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Some Remarks on p-Local Cores in Linear Groups

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In this paper, we prove the following result:

THEOREM A. *Let G be a finite group, p be a prime, F be an algebraically closed field of characteristic p. Let M be an FG-module such that $C_G(M)$ is a p-group. Let x be an element of G such that $M(1-x)^{(p-1)/2}=0$. Then $O_{p'}(C_G(x))=O_{p'}(G)$.*

Of course, it quickly follows that the corresponding result is true if F is of characteristic p, but not algebraically closed, or if F is a field of characteristic 0 and x is a p-element having minimum polynomial of degree $(p-1)/2$ or less on the FG-module M such that $C_G(M)$ is a p-group.

The main idea behind the proof of Theorem A is to apply the ideas used in Nagao's proof of Brauer's second main theorem [1] in a situation when we are dealing with a p-element having a minimum polynomial of small degree.

We start with two general lemmas. The first is probably well known, and the second is a straightforward adaptation of Nagao's argument to our situation.

LEMMA 1. *Let H be a finite group, p be a prime. Then for any $x \in O_p(H)$ we have $O_{p'}(H)=O_{p'}(C_H(x))$.*

Proof. Certainly, we have $[O_{p'}(H), x]=1_H$, $O_{p'}(H) \leqslant O_{p'}(C_H(x))$. To prove the reverse inclusion, we may suppose that $O_{p'}(H)=1_H$, and we do so. Let $A=O_{p'}(C_H(x))$, $B=O_p(H)$. Then $A \times \langle x \rangle$ normalizes B, and $[A, C_B(x)] \leqslant O_{p'}(C_H(x)) \cap O_p(C_H(x))=1_H$. By a well-known lemma of J. G. Thompson, $[A, B]=1_H$.

Now let $E=E(H)$ (the central product of the components of H). Then $[E, x] \leqslant E \cap O_p(H)=1_H$, so $E \leqslant C_H(x)$. Hence E and $O_{p'}(C_H(x))$ normalize each other, so $[E, O_{p'}(C_H(x))] \leqslant E \cap O_{p'}(C_H(x))$. Since $O_{p'}(H)=1_H$, each component of H has order divisible by p, so it quickly follows now that

0021-8693/83 $3.00

$[E, O_{p'}(C_H(x))] = 1_H$. We now have $[O_{p'}(C_H(x)), F^*(H)] = 1_H$, so $O_{p'}(C_H(x)) \leqslant Z(F(H))$, and $O_{p'}(C_H(x)) = 1_H$, since $O_{p'}(H) = 1_H$.

LEMMA 2. *Let H be a finite group, p be a prime, x be an element of order p in H. Let M be an FH-module in the p-block B, where F is a field of characteristic p containing a splitting field for H and its subgroups. Then we may write $M_{C_H(x)} = L \oplus N$, where L has composition factors in p-blocks of $C_H(x)$ which are dominated by B in the sense of the Brauer homomorphism and where $N_{\langle x \rangle}$ is projective.*

Proof. Let $\tau: Z(FH) \to Z(FC_H(x))$ denote the Brauer homomorphism, let e be the block idempotent of $Z(FH)$ associated with B, and let $f = e\tau$. Then $M_{C_H(x)} = M_{C_H(x)} f \oplus M_{C_H(x)}(e - f)$ (because $Me = M$). Let $L = M_{C_H(x)} f$, $N = M_{C_H(x)}(e - f)$. It only remains to prove that $N_{\langle x \rangle}$ is projective.

Now let K be a class sum in $Z(FH)$. Then we may write $K = K\tau + U + x^{-1}Ux + \cdots + x^{-(p-1)}Ux^{p-1}$ for some U in FH. Thus we may write $e = f + Z + x^{-1}Zx + \cdots + x^{-(p-1)}Zx^{(p-1)}$ for some Z in FH. Now $N(e - f) = N$ (more precisely, $n(e - f) = n$ for each n in N). The mapping $\phi: N \to N$ given by $n\phi = nZ(e - f)$ satisfies $i_N = \phi + x^{-1}\phi x \cdots + x^{-(p-1)}\phi x^{(p-1)}$, and so, by the criterion of D. G. Higman, $N_{\langle x \rangle}$ is projective.

