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EDGE COSYMMETRIC GRAPHS

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This study explores the structure of graphs which together with their complements are edge symmetric.

A graph G with vertex set $V(G)$ and edge set $E(G)$ is *edge symmetric* (*vertex symmetric*) if the induced edge-automorphism group $\mathscr{A}^*(G)$ (the automorphism group $\mathscr{A}(G)$) acts transitively on $E(G)$ (on $V(G)$). In [3] we adopted the following definition: a graph G is *uniquely edge extendible* if for all $e, f \in E(\bar{G})$, $G+e \cong G+f$. A connection between edge symmetric graphs and uniquely edge extendible graphs was also established in [3]: a graph G is edge symmetric if and only if its complement $\bar{G}$ is uniquely edge extendible. Clearly, $\bar{G}$ is uniquely edge extendible if and only if $G-x \cong G-y$ for all $x, y \in E(G)$. In view of this, any graph G for which $G-x \cong G-y$ and $G+e \cong G+f$ for all $x, y \in E(G)$ and $e, f \in E(\bar{G})$ has the property that G is edge symmetric and uniquely edge extendible. Consequently, the investigation of conditions under which a graph and its complement are both edge symmetric, or both uniquely edge extendible, reduces to the study of graphs which are simultaneously edge symmetric and uniquely edge extendible. We call a graph G *edge cosymmetric* if G and its complement $\bar{G}$ are both edge symmetric. Thus, an edge cosymmetric graph G is characterized by the property that only one graph is produced by the addition of any edge to G, and only one graph is produced by the removal of any edge from G. We follow the terminology of [1], and note that edge symmetry has also been referred to as line-symmetry in [6, 7, 8].

In [4], we studied properties of edge symmetric graphs and obtained some classifications. A few definitions are in order before stating a result which appears in [4]. For a bipartite edge symmetric graph G, a *transitive bipartition* of G is a

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bipartition of G where $\mathscr{A}(G)$ acts transitively on each partite set. A graph G is ***biregular*** if the degree set of G has cardinality two.

Theorem 1. *Let G be an edge symmetric graph which has no isolated vertices. Then G satisfies exactly one of the following.*

(i) *G is biregular and bipartite, and G has a unique transitive bipartition where a set of vertices of equal degree constitutes each of the partite sets.*

(ii) *G is regular and bipartite (but may or may not be vertex symmetric).*

(iii) *G is regular and vertex symmetric, but not bipartite.*

It should be mentioned that Theorem 1 is substantially the same as a result of Dauber and Harary [6, 8].

There exist edge cosymmetric graphs with exactly one isolated vertex, as our next result indicates. This theorem also serves the dual purpose of characterizing all biregular edge cosymmetric graphs.

Theorem 2. *A biregular graph G of order $p \geq 3$ is edge cosymmetric if and only if either $G \cong K(1, p-1)$ or $G = K_1 \cup K_{p-1} = \overline{K(1, p-1)}$.*

Proof. Clearly, $K(1, p-1)$ is edge cosymmetric. Let G be a biregular edge cosymmetric graph. We claim that either G or $\bar{G}$ must have an isolated vertex. Suppose to the contrary that neither G nor $\bar{G}$ has an isolated vertex. By Theorem 1 G is bipartite with a transitive bipartition $V(G) = V_1 \cup V_2$ where all vertices in V_1 have the same degree say a, and all vertices in V_2 have the same degree, say b. Then $\bar{G}$ is also a biregular bipartite edge cosymmetric graph. Since V_1 and V_2 are partite sets in both G and $\bar{G}$, it follows that $|V_1| = |V_2| = 1$ implying that $a = b = 1$. This contradicts the hypothesis that G is biregular. Thus we really need consider only two cases.

Case 1. G has an isolated vertex v_0.

Let $V(G) = V_1 \cup V_2$ be the unique transitive bipartition of G. Without loss of generality we may assume that $v_0 \in V_1$. Then v_0, being isolated in $\bar{G}$, is of degree $p-1$ in G, so v_0 is adjacent in G to every other vertex of V_1. Moreover, V_1 being a partite set of G forces $|V_1| = 1$, and $V_1 = \{v_0\}$. This implies that every vertex in V_2 has degree one in G since they are adjacent only to the single vertex v_0 of V_1. Thus $G \cong K(1, n)$ where $n = |V_2| \geq 2$ in order that G be biregular. Since $|V_1| = 1$, $n = p-1$.

Case 2. G has an isolated vertex.

