorial in $R[\{X_\alpha\}]$, and $IR\langle\{X_\alpha\}\rangle = (R\langle\{X_\alpha\}\rangle :_{R\langle\{X_\alpha\}\rangle} u)$. Clearly, $I \subseteq MR[\{X_\alpha\}]$ and $I \nsubseteq NR[\{X_\alpha\}]$ for each maximal t-ideal N of R with $N \neq M$. Moreover, I is contained in at most finitely many maximal t-ideals P with $P \cap R = 0$. We shall show how to enlarge I so as to avoid each such P. By Theorem 3.3, we have that $PR\langle\{X_\alpha\}\rangle = R\langle\{X_\alpha\}\rangle$, and P is v-finite. Therefore, since $R[\{X_\alpha\}]_P$ is a DVR, we may pick $h \in R[\{X_\alpha\}] \setminus P$ with $hP^n \subseteq I$. Hence $hR\langle\{X_\alpha\}\rangle = hP^nR\langle\{X_\alpha\}\rangle \subseteq IR\langle\{X_\alpha\}\rangle$, and there is an element $g \in R[\{X_\alpha\}]$ with $c(g)_v = R$ and $gh \in I$. In particular, $g \notin MR[\{X_\alpha\}]$, so that the divisorial ideal $(I :_{R[\{X_\alpha\}]} g)$ is contained in $MR[\{X_\alpha\}]$. Moreover, $h \in (I :_{R[\{X_\alpha\}]} g) \setminus P$. Hence $(I :_{R[\{X_\alpha\}]} g)$ is a divisorial ideal contained in $MR[\{X_\alpha\}] \setminus P$. This process may be continued finitely many times to produce a divisorial ideal which is contained in $MR[\{X_\alpha\}]$ but in no other maximal t-ideal of $R[\{X_\alpha\}]$. Thus $R[\{X_\alpha\}]$ is a $t\#$-domain.

To prove the final statement, assume that $R[\{X_\alpha\}]$ is a $t\#$-domain. Let $M \in t\text{-Max}(R)$. By Theorem 1.6, there is a finitely generated ideal I of $R[\{X_\alpha\}]$ such that $I \subseteq MR[\{X_\alpha\}]$

and I is contained in no other maximal t-ideal of $R[\{X_\alpha\}]$. Clearly, $c(I) \subseteq M$ and $c(I)$ is contained in no other maximal t-ideal of R. Another application of Theorem 1.6 completes the proof. □

It is well known that a domain R is a PVMD if and only if $R[\{X_\alpha\}]$ is a PVMD. Thus for a PVMD R the conditions of Theorem 3.6 are each equivalent to R being a $t\#$-domain. It follows that if R is a Prüfer domain, then R is a #-domain if and only if $R[\{X_\alpha\}]$ is a $t\#$-domain.

Now recall that it is possible for a polynomial ring over a Mori domain to fail to be Mori [22]. In view of the fact that a Mori domain is automatically a $t\#$-domain (Corollary 1.3), we see by Theorem 3.4 that if R is a Mori domain, then $R[\{X_\alpha\}]$ is a $t\#$-domain even though $R[\{X_\alpha\}]$ may not be a Mori domain.

It is an open question whether R v-coherent implies that $R[\{X_\alpha\}]$ is v-coherent. We are therefore unable to determine whether the last statement of Theorem 3.6 remains true if we assume only that R is v-coherent. It is true, however, that v-coherence of $R[\{X_\alpha\}]$ implies that of R, as the following result shows.

Proposition 3.7. *If $R[\{X_\alpha\}]$ is v-coherent, then R is v-coherent.*

Proof. Let I be a finitely generated ideal of R. We have $(I[\{X_\alpha\}])^{-1} = I^{-1}[\{X_\alpha\}]$; by hypothesis, this produces a finitely generated fractional ideal J of $R[\{X_\alpha\}]$ with $I^{-1}[\{X_\alpha\}] = J_v$. We may assume $1 \in J$. Moreover, since $R[\{X_\alpha\}] \subseteq I^{-1}[\{X_\alpha\}] \subseteq K[\{X_\alpha\}]$, we have $R[\{X_\alpha\}] \subseteq J \subseteq K[\{X_\alpha\}]$. Hence $c(J)$ is a finitely generated ideal of R with $1 \in c(J)$. We claim that $c(J)_v = I^{-1}$. Note that $J \subseteq c(J)[\{X_\alpha\}] \subseteq I^{-1}[\{X_\alpha\}]$. Hence

$$J_v \subseteq (c(J))[\{X_\alpha\}])_v = c(J)_v[\{X_\alpha\}] \subseteq I^{-1}\{X_\alpha\}] = J_v,$$

and the claim follows. □

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