The way of Diophantus: Some clarifications on Diophantus’ method of solution

Jean Christianidis

Department of History and Philosophy of Science, University of Athens, 157 71 University Campus, Athens, Greece

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In memory of Isabella Grigorievna Bashmakova

Abstract

In the introduction of the Arithmetica Diophantus says that in order to solve arithmetical problems one has to “follow the way he (Diophantus) will show.” The present paper has a threefold objective. Firstly, the meaning of this sentence is discussed, the conclusion being that Diophantus had elaborated a program for handling various arithmetical problems. Secondly, it is claimed that what is analyzed in the introduction is definitions of several terms, the exhibition of their symbolism, the way one may operate with them, but, most significantly, the main stages of the program itself. And thirdly, it is argued that Diophantus’ intention in the Arithmetica is to show the way the stages of his program should be practically applied in various arithmetical problems.

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A remarkable increase in the production of historical studies on mathematics and the sciences of Late Antiquity has occurred in recent years. To a certain extent this proliferation is due to the fact that Late Antiquity had not formerly been studied as thoroughly as, for example, the Classical Greek and Hellenistic periods. Preconceptions and fixed views that for many decades dominated the historiography of science—promoting the prime cliché that the period of Late Antiquity is no more than a period of decline in the sciences—naturally favored this comparatively moderate interest. These preconceptions have their roots, on the one hand, in similar conceptions prevailing in the general history
of culture\(^1\) and, on the other, in the fact that the history of the sciences, and especially that of mathematics, was for a long time directed toward studying the achievements of the past mainly for the role they played in the evolution of one or another discipline and much less toward comprehending those achievements within the cultural framework of the era and society in which they were produced. These preconceptions are now being reconsidered and, to a large extent, contested. One result is the recently observed flourishing in the historiography of mathematics and the sciences of Late Antiquity.

At the same time, increasing interest in the work of Diophantus, one of the most enigmatic mathematical figures of the period of Late Antiquity, has also arisen. We must note, however, from the start, that this interest is not a product of the flourishing of the historiography of Late Antiquity. Its origin is quite different, primarily the discovery and subsequent publication of the four Arabic books of the Arithmetica [Rashed, 1984a; Sesiano, 1982] and also the development of new interpretations of the character of the work and accurate descriptions of the goal and mathematical practice of its author [Bashmakova, 1966, 1981; Christianidis, 1998; Sesiano, 1999; Thomaidis, 2005; Vitrac, 2005]. Furthermore, the Diophantine bibliography has recently been enriched with a number of other studies on various subjects concerning particular aspects of Diophantus’ work [Christianidis, 1991, 2004; Knorr, 1993; Schappacher, 1998; Sesiano, 2004].

The proliferation of the Diophantine bibliography is undoubtedly a welcome event. New questions have been formulated, more convincing arguments concerning older issues have been proposed, and, in general, some interesting interpretations have been put forth for various aspects of Diophantus’ work. However, the new scholarship has not necessarily resolved the open questions of the past, such as, for example, the question of the precise date of Diophantus. There are, today, three answers to this question, each with its own recommendations and drawbacks. At the end of the 19th century, Tannery dated Diophantus to the middle of the 3rd century A.D. [Tannery, 1896]; thenceforth this chronology became broadly accepted. At the end of the 20th century, Knorr suggested another dating, according to which Diophantus lived at the same time as Hero of Alexandria, i.e., two centuries earlier [Knorr, 1993]. More recently, Schappacher, pointing to the fact that Diophantus was mentioned neither by the authors of the first three centuries A.D. nor even by those active in the first decades of the 4th century (e.g., Pappus, Iamblichus), but for the first time by Theon of Alexandria at the end of the 4th century, concluded that the date of Diophantus could not have much preceded that of Theon [Schappacher, 1998].

Another unresolved question about Diophantus is that concerning the relationship between his mathematical activity and algebra. In pursuing this question, some modern historians of mathematics have contested the old and broadly accepted view that the work of Diophantus belongs to the history of algebra.\(^2\) Rashed, for example, claims that the relationship between the Arithmetica and algebra was established only when the Arab mathematicians—who, according to Rashed, first created algebra—began to study the work of Diophantus algebraically. Rashed rejects entirely any attempt to assign an algebraic understanding to Diophantus himself, claiming that “l’ouvrage de Diophante . . . est en fait un ouvrage d’arithmétique” [Rashed, 1979, 196]. A similar view, although in a slightly more moderate tone, was expressed recently by Vitrac, in a study that inquires “Can we talk about algebra in ancient Greek mathematics?” [Vitrac, 2005]. Although Vitrac admits that Diophantus outlined a protoalgebraic form of calculation, the conclusion of his study is that the decisive step in the development of algebra was made in the first half of the 9th century, in the countries of Islam. Hence, he declines to speak of Greek algebra, considering the term “algebra” to be applicable only to the Arabic contribution.

Underlying these more recent historical conclusions is the interpretation of the term “algebra” in a rather narrow sense, i.e., as a theory of equations. Characterizing Arabic algebra in such a manner, these scholars conclude that

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1. So, for example, in older handbooks, the period in question is often referred to in entirely negative terms, such as “the end of Antiquity” or “the decline of the ancient Greek culture,” and, less often, by the term “Late Antiquity,” which could convey a positive connotation as well. See [Marrou, 1977].

2. The roots of the older view have been traced back to the Arabs and the algebraists of the Renaissance. Characteristic of these early opinions is that expressed by Rafael Bombelli in the preface of his Algebra (1572): “I have decided first to consider the majority of the authors who up to now have written about [algebra] . . . They are very many, and among them certainly Mohammed ibn Musa, an Arab, is believed to be the first, and there is a little book of his, but of very small value. I believe that the word ‘algebra’ came from him, because some years ago, Brother Luca [Pacioli] . . . said that the word ‘algebra’ was Arabic . . . and that the science came from the Arabs. Many who have written after him believed and said likewise, and in recent years, a Greek work on this discipline has been discovered in the Library of our Lord in the Vatican, composed by a certain Diophantus of Alexandria, a Greek author, who lived at the time of Antoninus Pius” [Fauvel and Gray, 1987, 262–263]. On the dating of Diophantus by Bombelli see [Tannery, 1879, 67–69].
algebra did not exist prior to its Arabic realization. Interpretation of a work such as Diophantus’ *Arithmetica* would greatly benefit, however, if we were to understand the algebraic mode of thought as an enterprise of problem-solving, which has its own particular features. In a very interesting recent paper, Oaks and Alkhateeb note the following criteria to be essential for an algebraic solution of a problem: (1) establishment of an equation in terms of algebraic numbers; (2) simplification of the equation to one of the standard types; and (3) application of the proper procedure to arrive at the answer to the problem [Oaks and Alkhateeb, 2005, 403]. This description could apply as well to premodern algebraic practice in general, such as Viète’s algebraic program, the Italian algebra of the Renaissance, and Arabic algebra.³ In the course of this paper I will consider whether also Diophantus’ work meets these criteria.

