Dynamics of the thermohaline circulation under uncertainty

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Abstract

The ocean thermohaline circulation under uncertainty is investigated by a random dynamical systems approach. It is shown that the asymptotic dynamics of the thermohaline circulation is described by a random attractor and by a system with finite degrees of freedom.

Keywords: Stochastic PDEs; Random dynamical systems; Random attractors; Finite-dimensional behavior; Geophysical flows

1. Introduction

The ocean thermohaline circulation (THC) involves water masses sinking at high latitudes and upwelling at lower latitudes. The process is maintained by water density contrasts in the ocean, which themselves are created by atmospheric forcing, namely, heat and freshwater exchange via evaporation and precipitation at the air–sea interface. Thus the ocean THC is driven by fluxes of heat and freshwater through the air–sea interface. During the
THC, water masses carry heat or cold around the globe. Thus, it is believed that the global ocean THC plays an important role in the climate [21].

The formulation and analysis of mathematical models is central to the progress of better understanding of the THC dynamics and its impact on climate change. Apart from a detailed modeling of the climate system using coupled general circulation models, sometimes simplified climate models could give insight into the general characteristics of the climate system. The most simplified climate models can be described in terms of stochastic differential equations [3,14,15]. These stochastic climate models can be viewed as comprehensive paradigms or metaphors for particular features of the climate system.

We consider a two-dimensional thermohaline ocean circulation model in the latitude-depth (meridional) plane, in terms of the stochastic Navier–Stokes fluid equations (vorticity form) and the transport equations for heat and salinity, together with air–sea flux or Neumann boundary conditions. The noise in the Navier–Stokes equations is due to various fluctuations such as random wind stress forcing.

We intend to investigate the characteristics of the THC dynamics when some random effect is taken into account. We use a random dynamical systems approach [2].

This paper is organized as follows. In the next section we present the THC model, and discuss the well-posedness of this model in Section 3. Section 4 is devoted to the investigation of the dynamical behavior of this model: random attractor and finite dimensionality.

2. A model for the thermohaline circulation

We consider the ocean thermohaline circulation in a bounded domain, i.e., a square

\[ D = \{(y, z) : -l \leq y \leq l, \ 0 \leq z \leq d\}, \]

on the meridional, latitude-depth \((y, z)\)-plane, as used by various authors [4,10,18,19,23]. It is composed of the Boussinesq version of the Navier–Stokes equations for oceanic fluid velocity \((v(y, z, t), w(y, z, t))\) and transport equations for the oceanic salinity \(S(y, z, t)\) and the oceanic temperature \(T(y, z, t)\) in dimensional form:

\[
\begin{align*}
    v_t + vv_y + vw_z &= -p_y + v\Delta v + \text{noise}, \\
    w_t + vw_y + ww_z &= -p_z - g(\alpha_S S - \alpha_T T) + v\Delta w + \text{noise}, \\
    v_y + w_z &= 0, \\
    T_t + vT_y + wT_z &= \kappa_T \Delta T, \\
    S_t + vS_y + wS_z &= \kappa_S \Delta S,
\end{align*}
\]

where \(\alpha_S\) and \(\alpha_T\) are the coefficients of volume expansion for salt and heat, respectively; \(g\) is the gravitational acceleration; \(v\) is the viscosity; and \(\kappa_S\) and \(\kappa_T\) are salt and heat diffusivities, respectively. The density is \(\rho = \rho_0(1 + \alpha_S S - \alpha_T T)\) with \(\rho_0\) the mean sea water density. The noise in the Navier–Stokes equations is due to various fluctuations such as random wind stress forcing [13,16,24]. Presumably, the noise also affects the transport of heat and salinity to some extent, but we will ignore this effect.
As discussed in [18, 23], this may be regarded as a zonally averaged model of the world ocean. The effect of the rotation can be parameterized in the magnitude of the viscosity and diffusivity terms. Introducing the stream function $\psi(y, z, t)$ for the velocity field,

$v = -\psi_z, \quad w = \psi_y,$

we can rewrite the above model in the vorticity form with only three unknowns $\psi, T, S$:

\begin{align*}
\Delta \psi_t + J(\psi, \Delta \psi) &= g(\alpha_T T_y - \alpha_S S_y) + \nu \Delta^2 \psi + \dot{W}_1, \quad (1) \\
T_t + J(\psi, T) &= \kappa_T \Delta T, \quad (2) \\
S_t + J(\psi, S) &= \kappa_S \Delta S, \quad (3)
\end{align*}

where $\dot{W}_1(y, z, t)$ is a Wiener process defined on an underlying probability space $(\Omega, \mathcal{F}, P)$. The noise is described by the generalized time-derivative of the Wiener process. The fluctuating noise in the oceanic fluid equation is usually of a shorter time scale than the response time scale of the large scale oceanic THC. We thus assume the noise is white in time (uncorrelated in time) but it is allowed to be colored in space, i.e., it may be correlated in space variables. Note that the pressure field $p$ is eliminated in the vorticity form of the Navier–Stokes equations.

