



ELSEVIER

Contents lists available at ScienceDirect

European Journal of Combinatorics

journal homepage: www.elsevier.com/locate/ejc

Finite planar emulators for $K_{4,5} - 4K_2$ and $K_{1,2,2,2}$ and Fellows' Conjecture

Yo'av Rieck^a, Yasushi Yamashita^b^a Department of Mathematical Sciences, University of Arkansas, Fayetteville, AR 72701, United States^b Department of Information and Computer Sciences, Nara Women's University Kitauoya Nishimachi, Nara 630-8506, Japan

ARTICLE INFO

Article history:

Received 3 February 2009

Accepted 26 June 2009

Available online 20 August 2009

ABSTRACT

In 1988 Fellows conjectured that if a finite, connected graph admits a finite planar emulator, then it admits a finite planar cover. We construct a finite planar emulator for $K_{4,5} - 4K_2$. Archdeacon [Dan Archdeacon, Two graphs without planar covers, J. Graph Theory, 41 (4) (2002) 318–326] showed that $K_{4,5} - 4K_2$ does not admit a finite planar cover; thus $K_{4,5} - 4K_2$ provides a counterexample to Fellows' Conjecture.

It is known that Negami's Planar Cover Conjecture is true if and only if $K_{1,2,2,2}$ admits no finite planar cover. We construct a finite planar emulator for $K_{1,2,2,2}$. The existence of a finite planar cover for $K_{1,2,2,2}$ is still open.

© 2009 Elsevier Ltd. All rights reserved.

1. Introduction

We begin by defining the main concepts used in this paper. All graphs considered are assumed to be finite and simple. A map between graphs is assumed to map vertices to vertices and edges to edges. Let \tilde{G} and G be graphs. We say that \tilde{G} is a *cover* (resp. *emulator*) of G if there exists a map $f : \tilde{G} \rightarrow G$ so that f is surjective and for any vertex \tilde{v} of \tilde{G} , the map induced by f from the neighbors of \tilde{v} to the neighbors of $f(\tilde{v})$ is a bijection (resp. surjection). A cover (resp. emulator) is called *regular* if there is a subgroup $\Gamma \leq \text{Aut}(\tilde{G})$ (the automorphism group of \tilde{G}) so that $G \cong \tilde{G}/\Gamma$, and f is equivalent to the natural projection. In this paper, regular covers and emulators are only used when citing results of Negami and Kitakubo; for detailed definitions see [10] (for covers) and [8] (for emulators). We note that Kitakubo used the term *branched covers* for emulators.

Let $i : S^2 \rightarrow \mathbb{R}P^2$ be the projection from the sphere to the projective plane given by identifying antipodal points. If a graph G embeds in $\mathbb{R}P^2$, then $i^{-1}(G)$ is a planar double cover of G . Conversely,

E-mail addresses: yoav@uark.edu (Y. Rieck), yamasita@ics.nara-wu.ac.jp (Y. Yamashita).

in [10] Negami proved that if a connected graph G admits a finite planar *regular* cover, then G embeds in $\mathbb{R}P^2$. Negami conjectured that this holds in general:

Conjecture 1 (*Negami's Planar Cover Conjecture*). *A connected graph has a finite planar cover if and only if it embeds in the projective plane.*

Kitakubo generalized Negami's theorem, showing that if a graph has a finite planar regular emulator, then it embeds in the projective plane. (The authors gave a further generalization in [12].) The following conjecture appears in [11, Conjecture 2], where Negami attributes it to Kitakubo:

Conjecture 2. *A connected graph has a finite planar emulator if and only if it embeds in the projective plane.*

Prior to Kitakubo, planar emulators were studied by Fellows [3,4], who posed the conjecture below; see, for example, [6, Conjecture 4] or [11]:

Conjecture 3 (*Fellows*). *A connected graph has a finite planar emulator if and only if it has a finite planar cover.*

In [6] Hliněný constructed a graph that admits an emulator that embeds in the genus 3 surface, but does not admit a cover that embeds there.

