Discrete Mathematics 310 (2010) 838–842

Contents lists available at ScienceDirect

Discrete Mathematics

journal homepage: www.elsevier.com/locate/disc

Every tree is a large subtree of a tree that decomposes K_n or $K_{n,n}$

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ARTICLE INFO

Article history:
Received 8 December 2008
Received in revised form 20 September 2009
Accepted 22 September 2009
Available online 2 October 2009

Keywords:
Graph labelings
Graph decompositions
Combinatorial nullstellensatz

ABSTRACT

Let T be a tree with m edges. A well-known conjecture of Ringel states that T decomposes the complete graph K_{2m+1}. Graham and Häggkvist conjectured that T also decomposes the complete bipartite graph $K_{m,m}$. In this paper we show that there exists an integer n with $n \leq \lceil (3m-1)/2 \rceil$ and a tree T_1 with n edges such that T_1 decomposes K_{2n+1} and contains T. We also show that there exists an integer n' with $n' \geq 2m-1$ and a tree T_2 with n' edges such that T_2 decomposes $K_{n',n'}$ and contains T. In the latter case, we can improve the bound if there exists a prime p such that $\lceil 3m/2 \rceil \leq p < 2m-1$.

1. Introduction

A *decomposition* of a graph G is a partition $\mathcal{P}$ of its set of edges. When the graph induced by each part of $\mathcal{P}$ is isomorphic to a graph H, we say that H *decomposes* G and write $H|G$.

A famous conjecture of Ringel from 1963 states that every tree with m edges decomposes the complete graph K_{2m+1} [13]. In spite of the hundreds of papers that have appeared in the literature on the subject (see the dynamic survey of Gallian [5]), Ringel's conjecture is still wide open. Graham and Häggkvist proposed the following generalization of Ringel's conjecture; see, *e.g.*, [6]:

Conjecture 1 (*Graham and Häggkvist*). *Every tree with m edges decomposes every $2m$-regular graph and every bipartite m-regular graph.*

Conjecture 1 in particular asserts that every tree with m edges decomposes the complete bipartite graph $K_{m,m}$. In the sequel we will refer to this particularization of Conjecture 1.

Both conjectures are known to hold for caterpillars, for trees of diameter at most five and for various particular families of trees.

In one of the early papers on the subject, Kotzig [10] showed that the substitution of an edge by a sufficiently large path in an arbitrary tree results in a tree T for which Ringel's conjecture holds. Thus every tree is homeomorphic to a tree for which the conjecture holds. On the other hand Kézdy [8] showed that the addition of an unspecified number of leaves to a vertex of a tree results in a tree with n edges that decomposes K_{2n+1}. An analogous result for the decomposition of $K_{n,n}$ was proved in [11]. Therefore, every tree contains the base tree of some tree for which both conjectures hold (the base tree of a tree is obtained by removing all its leaves). However, neither result gives a quantitative estimate of the number of additional vertices that will suffice to make a tree decompose the appropriate complete graph.

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doi:10.1016/j.disc.2009.09.021

In this paper we consider an approximation to both conjectures and prove that every tree is a large subtree of two trees for which the conjectures hold respectively. We prove:

Theorem 1. *Let T be a tree with m edges.*
(i) *For every odd $n \geq 2m-1$, there exists a tree T' with n edges that decomposes $K_{n,n}$ and contains T.*
(ii) *For every prime $p \geq \lceil 3m/2 \rceil$, there exists a tree T' with p edges that decomposes $K_{p,p}$ and contains T.*

Theorem 2. *Let T be a tree with m edges. For every $n \leq \lceil (3m-1)/2 \rceil$, there exists a tree T' with n edges that decomposes K_{2n+1} and contains T.*

2. The tools

The classical approach to the decomposition problem of graphs uses labeling techniques that aim to find *cyclic* decompositions. A tree T with m edges cyclically decomposes K_{2m+1} if there is an injection $\phi : V(T) \to [0, 2m]$ such that the translations $\phi(v) + k (\text{mod } 2m+1)$ give $2m+1$ edge-disjoint copies of T. Similarly, T cyclically decomposes $K_{m,m}$ if there is a map $\phi : V(T) \to [0, m-1]$ that is injective on each partite set of T such that the translations $\phi(v) + k (\text{mod } m)$ produce m edge-disjoint copies of T.