This is completely parallel to case 1 ending with the conclusion that $\bar{G} \cong K(1, p-1)$, or equivalently, that $G \cong K_1 \cup K_{p-1}$. □

Theorem 3. *If a graph G of order p has an isolated vertex, then G is edge cosymmetric if and only if either $G \cong K_1 \cup K_{p-1}$ or $G \cong \overline{K_p}$.*

Proof. Clearly all graphs of the forms specified are edge cosymmetric. Conversely, suppose that G is an edge cosymmetric graph with $i \geq 1$ isolated vertices. Let $i = 1$. If $p = 1$, then $G \cong \overline{K_1}$ and we are done. For $p > 1$, G has at least one vertex of positive degree, and therefore, G is biregular and edge cosymmetric. Using Theorem 2 and the existence of an isolated vertex, $G \cong K_1 \cup K_{p-1}$.

Let $i \geq 2$. We seek to show that $\bar{G}$ is complete, so suppose to the contrary that $\bar{G}$ is not complete. Then $p \geq 4$ and G has at least one vertex of positive degree n, as well as two vertices of degree zero. Then $\bar{G}$ has at least two vertices of degree $p-1$ and at least one vertex of degree $p-1-n$. Also, $\bar{G}$ is edge symmetric and has no isolated vertices since a graph and its complement cannot both have isolated vertices. By Theorem 1, $\bar{G}$ must be bipartite with a transitive bipartition in which all vertices of $\bar{G}$ of degree $p-1$ constitute one partite set V_1. Because all vertices of degree $p-1$ are adjacent to each other, we must have $|V_1| = 1$, contradicting $i \geq 2$. Thus $\bar{G} \cong K_p$, i.e. $G \cong \overline{K_p}$. □

We also have a characterization of the regular bipartite edge cosymmetric graphs. To prepare for that result we have a theorem about regular edge cosymmetric graphs which was inspired by the observation that the cartesian product $K_n \times K_n$ and the line-graph $L(K_n)$ are edge cosymmetric for every integer $n \geq 2$, and these graphs are also strongly regular. (A regular graph G which is neither complete nor empty is said to be *strongly regular* if there exist non-negative integers s and t such that for every pair of adjacent vertices x and y of G, x and y have exactly s common neighbors and for every pair of non-adjacent vertices u and v of G, u and v have exactly t common neighbors.) For vertices $u, v \in V(G)$, let $A(u, v) = \{z \in V(G) \mid zu,\ zv \in E(G)\}$ denote the set of vertices which are adjacent to both u and v.

Lemma 4. *If G is a regular edge cosymmetric graph which is neither complete nor empty, then G is strongly regular.*

Proof. Let G be a regular edge cosymmetric graph which is neither complete nor empty. Select $e_0 = u_0v_0 \in E(G)$ and $f_0 = x_0y_0 \in E(\bar{G})$. Let u and v be arbitrary adjacent vertices of G, with $e = uv$. Then there exists $\pi \in \mathscr{A}(G)$ such that $\hat{\pi}(e) = e_0$ where $\hat{\pi} \in \mathscr{A}^*(G)$ is the edge automorphism induced by π. Since $\hat{\pi}(e) = e_0$ implies $\{\pi(u), \pi(v)\} = \{u_0, v_0\}$ it follows that $\pi(A(u, v)) = A(u_0, v_0)$, and since π is injective, $|A(u, v)| = |A(u_0, v_0)|$. For arbitrary non-adjacent vertices x and y of G with $f = xy \in E(\bar{G})$ there exists $\alpha \in \mathscr{A}(\bar{G}) = \mathscr{A}(G)$ such that $\hat{\alpha}(f) = f_0$. Again $|A(x, y)| = |A(x_0, y_0)|$ since α is injective. Thus G is strongly regular with parameters $s = |A(u_0, v_0)|$ and $t = |A(x_0, y_0)|$. □

We have not been able to prove the natural conjecture suggested by Lemma 4, namely, that all strongly regular graphs are edge cosymmetric but neither do we have a counter-example. However Lemma 4 is useful in the characterization of the regular bipartite edge cosymmetric graphs.