In this note we prove:

Theorem 4. *The graphs $K_{4,5} - 4K_2$ and $K_{1,2,2,2}$ admit finite planar emulators.*

Archdeacon [2] proved that $K_{4,5} - 4K_2$ does not admit a finite planar cover. Together with Theorem 4, we get:

Corollary 5. *The graph $K_{4,5} - 4K_2$ gives a counterexample to Conjectures 2 and 3.*

It is known that $K_{1,2,2,2}$ does not embed in $\mathbb{R}P^2$ [5]. Hence, if it admits a finite planar cover, Negami's Planar Cover Conjecture is false. The work of Archdeacon, Fellows, Hliněný, and Negami shows that the converse also holds, and Negami's Planar Cover Conjecture is in fact equivalent to $K_{1,2,2,2}$ having no finite planar cover; see, for example, [11] or [7] and references therein. At the time of writing, the existence of a finite planar cover to $K_{1,2,2,2}$ remains an intriguing open question. However, Theorem 4 shows that $K_{1,2,2,2}$ does admit a finite planar emulator. Perhaps this should not be seen as evidence against Negami's Planar Cover Conjecture. Perhaps this should be seen as evidence that finite planar emulators are ubiquitous (although clearly not all graphs have finite planar emulators). We note that if Negami's Planar Cover Conjecture holds, then the existence of a finite planar cover can be decided in linear time [9]; the set of forbidden minors is given by Archdeacon [1]. By Robertson and Seymour [14] there is a set of forbidden minors for existence of a finite planar emulator; therefore [13] existence of such an emulator can be decided in polynomial time.

Question 6. *What graphs admit finite planar emulators? What is the set of forbidden minors? Construct an algorithm to decide if a given graph admits a finite planar emulator. What is the complexity of this problem?*

In Section 2 we explicitly show an emulator with 50 vertices for $K_{4,5} - 4K_2$ and in Section 3 we explicitly show an emulator with 266 vertices for $K_{1,2,2,2}$, thus proving Theorem 4. The emulator for $K_{1,2,2,2}$ is symmetric and quotients out to an emulator with 133 vertices that embeds in $\mathbb{R}P^2$.

2. A finite planar emulator for $K_{4,5} - 4K_2$

An emulator for $K_{4,5} - 4K_2$ is given in Fig. 2. We explain how to read this graph. $K_{4,5} - 4K_2$ is constructed as follows: start with the 1-skeleton of a cube (Fig. 1) and add a ninth vertex, denoted v , that is connected to the vertices of the cube labeled 1, 3, 5, and 7. The graph shown in Fig. 2 maps to

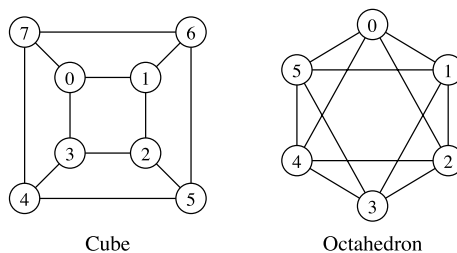


Fig. 1.

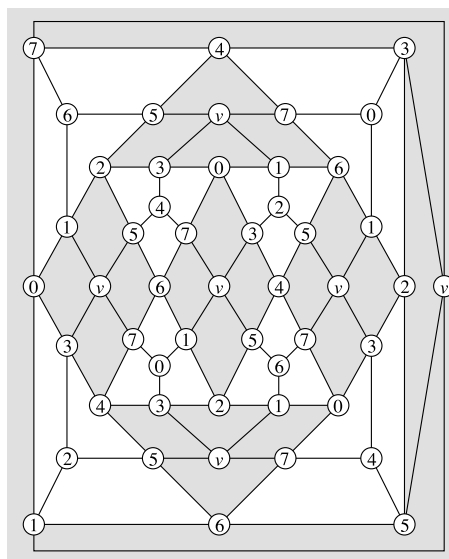


Fig. 2. A finite planar emulator of $K_{4,5} - 4K_2$.