Proof of Theorem A. The proof is broken into a number of easy steps. We suppose that Theorem A is false, and that a pair (G, M) has been chosen to violate the theorem with $|G| + \dim_F(M)$ as small as possible.

Step 1. By Lemma 1, we see immediately that $x \notin O_p(G)$.

Step 2. By the Hall–Higman theorem, $[O_{p'}(G), x] \leqslant C_G(M)$, so that $O_{p'}(G) \leqslant O_{p'}(C_G(x))$, as $C_G(M)$ is a p-group. It only remains to prove that $O_{p'}(C_G(x)) \leqslant O_{p'}(G)$.

Step 3. M is irreducible, and $C_G(M) = 1_G$.

Proof of Step 3. Suppose that M is not irreducible. Then, as $x \notin O_p(G)$, x acts nontrivially on some FG-composition factor of M, say V.

Let $K = C_G(V)$, $\bar{G} = G/K$ and $\bar{\ }$ denote images in $\bar{G}$. Then $[G, x] \nleqslant K$, since V is irreducible and $\bar{x}$ acts nontrivially on V. By the minimality of (G, M), $O_{p'}(C_{\bar{G}}(\bar{x})) \leqslant O_{p'}(\bar{G})$ (for $\dim_F(V) < \dim_F(M)$). Define the subgroup U of G by $\bar{U} = C_{\bar{G}}(\bar{x})$. Then $U < G$, so by the minimality of (G, M), $O_{p'}(C_u(x)) \leqslant O_{p'}(U)$, so that $O_{p'}(C_G(x)) \leqslant O_{p'}(C_u(x)) \leqslant O_{p'}(U)$. Now $\overline{O_{p'}(U)} \leqslant O_{p'}(C_{\bar{G}}(\bar{x}) \leqslant O_{p'}(\bar{G})$.

Define the subgroup T of G by $\bar{T} = O_{p'}(\bar{G})$. Suppose that $T\langle x \rangle \neq G$. Then $O_{p'}(C_G(x)) \leqslant O_{p'}(C_{T\langle x \rangle}(x)) \leqslant O_{p'}(T)$ by the minimality of (G, M). Since $T \lhd G$, we have $O_{p'}(C_G(x)) \leqslant O_{p'}(G)$, contrary to assumption. Thus $T\langle x \rangle = G$. By the Hall–Higman theorem $\bar{G} = \langle \bar{x} \rangle \times \bar{T}$, as $\bar{T}$ is a p'-group. This contradicts the fact that $\bar{x}$ is not central in $\bar{G}$. Hence we are forced to conclude that M is irreducible. If $C_G(M) \neq 1_G$, the argument above again leads to a contradiction, so we have $C_G(M) = 1_G$.

Step 4. Let $y = \langle O_{p'}(C_G(x))^g : g \in G \rangle$. Then $G = Y\langle x \rangle$, and M_Y is irreducible.

Proof of Step 4. Suppose that $Y\langle x \rangle \neq G$. Then by the minimality of (G, M) we have $O_{p'}(C_{Y\langle x \rangle}(x)) \leqslant O_{p'}(Y) \leqslant O_{p'}(G)$. Then we have $O_{p'}(C_G(x)) \leqslant O_{p'}(C_{Y\langle x \rangle}(x)) \leqslant O_{p'}(Y) \leqslant O_{p'}(G)$, contrary to assumption. Thus $G = Y\langle x \rangle$. We note also that x has order p, since $C_G(M) = 1_G$ and $M(1-x)^{(p-1)/2} = 0$. If M_Y is not irreducible, then we may write $M_Y = V \oplus Vx \cdots \oplus Vx^{p-1}$, where V is some irreducible submodule of M_Y. This contradicts the fact that x has minimum polynomial of degree $(p-1)/2$ or less on M.

Step 5. Let x have minimum polynomial degree n on M. Then the trivial module occurs at least n times as an $FC_G(x)$-composition factor of $\mathrm{Hom}_F(M, M)_{C_G(x)}$.