Boundary conditions for the oceanic fluid are no normal flow and free-slip (no-stress) on the whole boundary [17]:

$\psi = \Delta \psi = 0 \quad \text{on} \quad \partial D. \quad (4)$

The boundary conditions for temperature and salinity are Neumann type. On the air–sea interface $z = d$, the heat/temperature flux and freshwater/salinity flux are prescribed as

$T_z(y, d) = \lambda (\theta(y) - T), \quad S_z(y, d) = F(y). \quad (5)$

Here $\theta(y)$ is the prescribed (known) atmosphere surface temperature and $F(y)$ is the mean freshwater flux (known). Moreover, $\lambda = BT / (\rho_0 C_p \kappa_T)$, with $B_T$ being the surface exchange coefficient of heat and $C_p$ the heat capacity.

Zero flux boundary conditions are taken for $T$ and $S$ on fluid bottom $(z = 0)$ and on fluid side $(y = \pm l)$:

$T_z(y, 0) = 0, \quad S_z(y, 0) = 0, \quad (6)$

$T_y(\pm l, z) = 0, \quad S_y(\pm l, z) = 0. \quad (7)$

The THC model above involves stochastic and deterministic partial differential equations (PDEs) and Neumann boundary conditions.

3. Cocycle property

In this section we will show that (1)–(3) defines a well-posed model. First, we introduce some function spaces from the theory of partial differential equations.
Let $H^1(D)$ be the Sobolev space of function on domain $D$ with the first generalized derivative in $L_2(D)$, the function space of square integrable functions on $D$ with norm and inner product

$$
\|u\|_{L_2(D)} = \left( \int_D |u(x)|^2 \, dD \right)^{1/2}, \quad (u, v)_{L_2(D)} = \int_D u(x)v(x) \, dD, \quad u, v \in L_2(D).
$$

The space $H^1(D)$ is equipped with the norm

$$
\|u\|_{H^1(D)} = \|u\|_{L_2(D)} + \|\partial_1 u\|_{L_2(D)} + \|\partial_2 u\|_{L_2(D)}.
$$

Let the $\dot{H}^1(D)$ be the space of functions vanishing on the boundary $\partial D$ of $D$. The norm of this space is defined as

$$
\|u\|_{\dot{H}^1(D)} = \|\partial_1 u\|_{L_2(D)} + \|\partial_2 u\|_{L_2(D)}. \quad (8)
$$

Another Sobolev space is $\dot{H}^1(D)$ which is a subspace of $H^1(D)$ consisting of functions $u$ with zero mean: $\int_D u \, dD = 0$. A norm equivalent to the $H^1$-norm on $\dot{H}^1(D)$ is given by the right-hand side of (8). For the space of functions in $L_2(D)$ having the same property, we denote it as $\dot{L}_2(D)$.

There exists a continuous trace operator

$$
\gamma_{\partial D} : H^1(D) \rightarrow H^{1/2}(\partial D).
$$

Here $H^{1/2}(\partial D)$ is a boundary space, see Adams [1, Chapter 7]. Similarly, we can introduce trace operators that map onto a part of the boundary of $\partial D$, for instance, for the subset $\{(y, z) \in \bar{D} : z = 2 \}$. For this mapping we will write

$$
\gamma_{\partial D} : H^1(D) \rightarrow H^{1/2}(0, d). \quad (9)
$$

The adjoint operator

$$
\gamma_{\partial D}^* : (H^{1/2}(0, d))' \rightarrow (H^1(D))^'.
$$

is also continuous, where $'$ denotes the dual space for a given Banach space.

Using the vorticity $q = w_y - v_z = \Delta \psi$, we can homogenize boundary conditions to obtain:

$$
q_1 + J(\psi, q) = g(\sigma_T T_y - \sigma_S S_y) + v \Delta q + \tilde{V}_1, \quad (10)
$$

$$
T_1 + J(\psi, T) = \kappa_T \Delta T + \gamma_{\partial D}^* \{\lambda (\theta(y) - \gamma_{\partial D} T)\}, \quad (11)
$$

$$
S_1 + J(\psi, S) = \kappa_S \Delta S + \gamma_{\partial D}^* F(y). \quad (12)
$$

New homogeneous boundary conditions are:

$$
\psi = 0, \quad q = 0 \quad \text{on} \ \partial D,
$$

$$
T_\gamma(y, d) = 0, \quad S_\gamma(y, d) = 0,
$$

$$
T_\gamma(y, 0) = 0, \quad S_\gamma(y, 0) = 0,
$$

$$
T_\gamma(\pm l, z) = 0, \quad S_\gamma(\pm l, z) = 0.
$$
For convenience, we introduce the vector notation for unknown geophysical quantities
\[ u = (q, T, S). \] (13)