$K_{4,5} - 4K_2$; each vertex is shown as a small circle labeled by the vertex of $K_{4,5} - 4K_2$ it gets sent to. It can be checked directly that it is a finite planar emulator of $K_{4,5} - 4K_2$.

Remarks.

- (1) It is easy to see how the graph was put together. It is made of 8 white “triangles”, each triangle meeting 3 others (this pattern can be seen by taking the convex hull of the midpoints of the edges of a cube or an octahedron). Each triangle is simply a corner of the cube (3 squares).
- (2) Note that the graph presented in Fig. 2 is not a cover of $K_{4,5} - 4K_2$. For example, we can find a vertex with label 0 which is adjacent to two vertices labeled 3.

3. A finite planar emulator for $K_{1,2,2,2}$

An emulator for $K_{1,2,2,2}$ is derived from Fig. 3. We explain how to read this graph. $K_{1,2,2,2}$ is constructed as follows: start with the 1-skeleton of an octahedron (Fig. 1) and add a seventh vertex, denoted v , that is connected to all the vertices of the octahedron. The graph shown in Fig. 3 maps to the 1-skeleton of the octahedron; each vertex is shown as a small circle labeled by the vertex of the octahedron it gets sent to. It can be checked directly that it is a finite planar emulator of the 1-skeleton of the octahedron.

Note that some of the faces have been shaded (this includes the outside face). We add a vertex in each of these faces. These vertices all map to v and are connected to every vertex on the boundary of

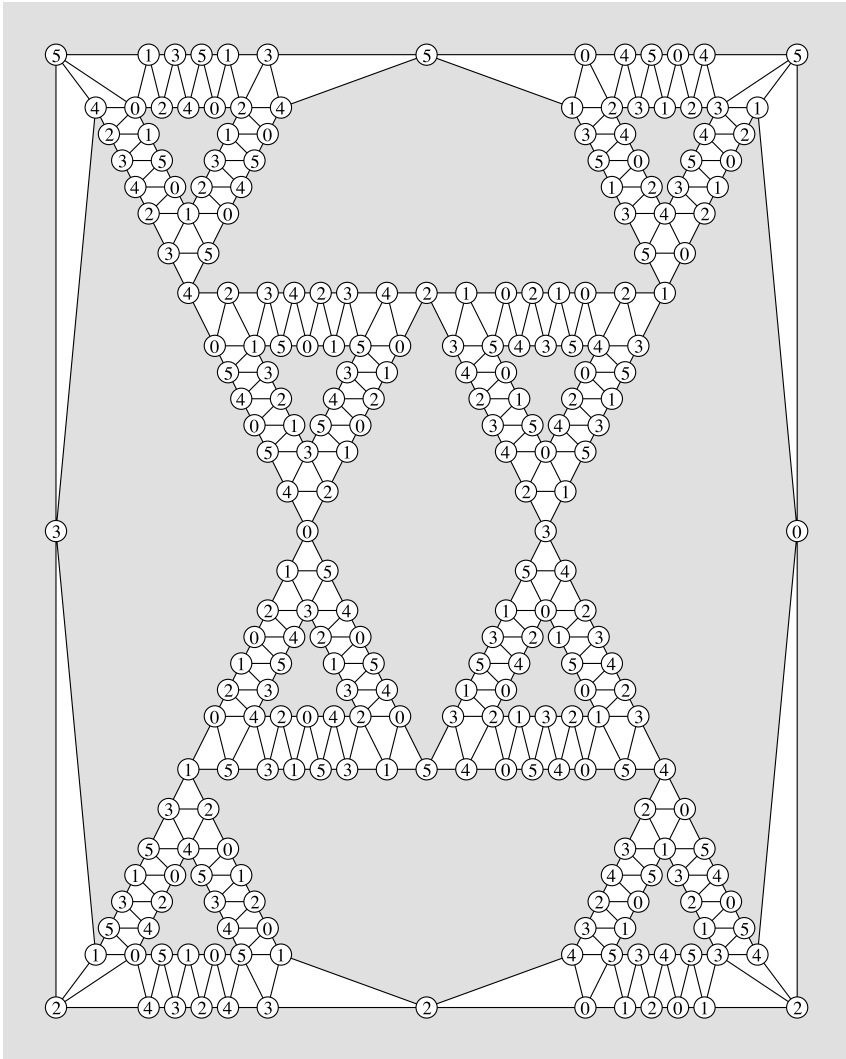


Fig. 3. A finite planar emulator of $K_{1,2,2,2}$.