As to the questions mentioned above, the variety of the answers arises from the way in which historians interpret or evaluate the historical data and conceive of how the various elements of the historical material are to be combined and adjusted to each other. In this sense, all answers have their own value, each contributing to the historical research, although the very existence of multiple answers preserves the vagueness and maintains a rather clouded vision regarding those questions.

In this paper, I will examine an issue persistent in Diophantic scholarship, in respect to which various historians of mathematics have expressed differing points of view. This issue concerns the question of whether or not Diophantus elaborated and employed a single general strategy for the treatment of arithmetical problems. To my mind, this issue cannot be adequately treated without prior clarification of another issue: the exact description and characterization of the mathematical practice of Diophantus. As mentioned in the beginning, the recent literature about Diophantus includes numerous studies dedicated to these issues. In this case, the spectrum of the proposed answers is rather wide and the final picture is even foggier than in the two aforementioned examples. The aim of this study is to provide certain clarifications regarding this group of questions.

### 1. The introduction of the *Arithmetica* in the modern Diophantic literature

The *Arithmetica* of Diophantus begins with an introduction of not insignificant length. It extends for seven full pages of the Tannery edition, that is to say, 18% of the first book of the *Arithmetica* [Tannery, 1893–1895, I, 2–16]. Scholars who approach the introduction usually assume that Diophantus exposes there the distinct parts of the machinery through which he treats and solves the arithmetical problems. Among these distinct parts, those most frequently mentioned are the definitions and the symbols (abbreviations) of the unknown and its powers, the operations between them, and the two rules, known by their Arabic names *al-jabr* and *al-muqābala*, by which an equation is transformed, as we would say today, into its final form. All the above are described in detail in the scholarship about Diophantus, and the disagreements among the historians focus primarily on the way in which Diophantus deals with one or another component of the machinery. In my opinion, the way in which historians of mathematics have, until today, treated the introduction of the *Arithmetica* can be characterized as follows: they have approached this part of the work of Diophantus from the perspective of the particular, in order to understand each concept separately and hence to determine how each distinct element of the machinery functions. The result of this approach has been to treat the introduction of the *Arithmetica*, on the one hand, and the way Diophantus resolves the problems, on the other, as the subject of separate inquiries.

In contrast, little attention has been paid to the question of how Diophantus perceives the coherence of the machinery, that is, how he understands the way in which the different elements are interconnected to form the machinery as a unified whole, which, when applied, would have allowed him to handle a large number of arithmetical problems. Elucidation of such a coherence would require consideration of both the nature and the characteristics of the Diophantine undertaking. Approaching the introduction with the aim of both understanding the distinct parts of the machinery and comprehending the coherence of the machinery itself avoids the danger of seeing no intrinsic, organic relation between the points that Diophantus expounds in the introduction to the *Arithmetica* and the way he deals

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³ Whatever the impact of the Arabic algebra and the consequent European traditions on Viète and on his immediate French predecessors, it has been convincingly argued that their work was strongly influenced by the work of Diophantus and by other works of the Greek mathematical tradition (such as the works of Pappus and Proclus). As J. Klein has put it, “Modern algebra and modern formalism grew out of Vieta’s occupation with Diophantus” [Klein, 1985, 22]. A similar view is held by K.H. Parshall, who writes, “Viète’s humanistic leanings predisposed him to reject the geometric variety of algebra with its Arabic line of descent” [Parshall, 1988, 153–154]. See also [Cifoletti, 1995; Klein, 1992, 150ff.; Mahoney, 1971].
with the arithmetical problems in the rest of the book. Such an approach has much to recommend it. The history of mathematics well illustrates the rubric that such works as the *Arithmetica*, which contain a series of exercises and their solutions, are customarily aimed at exhibiting general ways of handling and solving, not at presenting particular exercises. In other words, such works are not concerned with the problems themselves but with the methods by which they can be solved. This conclusion holds true in the case of the Babylonian and the Egyptian mathematical texts, the Chinese mathematical texts, and the arithmetical books of the Middle Ages and the Renaissance, and is therefore plausible in the case of the Diophantine work. However, if Diophantus’ intention in the *Arithmetica* is to indicate his strategy, i.e., the general way of treating and solving arithmetical problems, then it is reasonable to presume that he wrote the introduction to the work precisely with the intention to present his strategy. In fact, Diophantus draws a close relationship between the introduction of the *Arithmetica* and the way in which he treats the problems in his statement that the arithmetical problems “are solved if you follow the way I will show you” [Tannery, 1893–1895, I, 4.10–11; emphasis added]. The Greek term used here is ὑπάρξις, whose range of meanings includes “way,” “general method,” and “strategy.” Interpreting this term and tracing its role are essential to understanding Diophantus’ intentions in the *Arithmetica*, and a large part of what follows will be focused on these issues. For the present, I merely note that, in this statement, Diophantus promises to demonstrate, *in the introduction* of the *Arithmetica*, the way by which one will be able to solve arithmetical problems. And, as we will see, he fulfils this promise.

Thus I claim that (1) Diophantus developed a ὑπάρξις, i.e., a general way, a strategy, a program, to deal with arithmetical problems. (2) The introduction to the *Arithmetica* includes definitions of several terms, illustrations of their symbolism, description of the way in which one may operate with them, and, most significantly, *the main stages of the program itself*; (3) Diophantus’ intention in the *Arithmetica* is to show the way in which the main stages of his program are to be practically applied to various arithmetical problems. In what follows, I aim to support these claims. So, in Section 2, it is argued that a specific term, appearing in the very first line of the *Arithmetica*, namely the term ἡ θεωρία (heuresis), is used by Diophantus to denote a separate stage of the resolution of an arithmetical problem. In Section 3 that follows this stage is further clarified as the stage in which the establishment of an equation out of a problem takes place, and it is argued that the establishment of the equation together with its transformation into a simplified form constitutes the core of the strategy of resolution of Diophantus. Then it is shown how the above schema is applied in the case of a particular problem of the *Arithmetica*.

### 2. Η θεωρία: a separate stage in the process of solving a problem

#### 2.1. The proposed translations of θεωρία

The opening line of the *Arithmetica* provides the first and, indeed, a very revealing hint of Diophantus’ intention in the work. His intention, he says, is to teach a certain Dionysius τὴν ἡθεωρίαν τῶν ἐν τοῖς ὑμῖν ὑπολείμμασιν *[Tannery, 1893–1895, I, 2.3].* Tannery translates this phrase into Latin as “solutionem arithmeticares problematum,” indicating his interpretation of the Greek term θεωρίας as “solutio.”⁴ Heath’s English translation of the passage that contains this phrase is “Knowing, my most esteemed friend Dionysius, that you are anxious to learn *how to investigate* problems in numbers…” [Heath, 1964, 129; emphasis added]. In the introduction of his book, however, Heath presents another interpretation of the passage, proposing two possible translations of θεωρίας: “Knowing you, O Dionysius, to be anxious to learn the solution (or, perhaps, ‘discovery,’ θεωρίας) of problems in numbers…” [Heath, 1964, 9]. In the French translation by P. ver Eecke, the passage reads: “Sachant, mon très honoré Dionysius, que tu es zélé pour apprendre à trouver des problèmes sur les nombres…” [Diophante, 1959, 1; emphasis added].