Now we can define the linear differential operator from (10)–(12),
\[ Au = \begin{pmatrix} -\nu \Delta q \\ -\kappa_T \Delta T \\ -\kappa_S \Delta S \end{pmatrix}. \]

We assume that \( F \) and \( \theta \in L^2(0, d) \). Note that
\[ \frac{d}{dt} \int_D S \, dy \, dz = \int_0^d F(y) \, dy = \text{constant}. \]

It is reasonable (see [10]) to assume that
\[ \int_0^d F(y) \, dy = 0 \]
and thus \( \int_D S \, dy \, dz \) is constant in time and we may assume it is zero,
\[ \int_D S \, dy \, dz = 0. \]

Thus we have the usual Poincaré inequality for \( S \). However, this is not true for \( T \). Fortunately we can derive the following Poincaré inequality for \( T \),
\[ \| T \|^2 \leq c(\Omega) \left( \| \gamma_{\gamma=1} T \|_{L^2}^2 + \| \nabla T \|_{L^2}^2 \right), \] (14)
as in Temam [22], where \( c(\Omega) \) is a constant dependent on \( \Omega \).

Introduce the phase space for our system \( H = L^2(D) \times L^2(D) \times \dot{L}^2(D) \) with the usual \( L^2 \) inner product and \( V = H^1(D) \times H^1(D) \times \dot{H}^1(D) \).

It is obvious that the nonlinear operator \( G(u) := G_1(u) + G_2(u) \) where
\[ G_1(u)[y, z] = \begin{pmatrix} -J(\psi, q) \\ -J(\psi, T) \\ -J(\psi, S) \end{pmatrix}[y, z] \]
and
\[ G_2(u)[y, z] = \begin{pmatrix} g(\alpha_T T - \alpha_S S) \\ \gamma^\alpha_{\gamma=1} \left( \lambda_0(\theta(y) - \gamma^\alpha_{\gamma=1} T) \right) \\ \gamma^\alpha_{\gamma=1} F(y) \end{pmatrix}[y, z]. \]

Then the THC system can be rewritten as a stochastic evolution equation on \( V' \):
\[ \frac{du}{dt} + Au = G(u) + \dot{W}, \quad u(0) = u_0 \in H, \] (15)
where \( W = (W_1, 0, 0) \) is a Wiener process, with continuous trajectories on \( \mathbb{R} \) and with values in \( L^2(D) \). We assume that this Wiener process has zero mean vector and has covariance
operator $Q$. The spatially correlated white noise, $\dot{W}$, is the generalized temporal derivative of the Wiener process $W$. We further assume that the covariance operator $Q$ is of finite trace with respect to the space $L^2(D): \text{tr}_L^2 Q < \infty$. We can choose the canonical probability space $(\Omega, \mathcal{F}, \mathbb{P})$ where the set of elementary events, $\Omega$, consists of the paths of $W$ and the probability measure $\mathbb{P}$ is the Wiener measure with respect to covariance operator $Q$.

Through integration by parts and direct estimation or from [5] we have the following lemmas.

**Lemma 3.1.** The operator $G_1 : V \rightarrow H$ is continuous. In particular, we have

$$\langle G_1(u), u \rangle = 0.$$  

**Lemma 3.2.** The following estimation holds:

$$\| G_2(u) \|_V \leq c_1 \| u \|_V + c_2$$

for some positive constants $c_1, c_2$.

In order to convert the above stochastic evolution equation (15) into an evolution equation with random coefficients (which is easier to show to generate a random dynamical system), we consider a stationary Ornstein–Uhlenbeck process $\eta(t)$ solving the following linear stochastic evolution equation on $D$:

$$\frac{d\eta}{dt} - \nu(k+1)\Delta \eta = \dot{W}_1$$

with the homogeneous Dirichlet boundary condition $\eta = 0$ on $\partial D$. Here $k$ can be chosen as a very large controlling parameter.

**Lemma 3.3.** Suppose that the covariance $Q$ has a finite trace $\text{tr}_L^2 Q < \infty$. Then (16) has a unique stationary solution generated by

$$(t, \omega) \mapsto \eta(\theta t \omega),$$

where $\eta$ is a tempered random variable in $H^1(D)$ and has trajectories in the space $L^2_{\text{loc}}(\mathbb{R}; H)$. Moreover, $Z(\omega) = (\eta(\omega), 0, 0)$ is a random variable in $V$.

For the proof we refer to [8]. If we set

$$(\tilde{\eta}, T, S) = v := u - Z = (q - \eta, T, S),$$

then we obtain a random differential equation in $V'$,

$$\frac{dv}{dt} + Av = G_1(v) + \tilde{G}_2(v + Z(\theta t \omega)), \quad v(0) = v_0 \in H,$$

where $\tilde{G}_2 = G_2 + (J(\eta, q) + J(\Delta^{-1}\tilde{q}, \Delta \eta) + J(\eta, \Delta \eta) - \nu k \Delta \eta, 0, 0)$.