A ρ-valuation of a graph H on m edges is an injection $\rho : V(H) \to \mathbb{Z}_{2m+1}$ such that the induced edge labels $\rho_E(uv) := \rho(u) - \rho(v)$, for $uv \in E(H)$, satisfy

$$\rho_E(e) \neq \pm \rho_E(f) \pmod{2m+1},$$

for all distinct pairs of edges $e, f \in E(H)$. Rosa [14] proved that a graph H with m edges cyclically decomposes K_{2m+1} if and only if it admits a ρ-valuation.

Similarly, a modular bigraceful labeling of a bipartite graph H with m edges and partite sets A and B is a map $f : V(H) \to \mathbb{Z}_m$ that is injective in each stable set and has the property that the values $f(v) - f(u)$ with $u \in A$ and $v \in B$ are different for distinct edges. It is shown by Lladó and López [11] that if H admits a modular bigraceful map then it cyclically decomposes $K_{m,m}$.

To prove Theorems 1 and 2 we shall show that a tree T with m edges can be embedded in a tree of the stated size that admits either a modular bigraceful labeling or a ρ-valuation. One of the ingredients of our proofs is the polynomial method of Alon [1]. In particular we shall use the following theorem of Alon:

Theorem 3 (*[1]*). *Let F be an arbitrary field, and let $f = f(x_1, \ldots, x_n)$ be a polynomial in $F[x_1, \ldots, x_n]$. Suppose the degree $\deg(f)$ of f is $\sum_{i=1}^n t_i$, where each t_i is a nonnegative integer, and suppose the coefficient of $\prod_{i=1}^n x_i^{t_i}$ in f is nonzero. Then, if $S_1, \ldots, S_n$ are subsets of F with $|S_i| > t_i$, there exist $s_1 \in S_1, s_2 \in S_2, \ldots, s_n \in S_n$ such that*

$$f(s_1, \ldots, s_n) \neq 0.$$

Applications of the polynomial method to other related graph labeling problems can be found in [4,7–9].

We also use a well-known theorem of Kneser. Recall that the stabilizer $H(C)$ of a nonempty subset C in an abelian group $\mathcal{G}$ is defined by $H(C) = \{g \in \mathcal{G} : C + g = C\}$. In other words, $H(C)$ is the largest subgroup of $\mathcal{G}$ that has the property $H(C) + C = C$. If $\mathcal{G}$ is finite, then $|H(C)|$ divides both $|\mathcal{G}|$ and $|C|$, since $H(C)$ is a subgroup of $\mathcal{G}$ and C is a union of cosets of this subgroup.

Theorem 4 (*Kneser; see, e.g., [12]*). *If A and B are finite nonempty subsets of an abelian group satisfying $|A+B| \leq |A| + |B| - 1$, and H is the stabilizer of $A+B$, then*

$$|A+B| = |A+H| + |B+H| - |H|.$$

The next lemma, which is based on Kneser's Theorem, will be used later to prove the existence of appropriate labelings.

Lemma 1. *Let r be a positive integer and let X_1, X_2, Y be nonempty subsets of $\mathbb{Z}_r$ with $|X_1| \geq |X_2|$ and $|Y| > 1$. If the following condition holds:*

$$r - |X_1| - |X_2| = |Y| - 1, \tag{1}$$

then $|X_1 + Y| > |X_2|$.

Proof. If $|X_1 + Y| \leq |X_2|$, then we must have $|X_1 + Y| = |X_2| = |X_1| < |X_1| + |Y| - 1$. By Kneser's Theorem,

$$|X_1 + Y| = |X_1 + H| + |Y + H| - |H|,$$

where H is the stabilizer of $X_1 + Y$. From this relation and $|X_1 + Y| = |X_1|$ we deduce that $|Y + H| = |H|$ and therefore $|Y| \leq |H|$.