Four translations, four different versions of the word θεωρίας. Obviously something odd is going on here. Indeed, it is quite unusual for one term, appearing in a single passage, to be translated in four different ways. It is even more unusual for the same term to be translated in three different ways by a single author. All the more so when the author happens to be Sir Thomas Heath, whose translations are most often considered reliable and insightful.

Let me say from the outset that none of the above translations is fully satisfactory. First, the interpretation of θεωρίας as “solution” is not supported by any major dictionary of the Greek language, either in its classical or any later form.⁵

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⁴ See also “Index graecitatis apud Diophantum” [*Tannery, 1893–1895, II, 271*, s.v. θεωρίας.

⁵ See, for example, LSJ (= *Liddell and Scott*, 1996) and *Dimitrakos, 1936–1950*, s.v. θεωρίας.
The Greek word for “solution” is λύσις, its corresponding verb being λύειν. Diophantus uses a form of the verb λύειν a bit later in his introduction, when he says that the arithmetical problems “are solved (λύσται) if you follow the way I will show you” [Tannery, 1893–1895, I, 4.10–11]. Forms of the same verb are also used in a variety of other passages in the Arithmetica, for instance, in the expressions “and [the problem] is solved in the indeterminate” (καὶ λύεται ἐν τῷ ἀδριτὸ πρὸς τὸ ἀδριτὸ) [Tannery, 1893–1895, I, 232.4; 278.9; 282.11] and “the question would have been solved” (λύλυθην ἐν τῷ ζητούμενον) [Tannery, 1893–1895, I, 246.4]. It is clear, then, that if Diophantus had wished to speak about the “solution” of arithmetical problems in the opening line of the Arithmetica, he could have used the word λύσις. Instead, he uses the word εὑρεσις, etymologically unrelated to “solution.”

Second, although the interpretation of εὑρεσις as “discovery” is linguistically possible, it cannot be accepted in this particular context. The εὑρεσις about which Diophantus speaks is the εὑρεσις of the problems (τῶν...προβλημάτων), and it is difficult to understand what the mathematical meaning of “discovery of the problems” (i.e., “finding of the problems”) might be, other than the setting up of a problem. Heath himself seems reluctant to accept this interpretation, marking it as a less plausible alternative by setting it in parentheses. His ultimate preference for the rather neutral translation “how to investigate problems in numbers” indicates that he found the two aforementioned interpretations of εὑρεσις to be less convincing.

The εὑρεσις referred to by Diophantus could, however, be related to “solution” if we take the expression εὑρεσις of the problems to be an elliptical rendering of “εὑρεσις of the solution of the problems.” We will see that, on this interpretation, the meaning of the text becomes at once comprehensible and coherent, both mathematically and grammatically. Although εὑρεσις should not be considered, even in this case, to be identical with “solution,”6 the two terms are on this reading no longer unrelated. Εὑρεσις could be conceived, in this instance, as a sort of preparation for answering a problem, and, in this sense, it could be understood as a distinct part of the solution, a separate stage in the process of solving the problem.

2.2. The interpretation of εὑρεσις as “invention”

In the Greek mathematical literature, there is in fact another instance in which the word εὑρεσις denotes a part of the resolution of a problem; in a passage from the fourth book of Pappus’ Collection, where Pappus discusses the classification of geometric problems, we read: “But those problems which are solved when there is assumed toward their heurèsis one or several of the sections of the cone are called solid” [Hultsch, 1876–1878, I, 270.8–11; the same passage occurs also at 54.12–14]. Although interpretations of εὑρεσις as “solution” and “discovery” are found, in this case too, in proposed translations of the passage, its translation as “inventio” has also been suggested by Hultsch.7 At any rate, whatever the correct translation of the term might be (either “invention” or “discovery [of the solution]”), what is important to note is that the word εὑρεσις is apparently used by both Pappus and Diophantus to denote a distinct part in the process of solving a problem and not the process as a whole.

The translation of εὑρεσις as “invention” (Latin “inventio”) originates in ancient rhetoric. Among the many books on “invention” that were produced in Late Antiquity, almost all of them shared the title De inventione (Περί εὑρέσεως), the most popular being the De inventione of Marcus Tullius Cicero (106–43 B.C.). In the context of ancient rhetoric, “inventio” is the first of five distinct, sequential parts into which an oration is divided, the second part being the “dispositio.” “Invention,” in this context, refers to the working out of arguments to answer a question, while “disposition” is the organization of the arguments into an effective whole [Bernard, 2003a, 408; 2003b, 144; Slawinski, 1994, 76–77]. On this analogy, a correspondence could be established between the rhetorical “invention” and the mathematical εὑρεσις, with the proviso that εὑρεσις should be conceived in the sense proposed above. That is, εὑρεσις, as understood here, is a part of the resolution of a proposed problem, just as “invention” is the first part of the working out of an oration responding to a posed rhetorical question (Latin “questio”).

It is difficult to say whether Diophantus was influenced by the rhetoricians’ “invention” in his remarks on τὴν εὑρέσιν τῶν ἐν τοῖς ἀριθμοῖς προβλημάτων. Such a possibility can, at least, not be excluded. Besides, something similar has happened later in the history of mathematics. Indeed, the understanding of aspects of mathematical practice through rhetoric, which may seem surprising to the modern historian of mathematics, had actually become a

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6 “Solution,” here, is taken to indicate the process of solving, not merely the numerical result of solving.

7 See “Index graecitatis” [Hultsch, 1876–1878, III], s.v. εὑρεσις.
rather natural idea at the end of the 16th century. Thus, for example, the algebraic enterprises of Jacques Peletier and Guillaume Gosselin were greatly influenced by the rhetorical categories of “invention” and “disposition” [Cifoletti, 1992, 1995]. It is noteworthy that Gosselin was in his day, as Cifoletti writes, “l’expert le plus considérable de Diophante à Paris,” and in his De Arte magna (1577) “on y observe l’effet d’une étude minutieuse de l’Arithmetica de Diophante” [Cifoletti, 1995, 1409].

The question of the rhetoricians’ influence aside, it seems a reasonable suggestion that Diophantus uses the word ε/δαισιξίς to indicate a separate stage of the resolution of an arithmetical problem, a stage that could be seen as corresponding to the invention of a rhetorical question. Accordingly, translating the Diophantine ε/δαισιξίς as “invention,” just as Hultsch did with regard to the Pappian ε/δαισιξίς, is here recommended. The precise mathematical content of this stage in the particular context of Diophantus’ treatment of arithmetical problems will be clarified subsequently in this article.