The above Eq. (18) is a differential equation with random coefficients then it can be treated sample-wise for any sample $\omega$. We are looking for solution $v$ in $C([0, \tau]; H) \cap L^2(0, \tau; V)$.
for all $\tau > 0$. If we can solve this equation, then $u := v + Z$ defines a solution version of (15). For the well-posedness of the problem we now have the following result.

**Theorem 3.4** (Well-posedness). For any time $\tau > 0$, there exists a unique solution of (18) in $C([0, \tau]; H) \cap L^2(0, \tau; V)$. In particular, the solution mapping

$$\mathbb{R}^+ \times \Omega \times H \ni (t, \omega, v_0) \mapsto v(t) \in H$$

is measurable in its arguments and the solution mapping $H \ni v_0 \mapsto v(t) \in H$ is continuous.

**Proof.** By the properties of $A$ and $G_1$ (see Lemma 3.1), the random differential equation (18) is essentially similar to the 2-dimensional Navier–Stokes equation. Note that $\tilde{G}_2$ is only an affine mapping. Hence, we have existence and uniqueness and the above regularity assertions.

Now we can define a random dynamical system since the solution mapping

$$\mathbb{R}^+ \times \Omega \times H \ni (t, \omega, v_0) \mapsto v(t, \omega, v_0) =: \psi(t, \omega, v_0) \in H$$

is well-defined. First, we define a so-called metric dynamical system $(\Omega, \mathcal{F}, P, (\theta_t)_{t \in \mathbb{R}})$. $\theta_t : \Omega \to \Omega, t \in \mathbb{R}$ is a family of measure preserving transformations such that $(t, \omega) \mapsto \theta_t \omega$ is measurable, $\theta_0 = \text{id}$, $\theta_t \theta_s = \theta_{t+s}$, for all $s, t \in \mathbb{R}$. Furthermore, the shift $\theta_t$ is ergodic if we define it as

$$w(\cdot, \theta_t \omega) = w(\cdot + t, \omega) - w(t, \omega) \quad \text{for} \ t \in \mathbb{R},$$

which is called the Wiener shift. A random dynamical system $\psi(t, \omega, u)$ is well-defined now with the cocycle property

$$\psi(t + \tau, \omega, u) = \psi(t, \theta_t \omega, \psi(\tau, \omega, u)) \quad \text{for} \ t, \tau \geq 0,$$

$\psi(0, \omega, u) = u$

for any $\omega \in \Omega$ and $u \in H$. For more detail about random dynamical systems we refer to [2].

**4. Random dynamics**

In this section, we investigate random dynamics of the THC. First, we are going to show that the random THC model is dissipative, i.e., it has an random absorbing set in the following sense:

**Definition 4.1.** A random set $B = \{B(\omega)\}_{\omega \in \Omega}$ consisting of closed bounded sets $B(\omega)$ is called absorbing for a random dynamical system $\psi$ if we have for any random set $D = \{D(\omega)\}_{\omega \in \Omega}$, $D(\omega) \in H$ bounded, such that $t \mapsto \sup_{y \in D(\theta_t \omega)} \|y\|_H$ has a subexponential growth for $t \to \pm \infty$,

$$\psi(t, \omega, D(\omega)) \subset B(\theta_t \omega) \quad \text{for} \ t \geq t_0(D, \omega),$$

$$\psi(t, \theta_{-t} \omega, D(\theta_{-t} \omega)) \subset B(\omega) \quad \text{for} \ t \geq t_0(D, \omega).$$

(19)
B is called forward invariant if
\[ \phi(t, \omega, u_0) \in B(\theta_t \omega) \quad \text{if} \quad u_0 \in B(\omega) \quad \text{for} \quad t \geq 0. \]

We now consider the equations for temperature T and salinity S in THC system separately from the equation for vorticity q. For this we introduce the spaces
\[ \tilde{H} = \mathbb{L}^2(D) \times \mathbb{L}^2(D), \quad \tilde{V} = H^1(D) \times \tilde{H}^1(D). \]
We choose a subset of dynamical variables of our system (10)–(12),
\[ \tilde{v} = (T, S). \] (20)

Applying the chain rule to \( \| \tilde{v} \|^2_{\tilde{H}} \), we obtain by Lemma 3.1,
\[
\frac{d}{dt} \| \tilde{v} \|^2_{\tilde{H}} + 2\kappa_T \| \nabla T \|^2_{\mathbb{L}^2} + 2\kappa_S \| \nabla S \|^2_{\mathbb{L}^2} = 2\lambda \left( \theta(y), \gamma z \right)_{L^2} - 2\lambda \left( \gamma z = dT, \gamma z \right)_{L^2} + 2 \left( F(y), \gamma z = dS \right)_{L^2}, \]
(21)

Cauchy–Schwarz inequality yields the following estimates:
\[
2\lambda \left( \theta(y), \gamma z = dT \right)_{L^2} - 2\lambda \left( \gamma z = dT, \gamma z \right)_{L^2} \leq \lambda \left\| \theta(y) \right\|^2_{L^2} + (\alpha - 2)\lambda \left\| \gamma z = dT \right\|^2_{L^2},
\]
where \( \alpha \) is a positive constant that satisfies \( 2 > a > 2 - (2\kappa_T)/\lambda \).