Theorem 5. *Let G be a bipartite graph which is regular of degree $d \neq 0$. Then G is edge cosymmetric if and only if there exists a positive integer n such that either $G \cong nK_2$ or $G \cong K(n, n)$.*

Proof. Obviously the graphs nK_2 and $K(n, n)$ are edge cosymmetric. Suppose that G is a bipartite edge cosymmetric graph which is regular of degree $d \neq 0$. Then since $d \neq 0$ we know that G is non-empty. If G is complete, then G must be K_2 because K_n is not bipartite for $n > 2$ and K_1 is regular of degree zero. Thus we need only consider the case of G neither empty nor complete whence, by Lemma 4, G is strongly regular. So there exist non-negative integers s and t such that any two adjacent vertices of G have exactly s common neighbors and any two non-adjacent vertices of G have exactly t common neighbors. Let $V(G) = V_1 \cup V_2$ be a bipartition of G, with $m = |V_1|$ and $n = |V_2|$. Then q, the size of G, satisfies $md = q = nd$, so $m = n$. Observe that $d \leq n$. We now consider two cases.

Case 1. Assume that $d < n$.

In this case there exist vertices $x \in V_1$ and $y \in V_2$ which are non-adjacent. Being in different partite sets x and y cannot have any common neighbors. Thus $t = 0$. Also since $d \neq 0$ we can find vertices $u \in V_1$ and $v \in V_2$ which are adjacent. Then by looking at common neighbors of u and v we similarly conclude that $s = 0$. Thus no two vertices of G can have any common neighbors. Then the edges of G are independent and $G \cong nK_2$.

Case 2. Assume that $d = n$.

In this case every vertex in V_1 is adjacent to every vertex in V_2, so $G \cong K(n, n)$. □

The final category of possible edge cosymmetric graphs is that of the regular non-bipartite graphs for which, unfortunately, we do not have a characterization at this stage. However, we present a class of edge cosymmetric graphs which fall largely in this category.

Theorem 6. *If G is a disconnected graph of order p, then G is edge cosymmetric if and only if either there exist integers $n \geq 1$ and $t \geq 2$ such that $G \cong tK_n$, or $G \cong K_1 \cup K_{p-1}$.*

Proof. Let G be a disconnected edge cosymmetric graph of order p. Moreover if any component of G is trivial then by Theorem 3 either $G \cong \overline{K_p} = pK_1$ or $G \cong K_1 \cup K_{p-1}$. Thus we may assume that every component of G is non-trivial. Let $G_1, \ldots, G_t$, $t \geq 2$, be the components of G and let $V_i = V(G_i)$ for $1 \leq i \leq t$. We seek to show that each G_i is complete, so suppose to the contrary that one is not. Without loss of generality we will assume that G_1 is not complete. Then we can find vertices u and v in V_1 such that $e = uv$ is an edge of $\bar{G}$. Also if x is any vertex of G_2, then $f = ux$ is an edge of $\bar{G}$. Let α be any automorphism of $\bar{G}$. Then α is also an automorphism of G, so there is some index j, $1 \leq j \leq t$ such that $\alpha(V_1) = V_j$, since α as an automorphism of G must permute the components of G.

Thus $\{\alpha(u), \alpha(v)\} \subset V_j$ so $\hat{\alpha}(e)$ cannot be the edge f which is incident with vertices in two different components of G, and we have the contradiction that $\bar{G}$ is not edge symmetric. So, as claimed, every component of G is complete. Then it is readily seen that the edge symmetry of G forces all components of G to be of the same order. Thus $G \cong tK_n$ where $n = |V_1| = |V_2| = \cdots = |V_t|$. The converse is clear. □

We know of the existence of edge cosymmetric graphs whose structure is not specified by any of the preceding theorems. These are all graphs in the regular non-bipartite category since all biregular and all regular bipartite edge cosymmetric graphs are characterized in Theorems 2 and 5. In addition to the line graphs $L(K_n)$ and the cartesian products $K_n \times K_n$, the only other example we have of an edge cosymmetric graph which is not covered by any of our structural characterizations is C_5. This is an especially interesting example since it is a self-complementary edge symmetric graph which automatically makes it edge cosymmetric, and this suggests a whole new subcategory, namely all self-complementary edge symmetric graphs.

The graph C_5 is one of an infinite family of edge symmetric self-complementary graphs known as the Paley graphs [5, p. 14], which are constructed as follows. Let q be a prime power which is congruent to 1 mod 4. Let V_q be the field with q elements and let I_q be the set of non-zero squares in V_q. The Paley graph $P(q)$ has as its vertex set V_q and its edge set $E_q = \{xy \mid x, y \in V_q \text{ and } y - x \in I_q\}$. For $q = 5$, we have $P(q) = P(5) = C_5$. It is straightforward to verify that the Paley graphs all fall in the regular non-bipartite category of edge cosymmetric graphs. However, the only Paley graphs we can specifically identify are $P(5)$ and $P(9) = K_3 \times K_3$.

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