2.3. Arithmetica, a book teaching “inventions”?

Now, taking into consideration that the word ε/δαισιξίς is the opening nongrammatical word of the Arithmetica, it is quite reasonable to suggest that the main aim of Diophantus in the Arithmetica might have been precisely to explain how “invention” can be conducted in the case of arithmetical problems. If this suggestion is sound, then modern Diophantic scholarship, under the influence of Tannery’s and Heath’s misinterpretation of the term ε/δαισιξίς in this context, errs in the interpretation of Diophantus’ aims. It is neither the solution of arithmetical problems, as Tannery says, nor the investigation of problems in numbers, as Heath maintains, that Dionysius is anxious to learn from Diophantus. Instead, he is anxious to learn the invention of arithmetical problems, and this is not the same as the complete resolution or the investigation (whatever this means) of a problem. In my view, the ε/δαισιξίς of an arithmetical problem for Diophantus is no more than the establishment of an equation out of the problem and, in this sense, constitutes part of the resolution but does not coincide with the resolution in its wholeness. According to this interpretation, for which I argue later in the article (especially in Sections 3.2 and 5, where a particular example is discussed), what one expects to find in the work of Diophantus is (a) the description of the theoretical background suitable for conducting the invention of arithmetical problems, (b) an explanation of what the resolution of a problem further comprises, (c) a presentation of a body of techniques through which the invention can proceed, and (d) a series of demonstrations of all the above in particular examples. Diophantus deals with the first two of the above topics in the introduction of the Arithmetica, and with the latter two in the process of solving the problems.

From the aforesaid, the reader must have already realized that, on my view, the strategy of resolution of Diophantus comprises two main stages: the invention and the disposition. Those are the stages he deals with in detail in the problems, and the introduction of the Arithmetica is basically dedicated to the explanation of those stages. In fact, the aim of the Arithmetica, the very reason why Diophantus wrote this work, is to explain those two stages. It is for this reason that Diophantus does not fully pursue the final stages of the resolution. He is not interested in explaining in detail all the stages of the resolution of the problems, but is, rather, exclusively focused on those two stages: the invention and the disposition. I will return to these issues later.

3. The content of the introduction of the Arithmetica

3.1. The bipartite division of the introduction

In the introduction to the Arithmetica, as noted above, Diophantus refers to a ὁδὸς that one has to follow in order to solve arithmetical problems: ἵνα τῇ ὑποθεκτῇ τῆς ὅπως τὰ ὀντολογίας τῆς ὑποθεκτῆς παρατηρήσεως ὁδὸς, “[the arithmetical problems] are solved if you follow the way I will show you” [Tannery, 1893–1895, I, 4.10–11]. This phrase, which is the key to comprehending the structure of the introduction, in fact physically divides the text of the introduction into two distinct parts. The phrase that immediately precedes this statement reads as follows: “It is from the addition, subtraction, or multiplication of these numbers or from the ratios which they bear to one another or to their own sides respectively that most arithmetical problems are formed” [Heath, 1964, 130; Tannery, 1893–1895, I, 4.7–10; emphasis added]. In the first part of the introduction, lines 2.14 through 4.10, Diophantus discusses the enunciation of the problems, the way in which arithmetical problems are formed (πλ. ἐκείθεν, a rather curious word to be used in a mathematical book). In the second part of the introduction, lines 4.10–11 through 14.24, he fulfills the promise made in lines 4.10–11: he
indicates the ὀδός, that is to say, he exhibits the way of solving the problems and presents the main stages of the method. Hence, Diophantus refers in the first part of the introduction to the formulation of the problems, and in the second part to the way of solving them. Whereas in the Tannery edition, the key phrase, lines 4.10–11, is placed at the end of the section on the formulation of the problems—as perhaps it was also placed in the manuscripts on which he based his edition—it logically belongs at the beginning of the section that treats the working out of the problems.

This division of the introduction to the Arithmetica into two parts separated by the phrase ἐλεύθερα δὲ βαθιζόντος οὖν τὴν ὑποδεικνυμένην ὀδόν is not clearly acknowledged by the majority of the modern commentators of Diophantus. Jacob Klein and Jacques Sesiano are among the few exceptions. Recognition of the structure of the introduction is, however, of great importance in understanding why Diophantus uses two different series of names for the various kinds of numbers in his work. The reason is that each of these series has a different role and a different function. The first has to do with the enunciation of the problems, indicating the different kinds of the sought numbers, i.e., the requested numbers appearing in the enunciations of the problems, whereas the second has to do with the resolutions, indicating the unknown number and its powers.

3.2. The first part of the introduction: the formulation of the problems

In the first part of the introduction, Diophantus deals with the way of formulating the various arithmetical problems [Tannery, 1893–1895, I, 2.14–4.10]. He starts by reminding us of the traditional Greek definition of “number” (i.e., “all numbers are made up of some multitude of units”) and by mentioning the series of numbers going to the infinite. Then he states that these numbers include the following:

- Numbers resulting from the multiplication of a number by itself. Those numbers are called “tetragonoi” (“squares”), and the number from which each square derives is called a “pleura” (“side”) of the square.
- Numbers resulting from the multiplication of a square number by its side. Those numbers are called “kyboi” (“cubes”).
- Numbers resulting from the multiplication of a square number by itself.
- Numbers resulting from the multiplication of a square number by the cube of the same side.
- Finally, numbers resulting from the multiplication of a cube by itself.

The Greek text runs as follows (the page numbers and line numbers refer to the Tannery edition):

p. 2 ὄν μὲν τετραγόνων, οἷς εἶσον ἐξ ἀριθμοῦ τινός ἐφ’ ἐκατόν πολυπλασιασθέντος· οὕτως δὲ ὁ ἀριθμός καλεῖ·
20 τα πλευρὰ τοῦ τετραγῶνου· ὃν δὲ κύβον, οἷς εἶσον ἐκ τετραγόνων ἐπί τὰς αὐτὰς πλευρὰς πολυπλασιασθέντων.

p. 4 ὃν δὲ [δυναμοδυνάμεων], οἷς εἶσον ἐκ τετραγώνων ἐφ’ ἐκατοιχοὶ πολυπλασιασθέντων,
5 ὃν ὁ [δυναμοσυκόβισιν], οἷς εἶσον ἐκ τετραγώνων ἐπὶ τοὺς ἀπὸ τῆς αὐτῆς αὐτὸς πλευρὰς κύβους πολυπλα− 
σιασθέντων,