By the Gronwall inequality, we finally conclude that
\[
\| \tilde{v} \|^2_{\tilde{H}} \leq \left\| \tilde{v}(0) \right\|^2_{\tilde{H}} e^{-\alpha t} + \frac{c_5}{\alpha}, \]
(23)

Then we can easily see that the ball \( B(0, R_1) \), where \( R_1^2 = (2c_5)/\alpha \), absorbs \( \tilde{v} \) in the sense of Definition 4.1.

To prove the dissipativity of the dynamical system \( \varphi \), we have to study \( \| \tilde{q} \|_{L^2} \). From (18), we have
\[
\tilde{q}_t = -J(\psi, q) + g(\alpha_T T_y - \alpha_S S_y) + \nu \Delta \tilde{q} - \nu \Delta \eta.
\] (24)

Then
\[
\frac{1}{2} \frac{d}{dt} \| \tilde{q} \|^2_{L^2} = \left( J(\psi, q), \eta \right) - \nu \| \nabla \tilde{q} \|^2_{L^2} - \nu \Delta \tilde{q} - \nu \Delta \eta + g(\alpha_T T_y - \alpha_S S_y, \tilde{q}).
\] (25)
From the definition of $J(\cdot, \cdot)$ and the Cauchy–Schwarz inequality we have

$$
(J(\psi, q), \eta) = (J(\psi, \eta), q) \leq \|\nabla q\|_{L^2} \|q\|_{L^2} \|\nabla \eta\|_{L^2}
$$

$$
\leq \frac{\nu}{2} \|\tilde{q}\|_{H^1}^2 + \frac{1}{2\nu} \|\eta\|_{H^1}^2 + \delta(\nu) \|\tilde{q}\|_{L^2}^2
$$

Here $c_6$ is the constant in the Poincaré inequality for $\eta \in \dot{H}^1(0, d)$. And for any $\epsilon > 0$ we can find $c_7(\epsilon) > 0$ and $c_8(\epsilon) > 0$ such that

$$
g(\alpha T_y - \alpha_S S_y, \tilde{q}) \leq \tilde{g} \|\nabla \tilde{v}\|_{H^1} \|\tilde{q}\|_{L^2} \leq \frac{\epsilon}{4} \|\tilde{q}\|_{L^2}^2 + c_7(\epsilon) \tilde{g}^2 \|\nabla \tilde{v}\|_{L^2}^2,
$$

$$
-vk(\Delta \eta, \tilde{q}) = v k(\nabla \eta, \nabla \tilde{q}) \leq \frac{\epsilon}{4\lambda_1} \|\tilde{q}\|_{H^1}^2 + \lambda_1 c_8(\epsilon) \tilde{g}^2 \|\eta\|_{H^1}^2,
$$

where $\tilde{g}$ is determined by $\alpha T_y - \alpha_S S_y$ and $\lambda_1$ is the first eigenvalue of the operator $-\Delta$ on $(0, d)$ with Neumann boundary condition.

Collecting all these estimates, we have

$$
\frac{d}{dt} \|\tilde{q}\|_{L^2}^2 \leq -\gamma(\theta_t \omega) \|\tilde{q}\|_{L^2}^2 + \delta(\nu) \|\nabla \tilde{v}\|_{H^1}^2 + r(\theta_t \omega)
$$

$$
\leq -\gamma(\theta_t \omega) \|\tilde{q}\|_{L^2}^2 + r(\theta_t \omega) + \frac{\delta(\nu) \tilde{g}^2}{\alpha} - \frac{\delta(\nu)}{\alpha} \|\tilde{q}\|_{L^2}^2
$$

where $\gamma(\omega) = \lambda_1 v - \epsilon - \frac{1}{\nu} \|\eta\|_{H^1}^2$, $\delta(\nu) = c_7(\epsilon) \tilde{g}^2$ and

$$
r(\omega) = (2\alpha_S c_8(\epsilon) \tilde{g}^2 + v) \|\tilde{q}\|_{H^1}^2 + \frac{c_6}{4\nu} \|\eta\|_{H^1}^2.
$$

(26)

Then Gronwall inequality yields

$$
\|\tilde{q}(t, \omega, u_0)\|_{L^2}^2 \leq \|v_0\|_{H^1}^2 e^{-\int_0^t \gamma(\theta_t \omega) ds}
$$

$$
+ \int_0^t \left( \frac{\delta(\nu) \tilde{g}^2}{\alpha} + r(\theta_t \omega) - \frac{\delta(\nu)}{\alpha} \|\tilde{v}(s)\|_{H^1}^2 \right) e^{-\int_0^s \gamma(\theta_t \omega) ds} ds
$$

$$
\leq \|v_0\|_{H^1}^2 e^{-\int_0^t \gamma(\theta_t \omega) ds} + \int_0^t \left( \frac{\delta(\nu) \tilde{g}^2}{\alpha} + r(\theta_t \omega) \right) e^{-\int_0^s \gamma(\theta_t \omega) ds} ds
$$

$$
+ \frac{\delta(\nu)}{\alpha} \|\tilde{v}(0)\|_{H^1}^2 e^{-\int_0^t \gamma(\theta_t \omega) ds}.
$$