Now, since $|H|$ divides the left hand side of (1), $|H|$ must also divide $|Y| - 1$. Finally, $|Y| > 1$ implies that $|H| \leq |Y| - 1$, contradicting $|Y| \leq |H|$. □

3. Proof of Theorem 1

Here we consider an extension of the modular bigraceful labeling defined by Cámara, Lladó and Moragas [4], which takes values in an arbitrary abelian group. Let H be a bipartite graph with partite sets A and B, and let $(\mathcal{G}, +)$ be an abelian group. A map $f : A \cup B \to \mathcal{G}$ is *$\mathcal{G}$-bigraceful* if the restrictions of f to each stable set are injective maps and the induced values of f over the edges of H are distinct, where the induced value on an edge uv with $u \in A$ and $v \in B$ is $f(v) - f(u)$. Note that a modular bigraceful labeling of a tree with m edges is a $\mathbb{Z}_m$-bigraceful labeling.

We first show that a tree that admits a $\mathbb{Z}_n$-bigraceful map can be embedded in a tree with n edges that decomposes $K_{n,n}$.

Lemma 2. *Every tree T that admits a $\mathbb{Z}_n$-bigraceful map with n odd is a subtree of a tree T' with n edges that admits a modular bigraceful labeling.*

Proof. Let m be the number of edges of T. Let f be a $\mathbb{Z}_n$-bigraceful map of T. Clearly $n \geq m$. We define a sequence of trees $T_m, T_{m+1}, \dots, T_n$, with $T_m = T$ and $T_n = T'$, by adding one leaf at each step, and extend f on T' as a modular bigraceful map.

Suppose we have defined T_i and a $\mathbb{Z}_n$-bigraceful map f on T_i for some i such that $m \leq i < n$. Let A_i and B_i be the two stable sets of T_i with $|A_i| \geq |B_i|$ (we may assume this by exchanging f for $f_r = n + 1 - f$ if necessary). Let $A'_i = f(A_i)$, $B'_i = f(B_i)$, and $C_i = \{f(y) - f(x) : xy \in E(T_i), x \in A_i, y \in B_i\}$, and let $D_i = \mathbb{Z}_n \setminus C_i$. Since T_i is a tree, we have the following relation among these sets:

$$|A_i| + |B_i| = n - |D_i| + 1. \tag{2}$$

It suffices to prove that $|D_i + A'_i| > |B_i|$. In this case there exists $d \in D_i$ and some $a \in A'_i$ such that $d + a \in \mathbb{Z}_n \setminus B'_i$. Define $T_{i+1} = T_i + e_{i+1}$, where e_{i+1} joins the vertex in A_i labeled a to a new vertex v_{i+1} and $f(v_{i+1}) = d + a$; this gives the extension of f to T_{i+1}.

Since $|D_i| = n - |C_i| = n - i \geq 1$, either $|D_i| = 1$ or $|D_i| > 1$. In the former case ($i = n - 1$), since n (which equals $|A_{n-1}| + |B_{n-1}|$) is odd, $|D_{n-1} + A'_{n-1}| = |A_{n-1}| > |B_{n-1}|$. In the latter case, we apply Lemma 1 with $r = n$, $X_1 = A'_i$, $X_2 = B'_i$ and $Y = D_i$. The condition (1) of the lemma holds by (2). $\square$

In view of Lemma 2, and using the cyclic decomposition from [11], to prove the statement (i) of Theorem 1 it suffices to show that every tree T with m edges admits a $\mathbb{Z}_n$-bigraceful labeling for every odd $n \geq 2m - 1$. The next lemma shows that this is indeed the case.

Lemma 3. *Every tree T with m edges admits a $\mathbb{Z}_n$-bigraceful map for every n such that $n \geq m + \max\{|A|, |B|\} - 1$, where A and B are the partite sets of T.*

Proof. The proof is by induction on m, the result being obvious for $m = 1$. Let u be a leaf of T with neighbor v, let $T' = T - u$, choose an integer n such that $n \geq m + \max\{|A|, |B|\} - 1$, and let f be a $\mathbb{Z}_n$-bigraceful map on T'. Let $C = \{f(y) - f(x) : xy \in E(T'), x \in A, y \in B\}$ and $D = \mathbb{Z}_n \setminus C$. Since $|f(v) - D| = |D| = n - m + 1 \geq |A|$, there exists $d \in D$ such that $f(v) - d \notin f(A \setminus \{u\})$. Extending f to T by defining $f(u) = f(v) - d$ produces a $\mathbb{Z}_n$-bigraceful labeling of T. $\square$

Statement (ii) of Theorem 1 may give a better upper bound for the minimum n for which we can ensure that there is a tree T' with n edges containing a given tree T with the property that T' decomposes $K_{n,n}$. We use the following simple lemma.