Proof of Step 5. Consideration of the series $O < M(1-x)^{n-1} < M(1-x)^{n-2} \cdots < M(1-x) < M$ shows that $M_{C_G(x)}$ has composition length at least n. Now if the Brauer character for $M_{C_G(x)}$ is $\sum_{j=1}^k a_j \phi_j$, then $\sum_{j=1}^k a_j \geqslant n$, and the Brauer character for $\mathrm{Hom}_F(M, M)_{C_G(x)}$ is $\sum_{i=1}^k \sum_{j=1}^k a_i a_j \phi_i \bar{\phi}_j$, so that the trivial module occurs at least $\sum_{j=1}^k a_j^2$ times as a composition factor of $\mathrm{Hom}_F(M, M)_{C_G(x)}$, so certainly at least n times.

Step 6. x has minimum polynomial of degree at most $p-2$ on $\mathrm{Hom}_F(M, M)$.

Proof of Step 6. This can be verified directly, using a suitable basis for M.

Step 7. Let $\{e_i : 1 \leqslant i \leqslant k\}$ be the set of block idempotents of $Z(FG)$, where e_1 is the principal block idempotent. Let $V_i = \mathrm{Hom}_F(M, M)\, e_i$ for $1 \leqslant i \leqslant k$. Then $\dim_F(V_1) = 1$.

Proof of Step 7. By Brauer's third main theorem, Lemma 2, and Step 6, $V_{1C_G(x)}$ has all its composition factors in the principal p-block of $C_G(x)$. Hence $O_{p'}(C_G(x))$ acts trivially on V_1. Thus Y acts trivially on V_1 (recall that $Y = \langle O_{p'}(C_G(x))^g : g \in G \rangle$). Since M_Y is irreducible, we have $\dim_F(V_1) = 1$.

Step 8: *The final contradiction.* By Step 5 (since certainly $n > 1$) and Step 7, the trivial module must occur as a composition factor of $V_{iC_G(x)}$ for some $i > 1$. However, by Lemma 2, Brauer's third main theorem, and Step 6, $V_{iC_G(x)}$ does not have any composition factor in the principal block of $C_G(x)$. This contradiction completes the proof of Theorem A.

Some Corollaries and Applications

Let G be a finite group, p be a prime, F be a field of characteristic p, and V be a finite dimensional faithful FG-module. Let P be a p-subgroup of G. Let r be the dimension of the largest indecomposable summand of V_p. Then $[V, P; r] = 0$ (where $[V, P; 0] = V$, $[V, P; i+1] = [[V, P; i], P]$ for $i > 0$).

Let $L_i(P)$ denote the ith term of the lower central series of P. Lemma 3 is presumably a well-known result.

Lemma 3. *For $i \geqslant 1$, we have $[V, L_i(P)] \leqslant [V, P; i]$.*

Proof. We use induction on i. The result is true for $i = 1$. Suppose that the result has been established for $i = k$. Then we have

$$[V, L_k(P), P] \leqslant [[V, P; k], P] = [V, P; k+1].$$

Also, we have

$$[P, V, L_k(P)] \leqslant [[V, P], P; k] = [V, P; k+1].$$

By the three subgroups lemma,

$$[L_k(P), P, V] \leqslant [V, P; k+1].$$

Hence we have

$$[V, L_{k+1}(P)] \leqslant [V, P; k+1],$$

as required.

Another easy induction argument now yields

$$[V, L_i(P); k] \leqslant [V, P; ik] \qquad \text{for} \quad i, k \geqslant 1.$$

Corollary 4. *Let j be an integer with $j \geqslant 2r/(p-1)$. Then for any x in $L_j(P)$ we have $O_{p'}(C_G(x)) = O_{p'}(G)$.*

Proof. We have $[V, L_j(P); (p-1)/2] \leqslant [V, P; j((p-1)/2)] = 0$. In particular, each $x \in L_j(P)$ has minimum polynomial of degree $(p-1)/2$ or less on V. By Theorem A, $O_{p'}(C_G(x)) = O_{p'}(G)$ for each $x \in L_j(P)$.