(27)

We will show that the right-hand side of (27) is finite as $t \to \infty$. In fact, we have the following lemma.
Lemma 4.2. If the controlling parameter $k$ is large enough that
\[ \lambda_1 > \frac{\text{tr}_L Q}{(k + 1)\nu} \]
and $\epsilon < (\lambda_1 \nu)/2$, then
\[ E \gamma(\omega) > 0. \]

Proof. Itô formula applied to $\|\eta\|_{L^2}$ yields
\[ \|\eta(\theta_t \omega)\|_{L^2}^2 + 2(k + 1)\nu \int_0^t \|\eta(\theta_s \omega)\|_{H^1}^2 \, ds = \|\eta(\omega)\|_{L^2}^2 + 2 \int_0^t (\eta, dw)_{L^2} + \tau \text{tr}_{L^2} Q. \]
Hence we can easily get that $E\|\eta\|_{H^1}^2 \leq \frac{(\lambda_1 \nu)}{2}$. Then
\[ E \gamma(\omega) > 0. \]

Now we can estimate the $\|\tilde{q}\|_{H^2}^2$. First, we have
\[ \lim_{t \to \infty} \left(\|v_0\|_H^2 + \frac{\delta(\epsilon)}{\alpha}(c_5 t + \|\tilde{v}(0)\|_{H^2}^2)\right) e^{-\int_0^t \gamma(\theta_s \omega) \, ds} = 0, \quad \text{p.a.s.} \]
And note that $\|\tilde{v}\|_{H^2}^2$ is bounded by $\|\tilde{v}(0)\|_{H^2}^2 e^{-\alpha t} + c_5/\alpha$ which tends to $c_5/\alpha$ exponentially as $t \to \infty$. We replace $\omega$ by $\theta_{-t} \omega$ to construct the radius of the absorbing set. Then we have
\[ \lim_{t \to \infty} \int_0^t \left( r(\theta_{-t} \omega) + \frac{\delta(\epsilon)}{\alpha}\|\tilde{v}\|_{H^2}^2 \gamma(\theta_{-t} \omega) \right) e^{-\int_0^\tau \gamma(\theta_{-t} \omega) \, d\tau} \, ds \]
\[ = \lim_{t \to \infty} \int_0^{-t} \left( r(\theta_s \omega) + \frac{\delta(\epsilon)}{\alpha}\|\tilde{v}\|_{H^2}^2 \gamma(\theta_s \omega) \right) e^{-\int_0^\tau \gamma(\theta_s \omega) \, d\tau} \, ds =: R_2^2(\omega) < \infty. \]

Collecting all the estimates we have the following lemma.

Lemma 4.3. The random set $\{B(\omega)\}_{\omega \in \Omega}$ given by closed balls $B(0, R(\omega))$ in $H$ with center zero and radius $R^2(\omega) := R_1^2 + R_2^2(\omega)$ is an absorbing and forward invariant set for the random dynamical system $\varphi$ generated by (18).

For the application in the following we need the particular regularity of the absorbing set. To this end we introduce the function space
\[ \mathcal{H}^s := \{ u \in H : \|u\|_{H^s}^2 := \|A^{s/2}u\|_H^2 < \infty \}, \]
where $s \in \mathbb{R}$. The operator $A^s$ is the $s$th power of the positive and symmetric operator $A$. Note that these spaces are embedded into the Sobolev spaces $H^s$, $s > 0$. The norm of these spaces is denoted by $\| \cdot \|_{H^s}$.

For our aim we show that $v(t, \omega, D)$ is a bounded set in $\mathcal{H}^s$ for some $s > 0$. Consider $t \|v(t)\|_{H^s}^2$ and by chain rule we have
\[ \frac{d}{dt}(t \|v(t)\|_{H^s}^2) = \|v(t)\|_{H^s}^2 + t \frac{d}{dt} \|v(t)\|_{H^s}^2. \]
The second term in the above formula can be expressed as follows:

\[ t \frac{d}{dt} (A^{s/2} v, A^{s/2} v)_H = 2t \left( \frac{d}{dt} v, A^s v \right)_H = -2t (Au, A^s v)_H + 2t (G_1 (v), A^s v)_H + 2t \left( \hat{G}_2 (v + Z(\theta t \omega)), A^s v \right)_H. \]