the shaded cells. On the boundary of each shaded face we see all the labels, so each of the vertices that map to v has all the necessary neighbors. Finally, we see that each vertex in Fig. 3 is on the boundary of at least one shaded face; hence, every vertex has a neighbor that maps to v .

This completes our construction of a finite planar emulator of $K_{1,2,2,2}$.

Remark. By viewing S^2 as the boundary of the convex hull of the midpoints of the edges of the cube or the octahedron, we may draw the emulator for $K_{1,2,2,2}$ symmetrically, so that it is invariant under the antipodal involution. The quotient gives an emulator for $K_{1,2,2,2}$ that has 133 vertices and embeds in $\mathbb{R}P^2$. This symmetric presentation of the emulator for $K_{1,2,2,2}$ reveals another interesting property. By considering the eight triangular faces (each shown in Fig. 3 as a white triangle with a single shaded face), we can see that they are formed from the union of 4 great circles, one with the labels 0, 1, 2, one with the labels 2, 3, 4, one with the labels 1, 3, 5, and one with the labels 0, 4, 5. Note that if we make the octahedron in Fig. 1 with two alternating colors, the faces with these labels correspond exactly to the faces of one color.

Acknowledgment

We thank Mike Fellows and Petr Hliněný for helpful comments regarding [Question 6](#), and the referees for their work and remarks. The first author thanks Tsuyoshi Kobayashi and the Department of Mathematics of Nara Women's University for their hospitality during the time this research was conducted.

References

- [1] Dan Archdeacon, A Kuratowski theorem for the projective plane, *J. Graph Theory* 5 (3) (1981) 243–246.
- [2] Dan Archdeacon, Two graphs without planar covers, *J. Graph Theory* 41 (4) (2002) 318–326.
- [3] Michael Fellows, *Encoding Graphs in Graphs*. Ph.D. Thesis, Univ. of California, San Diego, 1985.
- [4] Michael Fellows, *Planar emulators and planar covers*, 1988.
- [5] Henry H. Glover, John P. Huneke, Chin San Wang, 103 graphs that are irreducible for the projective plane, *J. Combin. Theory Ser. B* 27 (3) (1979) 332–370.
- [6] Petr Hliněný, A note on possible extensions of Negami's conjecture, *J. Graph Theory* 32 (3) (1999) 234–240.
- [7] Petr Hliněný, 20 years of Negami's planar cover conjecture, in: *20th Workshop on Topological Graph Theory in Yokohama, 2008*, pp. 50–59.
- [8] Shigeru Kitakubo, Planar branched coverings of graphs, *Yokohama Math. J.* 38 (2) (1991) 113–120.
- [9] Bojan Mohar, Projective planarity in linear time, *J. Algorithms* 15 (3) (1993) 482–502.
- [10] Seiya Negami, The spherical genus and virtually planar graphs, *Discrete Math.* 70 (2) (1988) 159–168.
- [11] Seiya Negami, Topological graph theory from Japan, in: *Proceedings of the Workshop on Graph Theory and Related Topics, Sendai, 1999*, vol. 7, 2001, pp. 99–112.
- [12] Yo'av Rieck, Yasushi Yamashita, On Negami's planar cover conjecture. Available at <http://arxiv.org/abs/math.CO/0612342>, 2006.
- [13] Neil Robertson, P.D. Seymour, Graph minors. XIII. The disjoint paths problem, *J. Combin. Theory Ser. B* 63 (1) (1995) 65–110.
- [14] Neil Robertson, P.D. Seymour, Graph minors. XX. Wagner's conjecture, *J. Combin. Theory Ser. B* 92 (2) (2004) 325–357.