The Tannery edition, as well as all Greek manuscripts of the Arithmetica, denote the numbers in the last three categories as “dynamodynamis,” “dynamokybos,” and “kybokybos.” In our reproduction of the text, I have placed those terms in brackets, having strong reasons to believe that the text is corrupted and that those three words should be suppressed. These reasons are as follows: (1) The terms “dynamodynamis,” “dynamokybos,” and “kybokybos” are introduced by Diophantus later, in the second part of the introduction, where he identifies them as “abbreviated designations” (συντομότερα ἐπωνυμία) of the numbers belonging to the three types above [Tannery, 1893–1895, I, 4.19–6.2]. (2) If the words “dynamodynamis,” “dynamokybos,” and “kybokybos” had appeared in this earlier series, they would have been part of both the earlier and the later series. In that case no sense can be made of Diophantus characterizing the terms of the second series as “abbreviated designations” of the terms of the first series. On the other
hand, if the terms “dynamodinamis,” “dynamokybos,” and “kybokybos” are deleted from the first series, then the first of these terms, for instance, can be recognized as an “abbreviated designation,” a much shorter way to describe the number “resulting from the multiplication of a square number by itself,” thus rendering the text of Diophantus consistent in respect to the two series. (3) As previously noted, terms of the first series are used in the enunciations of the problems of the Arithmetica. However, the terms “dynamodinamis,” “dynamokybos,” and “kybokybos” do not appear at all in the enunciations of the problems. On the contrary, the only terms appearing in the enunciations are “pleura,” “tetrâgonos,” and “kybos,” and of course the term “arithmos” (number) in the sense of the common noun of everyday language. (4) Finally, the emendation I suggest is not unlikely on general grounds, such as the smooth reading of the Greek text.  

3.3. The second part of the introduction: the way of solving the problems

3.3.1. Invention: the first stage of the method of Diophantus

In the second part of the introduction—the more extensive and interesting of the two, and the one in which Diophantus describes the ὀδὸς, that is, the general way one has to follow in order to solve arithmetical problems—Diophantus begins his description of the general method with a sentence that is revealing with respect to the first stage of the method. “It is a confirmed opinion that each of these numbers, after having received an abbreviated designation, constitutes an element of the arithmetical theory” (Εδοκιμάσθη οὖν ἕκαστος τούτων ὁ ἀριθμὸς συντομωτέραν ἐπωνυμίων κτισθένις τῆς ἀριθμητικῆς θεωρίας εἶναι) [Tannery, 1893–1895, I, 4.12–14]. Although often discussed in modern commentaries, this sentence is not always properly comprehended. Thus, Rashed suggests that, with this sentence, Diophantus indicates that his aim in the Arithmetica is to build, himself, an arithmetical theory: “Or, dans les Arithmétiques, le but du mathématicien est clair: édifier une théorie arithmétique” [Rashed, 1979, 197]. This interpretation suits well Rashed’s thesis that the Arithmetica is not an algebraic work but rather a work belonging to arithmetic, but it does not take into consideration the past tense of ἐδοκιμάσθη, with which, as Tannery has remarked, “notre auteur reproduit une tradition consacrée” [Tannery, 1884, 68]. That is, the arithmetical theory existed already, having been elaborated before Diophantus wrote his work [Christianidis, 1991, 240]. In my view, the arithmetical theory—to which the subsequent text of the introduction, through line 14.10, is devoted—constitutes a large part of the auxiliary material involved in the εὑρεσις (“invention”) of arithmetical problems. This material comprises, among other things, operations with species of the unknown number, arithmetical identities, proportions and their properties, etc.

Apart from the allusion to “the arithmetical theory” (τῆς ἀριθμητικῆς θεωρίας), lines 4.12–14 contain also the phrase συντομωτέραν ἐπωνυμίων (“abbreviated designation”), an expression critical to the way in which the εὑρεσις should proceed. These lines reveal Diophantus’ point that the terms included in the enunciation of a problem (the terms “pleura,” “tetrâgonos,” and “kybos”), i.e., the sought numbers of the problem, should acquire “abbreviated names,” should be translated into a technical language, that is, the language of the arithmetical theory. The result of the translation is the transfer of the problem into the framework of the arithmetical theory, and the outcome of this transfer is an equation. The language in which this equation is written is not the language used in the formulation of the problem, is not the common language (the “koinê”). It is a technical language, the language of the arithmetical theory, the language of the algebraic unknown and its powers.  

Misidentification of the algebraic unknown and its powers with the sought numbers, i.e., the requested numbers appearing in the enunciation of a problem, is a mistake made often by commentators on Diophantus, leading them into serious misunderstandings of the way by which he worked. The aforementioned textual corruption, by which the series of the sought numbers of the enunciation is made

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8 Distortion in an earlier manuscript having “passed” into all later manuscripts is by no means unusual in the history of the manuscript tradition of ancient Greek science. We must not forget that the copyists were engaged in copying, not understanding, and that they were often unable to distinguish whether a marginal note was intended to fill a lacuna or to complement preserved text. In the case of the Arithmetica, the oldest preserved manuscripts are dated not earlier than the 13th century, and they all originate from a lost archetype which, according to Tannery, was written in the 8th or 9th century [Heath, 1964, 15; Tannery, 1893–1895 II, xxiii]. The distortion which I point out here must have already existed in this archetype, which would explain its presence in all preserved manuscripts of the Arithmetica. For a similar thesis on these specific terms in the introduction to the Arithmetica, see [Ruska, 1917, 68–69; followed by Klein, 1992, 251 n. 177].

9 A similar phenomenon is observed with the use of the term māl in medieval Arabic algebra. When used in the enunciation of a problem, this term is a common noun meaning “quantity,” while, when used in the resolution, it is a technical term meaning the square of the algebraic unknown. See [Oaks and Alkhateeb, 2005].
to coincide more or less with the series of the powers of the algebraic unknown, may account in part for this mistake. Properly understood, the sentence “It is a confirmed opinion that each of these numbers, after having received an abbreviated designation, constitutes an element of the arithmetical theory” should be taken to indicate that the first stage of the method of Diophantus (that is the “invention”) consists in the translation of the problem into the language of the arithmetical theory, and in the transformation of the terms that appear in the enunciation into “elements of the arithmetical theory.”

Now, the basic technical terms of the language of the arithmetical theory are the terms “arithmos,” “dynamis,” “kybos,” “dynamodynamis,” “dynamokybos,” and “kybokybos,” as well as “monas.” More specifically:

- “Dynamis” is the term used for a square number (“tetragônos”) transferred to the theory.
- “Kybos” is the term used for a cubic number (“kybos”) transferred to the theory.
- “Dynamodynamis” is the term used for a number produced “from the multiplication of a square number by itself” transferred to the theory.
- “Dynamokybos” is the term used for a number produced “from the multiplication of a cube by the square of the same side” transferred to the theory.
- “Kybokybos” is the term used for a number produced “from the multiplication of a cube by itself” transferred to the theory.

The linguistic difference between the terms of the two series is apparent. It is also apparent that the terms of the second series have shorter names than the terms of the first. The only term that appears in both series, accompanying the enunciation of a problem as well as the resolutive procedure, is the term “kybos.” In interpretations of the Diophantine text, the use of “kybos” as a sought number and as an element of the arithmetical theory has given rise to confusion. The danger was perceived by the Arab translator of the Arithmetica, Qustâ ibn Lûqâ, who dealt with it by using two different terms, $mukaabab$ to indicate the cubic number of the common language (the cube that appears in the enunciation of a problem) and $kaab$ to indicate the technical term “cube” (the cube of the arithmetical theory) [Rashed, 1984a, lxxiii; Sesiano, 1982, 452–453, 66 n. 39].