Lemma 4. *A tree T with partite sets A and B such that $|A| \geq |B|$ has at least $|A| - |B| + 1$ leaves in A.*

Proof. Let $A' \subset A$ be the set of nonleaves in A, and let $T' = T - (A \setminus A')$. Then $|A'| + |B| - 1 = |E(T')| = \sum_{x \in A'} d(x) \geq 2|A'|$. Hence $|A'| \leq |B| - 1$, and T has at least $|A| - |A'| \geq |A| - |B| + 1$ leaves in A. $\square$

Lemma 5. *Let T be a tree with m edges. If p is a prime such that $p \geq \lceil 3m/2 \rceil$, then there is a $\mathbb{Z}_p$-bigraceful map of T.*

Proof. Let A and B be the partite sets of T, labeled so that $|A| \geq |B|$. By Lemma 4 there is a set $A_0 \subset A$ of leaves such that $|A'| = |A \setminus A_0| = |B|$. Let $T' = T - A_0$. Since $|B| \leq \lceil m/2 \rceil$ and $p \geq m + |B|$, it follows from Lemma 3 that there is a $\mathbb{Z}_p$-bigraceful map f' of T'. If $A_0 = \emptyset$ then $T' = T$ and we are done. Otherwise, let C' denote the set of edge values of f'. Thus C' is a subset of $\mathbb{Z}_p$ of cardinality $2|A'| - 1$.

Let $A_0 = \{a_1, \dots, a_k\}$. Let $b_{\sigma(i)}$ be the vertex in B adjacent to a_i, for $1 \leq i \leq k$. Consider the polynomial $P \in \mathbb{Z}_p[z_1, \dots, z_k]$ defined as

$$P(z_1, \dots, z_k) = \prod_{1 \leq i < j \leq k} (z_i - z_j) \prod_{1 \leq i < j \leq k} (b'_{\sigma(i)} - z_i - (b'_{\sigma(j)} - z_j)) \prod_{1 \leq i \leq k} \prod_{a \in A'} (b'_{\sigma(i)} - z_i - a'),$$

where $b'_{\sigma(i)} = f'(b_{\sigma(i)})$ and $a' = f'(a)$. We can write

$$P = (-1)^{k(k-1)/2 + |A'|} \prod_{1 \leq i < j \leq k} (z_i - z_j)^2 \prod_{1 \leq i \leq k} z_i^{|A'|} + \text{ terms of lower degree.}$$

It is known that the coefficient of the monomial $\prod_{i=1}^{k} z_i^{k-1}$ in the expansion of $\prod_{1\le i<j\le k}(z_i-z_j)^2$ is $(-1)^{\binom{k}{2}}k!$ (see, *e.g.*, [2]), which is nonzero modulo p. Therefore P has a monomial

$$z_1^{k+|A'|-1}\cdots z_k^{k+|A'|-1}$$

of maximum degree with nonzero coefficient. Let $D=\mathbb{Z}_p\setminus C'$. Note that $|D|=p-|C'|\ge\lceil 3(2|A'|+k-1)/2\rceil-2|A'|+1\ge|A'|+k$. By Alon's Theorem, there exist $d_1,\dots,d_k\in D$ such that $P(d_1,\dots,d_k)\neq 0$. Extend f' on T' to f on T by defining $f(a_i)=f'(b_{\sigma(i)})-d_i$. Since $\prod_{1\le i\le k}\prod_{a\in A'}(b'_{\sigma(i)}-d_i-a')\neq 0$, the values of f on A_0 are different from the ones on A'; since $\prod_{1\le i<j\le k}(b'_{\sigma(i)}-d_i-(b'_{\sigma(j)}-d_j))\neq 0$, these values are pairwise distinct. Finally, since $\prod_{1\le i<j\le k}(d_i-d_j)\neq 0$, the edge values $d_1,\dots,d_k$ on the edges incident to $a_1,\dots,a_k$ are distinct and, since $d_i\in\mathbb{Z}_p\setminus C'$, they are different from the ones taken by f on T'. Thus f is a $\mathbb{Z}_p$-bigraceful map of T. □

Theorem 1(ii) follows from Lemmas 5 and 2, and using the cyclic decomposition from [11].