Notice that for \( \varepsilon > 0 \) there are constants \( c_9 - c_{14} \) such that

\[
\begin{align*}
(J(\eta, \tilde{q}), A^s v)_H &\leq c_9 \| v \|_V \| \eta \|_{1+s} \| v \|_{1+s} + \varepsilon \| v \|_{1+s}^2, \\
(J(\psi, \Delta \eta), A^s v)_H &\leq c_{11} \| \Delta \eta \|_1 \| \psi \|_{1+s} \| v \|_{1+s} + \varepsilon \| v \|_1^2, \\
(J(\eta, \Delta \eta), A^s v)_H &\leq c_{13} \| \Delta \eta \|_1 \| \eta \|_{1+s} \| v \|_{1+s} + \varepsilon \| v \|_1^2,
\end{align*}
\]

and

\[ \int_0^t \| v \|_2^2 \, ds \leq c_{15} \int_0^t \| v \|_V^2 \, ds, \quad s \leq 1 \]

for the embedding constant \( c_{15} \) between \( \mathcal{H}^s \) and \( V \). Then by the similar arguments as in [5] or [11], we have the following dissipative property of THC.

**Lemma 4.4.** For the random dynamical system \( \varphi \) generated by (18), there exists a compact random set \( B = \{ B(\omega) \}_{\omega \in \Omega} \) which satisfies Definition 4.1.

We define

\[ B(\omega) = \overline{\varphi(1, \theta_{-1} \omega, B(0, R(\theta_{-1} \omega)))} \subset \mathcal{H}^s, \quad 0 < s < \frac{1}{4}. \]  

(28)

In particular, \( \mathcal{H}^s \) is compactly embedded in \( H \).

Now we show that the dynamics of THC is described by a random attractor. First we recall the following basic concept; see [6].

**Definition 4.5.** Let \( \varphi \) be a random dynamical system. A random set \( A = \{ A(\omega) \}_{\omega \in \Omega} \) consisting of compact nonempty sets \( A(\omega) \) is called random global attractor if for any random bounded set \( D \) we have for the limit in probability

\[ (\mathbb{P}) \lim_{t \to \infty} \text{dist}_H (\varphi(t, \omega, D(\omega)), A(\theta t \omega)) = 0 \]

and

\[ \varphi(t, \omega, A(\omega)) = A(\theta t \omega) \]

for any \( t \geq 0 \) and \( \omega \in \Omega \).

The following theorem [6] gives a condition of the existence of random attractor.
Theorem 4.6. Let $\phi$ be a random dynamical system on the state space $H$ which is a separable Banach space such that $x \rightarrow \phi(t, \omega, x)$ is continuous. Suppose that $B$ is a set ensuring the dissipativity given in Definition 4.1. In addition, $B$ has a subexponential growth (see Definition 4.1) and is regular (compact). Then the dynamical system $\phi$ has a random attractor.

Then from the above analysis we have the following result to the random system $\phi$ generated by (18), and, via the transformation (17), thus to the original stochastic THC model.

Theorem 4.7 (Random attractor). The THC model (1)–(3) has a random attractor in the phase space $H = L^2(D) \times L^2(D) \times \dot{L}^2(D)$.

It is well known that, the global attractor for certain deterministic infinite-dimensional systems has finite dimension [22]. This result leads to the fact that asymptotic behavior of these systems can be described using finite-dimensional systems. And the similar theory has been developed for random dynamical systems; see, for instance, [7,9,20]. However, for the THC system we will apply another approach to prove the fact that the random attractor of (1)–(3) has only finitely many degrees of freedom. Namely we will use the concept of determining functionals in probability as in Chueshov et al. [5]. This concept was introduced by Foias and Prodi [12] for deterministic systems.

Definition 4.8. We call a set $L = \{l_j: j = 1, \ldots, N\}$ of linear continuous and linearly independent functionals on a space $X$ continuously embedded in $H$ (for instance $X = H^s$ or $V$) asymptotically determining in probability if

$$\lim_{t \to \infty} \frac{1}{t+1} \int_t^{t+1} \max_j \|l_j(\phi(\tau, \omega, v_1) - \phi(\tau, \omega, v_2))\|^2 d\tau = 0$$

for two initial conditions $v_1, v_2 \in H$ implies

$$\lim_{t \to \infty} \|\phi(t, \omega, v_1) - \phi(t, \omega, v_2)\|_H = 0.$$

We introduce a constant $\varepsilon_L$ to describe the qualitative difference of the space $H$ and $X$ for some set of functionals

$$\|u\|_H \leq C_L \max_{l_i \in L} |l_i(u)| + \varepsilon_L \|u\|_X, \quad C_L > 0. \tag{29}$$

We cite the following theorem from [5].