Remark. Statements corresponding to Corollary 4 can be made about elements of $P^{(k)}$ for sufficiently large k by making use of the fact that $P^{(k)} \leqslant L_{2^k}(P)$ for $k \geqslant 0$.

We also remark that for j as in Corollary 4, we may choose an element z in $Z(P)^{\#} \cap L_j(P)$, if $L_j(P) \neq 1_G$. By the Hall–Higman theorem, for any

x in P we have $O_{p'}(C_G(x)) \leqslant C_G(z)$. Thus $O_{p'}(C_G(z)) = O_{p'}(G)$ and $\langle O_{p'}(C_G(x)) : x \in P \rangle \leqslant C_G(z)$.

COROLLARY 5. *Let y be a p-element of G, Q b a p-subgroup of G with $y \in C_G(Q)$, and let $W = C_V(Q)$. Then if $W(1-y)^{(p-1)/2} = 0$,* we have

$$O_{p'}(C_G(y)) \cap C_G(Q) \leqslant O_{p'}(C_G(Q)).$$

Proof. Let $H = C_G(Q)$. Then W is H-invariant, and $C_H(W)$ is a p-group. Since $y \in H$, we may apply Theorem A to conclude that $O_{p'}(H) = O_{p'}(C_H(y))$. Now $O_{p'}(C_G(y)) \cap C_G(Q) \leqslant O_{p'}(C_H(y)) = O_{p'}(H) = O_{p'}(C_G(Q))$.

COROLLARY 6. *Let H be a p-constrained finite group with $O_{p'}(H) = 1_H$, and such that $K = O_p(H)$ is elementary Abelian. Let $P \in$* Sylp(H) be of nilpotence class n. Let j be an integer $\geqslant 2n/(p-1)$, and let $\bar{\ }$ denote images in H/K. Then for any x in $L_j(P)$ we have $O_{p'}(C_{\bar{H}}(\bar{x})) = O_{p'}(\bar{H})$.

Proof. $[K, L_j(P); (p-1)/2] \leqslant L_{j((p-1)/2)}(P) = 1_H$. Thus $\bar{x}$ has minimum polynomial of degree $(p-1)/2$ or less on K (regarded as a vector space). Since $\bar{H}$ is faithfully represented on K, the result now follows from Theorem A.

COROLLARY 7. *Let H be a finite group, p be a prime, M be a faithful FH-module, where F is a field of characteristic p, W be an Abelian p-subgroup of H which is generated by elements having minimum polynomial of degree $(p-1)/2$ or less on M. Then for any subgroup, A, of W, we have $O_{p'}(C_H(A)) = O_{p'}(H)$.*

Proof. We use induction on $|H|$. If $W \leqslant Z(H)$, the corollary is obviously true. Suppose that $W \nleqslant Z(H)$, and let $w \in W \backslash Z(H)$ be a generator of W having minimum polynomial of degree $(p-1)/2$ or less on M. By the Hall–Higman theorem, $O_{p'}(C_H(A)) \leqslant C_H(w)$ for each subgroup A of W. By induction, we have $O_{p'}(C_H(A)) \leqslant O_{p'}(C_H(w))$ for each such subgroup (since $|C_H(w)| < |H|$). By Theorem A, $O_{p'}(C_H(w)) = O_{p'}(H)$. Also, by the Hall–Higman theorem, $O_{p'}(H) \leqslant C_H(W)$. Thus $O_{p'}(H) = O_{p'}(C_H(A))$ for each subgroup A of W.

Remark. Let H be a finite group, p be a prime, and suppose that M is a faithful KH-module, where K is a field of characteristic 0. Let $P \in$ Sylp(H) and let $x \in P$ be an element having $(p-1)/2$ or fewer eigenvalues on M. Let $W = \langle x^g \in P : g \in G \rangle$. Then W is Abelian (for we may suppose that elements of P are represented by monomial matrices, in which case all conjugates of x in P are represented by diagonal matrices). We can therefore apply Corollary 7 to the reduction $(\bmod\, p)$ of M.

REFERENCE

1. H. NAGAO, A proof of Brauer's theorem on generalized decomposition numbers, *Nagoya Math. J.* **22** (1963), 73–77.