At this point, let us recapitulate what I have said about the strategy of resolution of Diophantus. The first stage of the resolution is the invention. The invention is performed as a transition from one set of terms (the terms appearing in the enunciation of the problem) to another set of terms (the technical terms of the arithmetical theory). Thus, in the course of invention, the sought numbers (appearing in the enunciation of the problem as “number” or “pleura,” “tetragônos,” and “kybos”) are expressed as a function of the algebraic unknown and its powers. This process of translating the problem into the arithmetical theory is quite obvious in such Diophantic expressions as “let us put the side 1 arithmos” (τετράγωνος ἡ πλευρά ἀριθμός α) and “let us put the tetragonos 1 dynamis” (τετράγωνος ὁ τετράγωνος δύναμις α). In these examples, the words πλευρά and τετράγωνος are words of the common language, the language in which the problems are formulated. The words ἀριθμός and δύναμις, on the other hand, are terms of the theory. The first are requested numbers. The latter are kinds of the unknown. But how should such expressions be interpreted in a modern presentation that wishes to remain faithful to the spirit of the text? If we fail to recognize that, by using phrases such as the above, Diophantus in fact translates the terms of the problem into the technical language of the arithmetical theory, or, in other words, transforms the sought numbers into unknown ones, then the above expressions could be read as “let us put $x = x$” and “let us put $x^2 = x^2$,” which is, of course, entirely meaningless. To avoid this absurdity, some modern commentators on Diophantus have invented various clever ways of displaying the difference between the requested and the unknown numbers. So, in his French translation of the Arabic books of the Arithmetica, Rashed interprets the words “tetragônos” and “kybos” as “carré” and “cube,” respectively, and the terms “dynamis” and “kybos” of the arithmetical theory as “carré” and “cube,” writing the first letter of the terms of the second pair in italics in order to distinguish them from the terms of the previous pairing. In the English edition of the same books, Sesiano represents the words “tetragônos” and “kybos” as $□$ and $□$, while he uses the familiar exponential symbolism $x^2, x^3$, etc., for the terms “dynamis,” “kybos,” etc. Finally, in his French translation of the six Greek books of the Arithmetica, Paul ver Eecke uses the symbols $X, X^2, X^3$ in capitals for the words “pleura,” “tetragônos,” and “kybos."

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10 In the absence of the suggested emendation, there is no linguistic differentiation between the two series, apart from that represented by the first term (“dynamis” instead of “tetragônos”). But even in this case, my interpretation is by no means untenable: one can still see the two series of terms as distinct, even if their outward appearance is, largely speaking, identical.
and the symbols $x, x^2, x^3$, etc. in lower case letters for the terms “arithmos,” “dynamis,” “kybos,” “dynamodynamis,” “dynamokybos,” and “kybokybos.” If we use the latter symbolism, then the two series mentioned before could be read as “let us put $x = x$” and “let us put $x^2 = x^2$.” All three of these formulations—those of Rashed, Sesiano, and ver Eecke—maintain the distinction between the two series of names and emphasize the process of translating the problem into the language of the “arithmetical theory” within which the problem will be solved. Because of its relative simplicity, I have previously adopted the choice of P. ver Eecke [Christianidis, 1998]. In the resolutions, the terms “arithmos,” “dynamis,” “kybos,” “dynamodynamis,” “dynamokybos,” “kybokybos,” and “monas” do not appear, but in the form of their abbreviations, such as $\Delta Y$, $K^Y$, $\Delta Y$, $\Delta K^Y$, etc.

Now, after having presented the terms of the arithmetical theory—including the terms that indicate the corresponding $\delta\mu\nu\nu\mu\alpha$ $\omicron\nu\mu\alpha\mu$ (homonymous fractional parts)—as well as their abbreviations, Diophantus continues the description of the framework within which the “invention” of each problem proceeds, by presenting the operations between the terms: “After having exposed to you the designation of each number, I will now proceed to the multiplications between them” (Εκθέμενος οὕς οὐς τῇ ἐκάστου τῶν ἄριθμῶν ἐπωνυμίαν, ἐπὶ τοὺς πολύπλασιαμοὺς ἀριθμοὺς [Tannery, 1893–1895, I, 6.22–24]. The operations are explained in a fully methodical manner until line 14.10.

At this point in the text of the Arithmetica, we have been fully apprised of the framework within which the manipulation of an arithmetical problem must, according to Diophantus, take place. This framework is an “arithmetical theory,” consisting of its own terms and operations. And the first stage of the solving strategy of Diophantus is the translation of the problem into the language of the arithmetical theory. Diophantus calls this stage, as we have already noted, εὑρεσις (“invention”), the outcome of which is the transformation of the problem into an equation. The emergence of the equation marks the end of the invention, that is, the end of the first stage of the solving strategy, the end of the first stage of the ὁδὸς of Diophantus.

A point not always appreciated by modern interpreters of Diophantus is that the equation emerges only after the transfer of all the components of the problem (i.e., the elements and the structure) into the arithmetical theory has been completed, after the translation of all the elements contained in the formulation of the problem into the language of the theory has been performed. The text of Diophantus is entirely clear on this point: “After that, when a problem has led to an equation in which certain species are equal to species . . .” (Μετὰ δὲ ταῦτα ἐὰν ἀπὸ προβλήματος τινος γένηται εἴδη τινά ἱσα αἰσθέοι ένεται . . .) [Tannery, 1893–1895, I, 14.11–12; emphasis added]. This is the first time the word ἱσα (“equal”) appears in the text of the introduction. The fact that it appears after the words Μετὰ δὲ ταῦτα (“after that”) is not at all accidental: the equation appears after the process of assigning “abbreviated designations” to all the elements contained in the enunciation of the problem has been completed. Thomaidis [2005] is thoroughly misguided in his claim that the treatment of the arithmetical problems by Diophantus consists in transforming the equation from an “initial” form to a “manageable” form so that it may ultimately attain its “final” form. What Thomaidis calls an “initial” form of the equation does not, however, exist in Diophantus as an equation. Thomaidis, here, is in fact transforming into an equation an intermediate “moment” of the process of transferring the problem into the arithmetical theory, that is, an “instant” of the translation process. But, as we have noted, the equation appears only after that,” i.e., when the whole process of transference has been completed.