Theorem 4.9. Let $L = \{l_j: j = 1, \ldots, N\}$ be a set of linear continuous and linearly independent functionals on $X$. We assume that we have an absorbing and forward invariant set $B$ in $X$ such that for $\sup_{v \in B(\omega)} \|v\|_X^2$ the expression $t \rightarrow \sup_{v \in B(\omega)} \|v\|_X^2$ is locally integrable and subexponentially growing. Suppose there exist a constant $c_{16} > 0$ and a measurable function $l \geq 0$ such that for $v_1, v_2 \in V$ we have for $\tilde{G}(\omega, v) = G_1(v) + \tilde{G}_2(v + Z(\omega))$,
\[-A(v_1 - v_2) + \tilde{G}(\omega, v_1) - \tilde{G}(\omega, v_2), v_1 - v_2] \\
\leq -c_{16} \| v_1 - v_2 \|^2_H + I(v_1, v_2, \omega) \| v_1 - v_2 \|^2_H.
\]

Assume that
\[
\frac{1}{m} \mathbb{E} \left\{ \sup_{v_1, v_2 \in B(\omega)} \int_0^m \| \varphi(t, \omega, v_1, \varphi(t, \omega, v_2, \theta_t \omega) \| dt \right\} < c_{16} \varepsilon \mathcal{L}^2
\]
for some \( m > 0 \). Then \( \mathcal{L} \) is a set of asymptotically determining functionals in probability for random dynamical system \( \varphi \).

Now we go back to the random THC system. Note that for \( \varepsilon > 0 \) there are constants \( c_{17}(\varepsilon), \ldots, c_{22}(\varepsilon) \) such that
\[
\| J(\tilde{\varphi}_1, \tilde{\varphi}_1) - J(\tilde{\varphi}_2, \tilde{\varphi}_2), \tilde{\varphi}_1 - \tilde{\varphi}_2 \| \leq \frac{\varepsilon}{4} \| \tilde{\varphi}_1 - \tilde{\varphi}_2 \|^2_H + c_{18}(\varepsilon) \| \tilde{\varphi}_1 \|^2_{\mathcal{H}^1(D)} \| \tilde{\varphi}_1 - \tilde{\varphi}_2 \|^2_{L_2}.
\]
Similarly
\[
\| J(\tilde{\varphi}_1, T_1) - J(\tilde{\varphi}_2, T_2), T_1 - T_2 \| \leq \frac{\varepsilon}{4} \| v_1 - v_2 \|^2_H + c_{19}(\varepsilon) \| T_1 \|^2_{\mathcal{H}^1(D)} \| T_1 - T_2 \|^2_{L_2},
\]
\[
\| J(\tilde{\varphi}_1, S_1) - J(\tilde{\varphi}_2, S_2), S_1 - S_2 \| \leq \frac{\varepsilon}{4} \| v_1 - v_2 \|^2_H + c_{20}(\varepsilon) \| \tilde{\varphi}_1 \|^2_{\mathcal{H}^1(D)} \| \tilde{\varphi}_1 - \tilde{\varphi}_2 \|^2_{L_2},
\]
and
\[
\| J(\tilde{\varphi}_1, \Delta \eta) - J(\tilde{\varphi}_2, \Delta \eta), \tilde{\varphi}_1 - \tilde{\varphi}_2 \| \leq c_{21} \| v_1 - v_2 \|^2_H \| \eta \|^2_{\mathcal{H}^1(D)}.
\]

Then we have
\[
\{-A(v_1 - v_2) + \tilde{G}(\omega, v_1) - \tilde{G}(\omega, v_2), v_1 - v_2\}
\leq -c_{22} \| v_1 - v_2 \|^2_H + I(v_1, v_2, \omega) \| v_1 - v_2 \|^2_H.
\]

Here we can take \( k \) is large enough and \( \varepsilon \) is small enough such that \( Ec_{22} = 1 - \varepsilon - c_{21} E \| \eta \|^2_{\mathcal{H}^1(D)} > 0 \) and
\[
I(v_1, v_2, \omega) = c_{17}(\varepsilon) \| \tilde{\varphi}_1 \|^2_{\mathcal{H}^1(D)} + c_{18}(\varepsilon) \| T_1 \|^2_{\mathcal{H}^1(D)} + c_{19}(\varepsilon) \| S_1 \|^2_{\mathcal{H}^1(D)} + c_{20}(\varepsilon).
\]
Now we set \( X = \mathcal{H}^s, s \in (0, 1/4) \). In the above discussion we have shown that the set \( \mathcal{B} \), consisting of bounded sets, is forward invariant. Then we can apply Theorem 4.9 above to the random dynamical system \( \varphi \) generated by (18), and, via the transformation (17), get the following result to the original stochastic THC model.

**Theorem 4.10** (Finite degrees of freedom). The THC model (1)–(3) has finitely many asymptotic degrees of freedom, in the sense of having a finite set of linearly independent continuous functionals which is asymptotically determining in probability, on \( \mathcal{H}^s \) with \( 0 < s < 1/4 \).
5. Conclusion

We have investigated the dynamical behavior of a random thermohaline circulation model. We have shown that the random THC model is asymptotically described by a random attractor (Theorem 4.7). And this system has finite degree of freedom in the sense of having a finite set of determining functionals in probability (Theorem 4.10).

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References