4. Extension to ρ-valuation

Following the ideas of the proof of Theorem 1, we give an upper bound for the number of edges that have to be added to an arbitrary tree T to obtain a tree that admits a ρ-valuation in terms of the size of T.

For our present purposes we define a relaxation in the definition of a ρ-valuation. Given a graph H with m edges and given $n\ge m$, a ρ_n-valuation is an injection $\rho_n : V(H)\to\mathbb{Z}_{2n+1}$ such that the induced edge labels defined as before (but now taking the differences modulo $2n+1$) are distinct.

Lemma 6. *Every tree T with m edges has a ρ_n-valuation for every $n\ge\lceil(3m-1)/2\rceil$.*

Proof. Let $T_1,T_2,\dots,T_m$ be trees such that $T_m=T$, T_1 has one edge v_0v_1, and T_{i+1} is obtained from T_i by adding a leaf v_{i+1} adjacent to some $u\in V(T_i)$. Define a ρ_n-valuation of T inductively as follows. Define $f(v_0)=x_0\in\mathbb{Z}_{2n+1}$, $f(v_1)=x_1\in\mathbb{Z}_{2n+1}$ arbitrarily, with $x_0\neq x_1$. Suppose f is defined on T_i for $1\le i<m$, and let $V_i=f(V(T_i))$, $C_i=\{\pm(f(x)-f(y)) : xy\in E(T_i)\}\cup\{0\}$, and $D_i=\mathbb{Z}_{2n+1}\setminus C_i$. Since $|D_i+f(u)|=|D_i|=2n+1-2i-1\ge m+1>|V_i|$, there exists $d\in D_i$ such that $d+f(u)\in\mathbb{Z}_{2n+1}\setminus V_i$. Thus we can define $f(v_{i+1})=d+f(u)$. At the end, we have a ρ_n-valuation of T. □

Lemma 7. *Every tree T of size m that admits a ρ_n-valuation for $n\ge m$ can be embedded into a tree T' of size n that admits a ρ-valuation.*

Proof. If $n=m$ we are done. Otherwise, let f be the ρ_n-valuation of T. We define a sequence of trees $T_m,T_{m+1},\dots,T_n$ with $T_m=T$ and $T_n=T'$, by adding one leaf at each step, and extend f to T' as a ρ-valuation.

Suppose we have defined T_i and a ρ_n-valuation f on T_i for some i such that $m\le i<n$. Let $V_i=f(V(T_i))$, $C_i=\{\pm(f(x)-f(y)) : xy\in E(T_i)\}\cup\{0\}$, and $D_i=\mathbb{Z}_{2n+1}\setminus C_i$.

Since T_i is a tree, we have the following relation:

$$2|V_i|-1=2n+1-|D_i|. \tag{3}$$

Since $|D_i|=2n+1-|C_i|=2n-2i\ge 2$ we can apply Lemma 1 with $r=2n+1$, $X_1=X_2=V_i$, and $Y=D_i$ to obtain $|D_i+V_i|>|V_i|$. By (3), condition (1) of Lemma 1 holds. Therefore there exists $d\in D_i$ and some $a\in V_i$ such that $d+a\in\mathbb{Z}_{2n+1}\setminus V_i$. Let $T_{i+1}=T_i+e_{i+1}$ where e_{i+1} joins the vertex in V_i labeled with a to a new vertex v_{i+1}. By defining $f(v_{i+1})=d+a$ we extend f to a ρ_n-valuation of T_{i+1}. By iterating this procedure we eventually get a ρ-valuation of a tree T' that contains T as a subtree. □

Theorem 2 is a direct consequence of Lemmas 6 and 7, and the fact that a graph with m edges cyclically decomposes K_{2m+1} if and only if it admits a ρ-valuation (Rosa [14]).

Another related result is given by Van Bussel [3, Theorem 1]; it implies that every tree with m edges has a ρ_n-valuation, with $n=2m-\mathrm{diam}(T)$. Since a random tree has diameter of order $\sqrt{n}$, this lower bound is in general worse than the one obtained in Theorem 2 (see also Lemma 6).

Acknowledgements

This work was supported by the Ministry of Science and Education of Spain under project MTM2005-08990-C02-01 and by the Catalan Research Council under grant 2005SGR00256. The authors also want to thank the anonymous referees for valuable suggestions that were helpful in improving the presentation of this paper.

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