The formation of the equation marks the end of the “invention,” the end of the first stage of the general method of Diophantus. This stage constitutes the most sophisticated part of Diophantus’ method, the part whose accomplishment requires the greatest skill and artfulness. Sesiano’s assertion that “c’est dans l’expression des grandeurs cherchées en fonction de l’inconnue que se manifeste l’art de Diopante” [Sesiano, 1999, 34] is accurate and to the point. The major part of the introduction, lines 4.12 to 14.10 in the Tannery edition, is dedicated to the description of this stage. This stage is, moreover, also the most important part of the method. This is why Diophantus chose to begin the introduction, and therefore the Arithmetica itself, by denominating precisely this stage and by identifying his intention in the Arithmetica to be teaching the reader how to perform τὴν εὑρεσιν τῶν ἐν τοῖς ἄριθμοῖς προβλημάτων.

3.3.2. Disposition: the second stage of the method of Diophantus

The second stage of Diophantus’ method, the “disposition,” is described only briefly in the introduction (lines 14.11–24 in the Tannery edition) and is not even named by Diophantus. The name assigned to it comes from Viète’s predecessors, the French algebraists of the 16th century (see Cifoletti, 1992, 1995). Disposition is the transformation of the equation arrived at through invention, into its final form. The disposition of the equation is carried out with the application of two rules, known in the history of mathematics by their Arabic names: al-jabr and al-muqabala. At
the end of the disposition, the equation receives the form “one species = one species” or the form “two species = one species,” from which the unknown number is easily determined.

4. The last two stages of the procedure of resolution and the origin of the Arabic books of the *Arithmetica*

At the point in the procedure when the unknown number is determined, the role of the “arithmetical theory” has been completed. The content of the introduction of the *Arithmetica* has also been exhausted, as the aim of Diophantus in the introduction was to describe and explain the two basic stages of the solution strategy, i.e., the invention and the disposition. In completing the treatment of the disposition, moreover, Diophantus completes all that he wants to say, not only in the introduction but also in his entire work, concerning the way of treating the arithmetical problems. It is not irrelevant to this observation that problems are presented in the *Arithmetica* whose resolution stops suddenly upon finding the unknown number, i.e., upon the completion of the disposition (for example, problems 21, 22, 23, 27, 28, and 29 of the fifth Greek book), notwithstanding the fact that, subsequent to the invention and the disposition, the complete resolution of an arithmetical problem requires two further stages: (1) The calculation of the values of the requested numbers (i.e., finding the numbers called for by the statement of the problem), a calculation performed by means of the relations through which the requested numbers are expressed as a function of the unknown. (2) The test proof, i.e., the verification that the determined values of the requested numbers indeed satisfy the conditions of the problem. Diophantus usually engages in the first of these two stages only briefly, in some cases omitting it completely, and when he does not entirely omit the second of these stages, he merely points it out using such cliché phrases as “and the proof is clear,” “and they do the problem.” Historians of mathematics often account for Diophantus’ omission of, and mere hinting at, these stages by claiming that he wants to leave the detailed elaboration of the problems as homework for the reader. I concede that this point may bear a certain weight. However, on the interpretation that Diophantus’ main goal in the *Arithmetica* is not the very solution of the problems, and therefore the complete and detailed presentation of all stages of the procedure of resolutions but is, rather, to indicate how one is to work out the two basic stages of the resolutory procedure, i.e., the invention and the disposition, we may more reasonably account for Diophantus’ brief treatment of these latter two stages: since the goal of Diophantus is to exhibit the way in which one should work in order to transfer an arithmetical problem into the arithmetical theory and in order to deal with it using the tools of that theory, the detailed calculation of the sought numbers and the test proof have no bearing on this goal, are of no importance for Diophantus, and are not the object of his concern.

At this point, I would like to make a digression in order to discuss a matter related to the Arabic translation of books IV–VII of the *Arithmetica*. In the Arabic books of Diophantus, the last two stages of the resolution (the calculation of the sought numbers and the test proof) are exposed in detail, as they are not in the six preserved Greek books. According to Sesiano, this is “the most striking difference in form between the Greek and the Arabic texts” [Sesiano, 1982, 48]. This difference “in form” between the Greek and the Arabic books is patently not essential, in that it concerns the form of presentation and not the mathematical arguments used in each resolution. To be sure, Late Antique commentators were painstakingly occupied with matters concerning the form of presentation [Netz, 1998, 269], and, in part on these grounds, Sesiano has suggested that the Arabic manuscript of the *Arithmetica* is not the translation of the *Arithmetica* itself, but is the translation of a “Major Commentary” on the work of Diophantus, possibly the early fifth-century commentary of Hypatia. It is the preserved Greek books of the *Arithmetica*, Sesiano submits, that derive from the text of Diophantus [Sesiano, 1982, 71; 73–75]. I.G. Bashmakova, E. Slavutin, and B. Rozenfeld had suggested, even prior to Sesiano, that “it is only natural to suppose that she (Hypatia) was the author of the books whose Arabic translation we are now discussing. To be more exact, the Arabic manuscript is the translation of part of Diophantus’ work commented on by Hypatia, the original problems of Diophantus being interspersed with her own comments” [Bashmakova et al., 1981, 159]. The above disquisition on the goals of Diophantus in the *Arithmetica* reinforces these suggestions: if Diophantus restricted his intention in the *Arithmetica* to indicating the ways in which the invention and disposition should be conducted, there would have been a clear need for a commentary that adapted the text of Diophantus to a more classical form of exposition—by adding, for example, the missing or tacitly assumed parts, such as the detailed calculations of the requested numbers, the test proof, and the final statement. Such a need may have motivated Hypatia to write her commentary on the *Arithmetica*, and the Arabic translation, in turn, may well be a translation of that work.
5. The canon of Diophantus and its application to Problem II.8

On the basis of the preceding discussion, we are now in a position to present Diophantus’ general method of arithmetical problem solving. This method includes the parts presented in the table below.

<table>
<thead>
<tr>
<th>The canon of Diophantus for solving arithmetical problems</th>
</tr>
</thead>
<tbody>
<tr>
<td>(1) Invention — transfer of the problem (in its instantiated version) to the framework of the “arithmetical theory,” i.e., transformation of the problem into an equation</td>
</tr>
<tr>
<td>(2) Disposition — transformation of the equation into its final form, and finding the unknown number</td>
</tr>
<tr>
<td>(3) Computation of the sought numbers</td>
</tr>
<tr>
<td>(4) Test proof</td>
</tr>
</tbody>
</table>

The parts shown in the table above do not constitute the whole of the stages through which the resolution of an arithmetical problem is developed. In the *Arithmetica*, a problem is always presented in a general and abstract formulation, containing the requested numbers and also the given numbers, if any. Next comes, if necessary, the statement of the condition of possibility, ensuring the solvability of the problem in positive, rational numbers. There follows the instantiation of the problem, and then, but not always, the instantiated enunciation. This instantiated version of the general problem is the one that Diophantus solves using the canon.

Indicated by the numbers (1) and (2) in the table are those stages from the canon—and, in fact, from the whole of the stages through which the resolution of a problem passes—that are developed within the “arithmetical theory.” Those two stages, and in particular the first of them, constitute the core of the method of Diophantus: his intention, the very reason he wrote the *Arithmetica*, is to explain the functioning of those stages.

Let us now examine an application of the canon to a problem provided in the *Arithmetica*. I will use the famous Problem 8 of Book II. Incidentally, II.8 is the first really “Diophantine” problem of the *Arithmetica*, the problems of the first book being, in modern terms, determinate, and Problems II.1–7 being considered, following Tannery, to be interpolated from an ancient commentary. The text of the *Arithmetica* contains two resolutions of this problem, of which the second is indicated with the word ἄλλως (“otherwise”). The full text is presented in the table below:

<table>
<thead>
<tr>
<th>First resolution</th>
<th>Second resolution</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>General formulation</strong></td>
<td>To divide a proposed tetragōnos into two tetragōnoi.</td>
</tr>
<tr>
<td><strong>Instantiated formulation</strong></td>
<td>Let it be proposed to divide 16 into two tetragōnoi.</td>
</tr>
<tr>
<td><strong>Invention</strong></td>
<td>And let us put the first tetragōnos 1 dynamis; then the other will be 16 units minus 1 dynamis. Therefore, 16 units minus 1 dynamis must be equal to a tetragōnos. I form the tetragōnos from any number of arithmoi minus as many units as there are in the side of the 16 units; be it 2 arithmoi minus 4 units. The tetragōnoi itself will be 4 dymeis plus 16 units minus 16 arithmoi. These are equal to 16 minus one dynamis.</td>
</tr>
<tr>
<td><strong>Disposition</strong></td>
<td>Add to both sides the missing terms and take like from like. Then 5 dymeis equal 16 arithmoi, and the arithmos becomes 16/5.</td>
</tr>
<tr>
<td><strong>Computation of the sought numbers</strong></td>
<td>The one will therefore be 256/25, the other 144/25, and their sum is 400/25 or 16, and each is tetragōnos.</td>
</tr>
<tr>
<td><strong>Test proof</strong></td>
<td>And the proof is clear.</td>
</tr>
</tbody>
</table>

11 For the interpolated problems in the Greek text of Diophantus, see the Tannery edition; for the interpolations in the Arabic books, see [Sesiano, 1982, 50–57].
In the succeeding presentation, terms that have not yet received abbreviated designations and therefore do not belong to the arithmetical theory are symbolized by capital letters, while terms that have abbreviated designations, and therefore belong to the arithmetical theory are symbolized by lower case letters. Thus, the requested numbers are symbolized by $X$, $Y$, etc., while the unknown and its powers are symbolized by $x$, $x^2$, etc. I will start with the first resolution.

The problem is “To divide a proposed tetragônos into two tetragônoi.” This is the general formulation, followed by the instantiation: it is proposed that the particular tetragônos 16 be divided into two squares. Hence, the instantiated version of the generally formulated problem could be described, according to the agreed convention for the symbolism, as

$$X^2 + Y^2 = 16.$$

At this point, the invention, that is to say the transfer of the terms and of the structure of the problem into the arithmetical theory, begins. The transfer is performed by translating $X^2$ and $Y^2$ into the language of the arithmetical theory. First, Diophantus translates one of the two squares, $X^2$. He uses the phrase, “let us put the first tetragônos 1 dynamis,” and we can write this phrase as $X^2 = x^2$. Herewith, the first tetragônos has been transferred into the theory; it has become one dynamis. Now the second tetragônos must also be transferred. Diophantus writes: “The other [tetragônos] will be 16 units minus 1 dynamis. Therefore, 16 units minus 1 dynamis must be (δειξαι) equal to a tetragônos.” The verb δειξαι here means, precisely, that in order to transfer the problem (as a whole) into the arithmetical theory, and therefore to give it the form of an equation, we must previously express also the second tetragônos in the language of the theory. And to achieve this, we must express the side of the tetragônos in the language of the theory. Diophantus expresses the side of the second tetragônos by saying, “I form the tetragônos from any number of arithmoi minus as many units as there are in the side of the 16 units; be it 2 arithmoi minus 4 units.” We can write this phrase as $Y = 2x - 4$. Now, the second tetragônos has also been transferred into the theory, since it has become $4x^2 + 16 - 16x$. So, all the constituents of the problem have been transferred into the theory and now the problem (in its instantiated version) can acquire the form of an equation. In this case the equation is $4x^2 + 16 - 16x = 16 - x^2$. The invention finishes at this point.

The next stage is the disposition, that is, the transformation of the equation into its final form and its solution. Diophantus describes this stage quite briefly. The final equation is $5x^2 = 16x$, and therefore $x = 16/5$. The disposition here comes to an end.

Next, the requested tetragônoi $X^2$ and $Y^2$ are calculated from the value of $x$, and the test proof follows.

The starting point of the second resolution is, once again, the above instantiated formulation of the general problem. The invention here is to put $X = x$ and $Y = 2x - 4$; therefore the equation into which the instantiated problem is translated is $5x^2 + 16 - 16x = 16$. The disposition is here practically omitted and the solution of the equation is only mentioned. There follows a brief calculation of $X^2$ and $Y^2$ and a mere hint of the test proof.

If we now compare the two resolutions, we notice that both are developed according to the stages of the canon described before. We also notice that the invention is the most important part of the canon, the one that demands the greater elaboration. In contrast, the remaining parts of the canon are only briefly presented or merely mentioned. But do we really have here two different resolutions? Is the use of the word ἀλλως at the beginning of the second resolution justified? Could the second resolution be, in fact, identical with the first? And, if the answer is yes, could the text of the second resolution belong not to Diophantus himself, but be a later addition to his work by a less than astute commentator? These are some questions that scholars have raised, and on which we are now able to throw new light.

There is no doubt that, from a mathematical point of view, the two resolutions are identical. They nevertheless differ in that the invention is performed differently in each. In the first resolution, the translation of the requested squares into the language of the arithmetical theory is performed in successive steps. First, one of the requested squares is translated; then follows the translation of the other. In the second resolution, the two squares are translated simultaneously. Furthermore, in the first resolution, the equation into which the problem is transformed is $4x^2 + 16 - 16x = 16 - x^2$. In the second resolution, the equation is $5x^2 + 16 - 16x = 16$. It is obvious that we have here two different ways of performing the invention, and therefore the two resolutions are indeed different, though mathematically identical. It follows that the use of the word ἀλλως to indicate the second resolution is fully justified and that, in my view, there is no sound reason to dispute Diophantus’ authorship of the second resolution.
Comprehending the canon of Diophantus in the way proposed here reveals the relevance, the intrinsic relationship, between the introduction of the *Arithmetica* and the rest of the work. It also reveals Diophantus’ intention in his statement that the solution of arithmetical problems can be achieved “if you follow the way I will show you.” The σαμαία (“way”) is nothing other than the canon cited. But the clarification of the “way” does not ensure resolution of all the issues involved in Diophantus’ mathematical practice, because the “way” is the general method, the general strategy of arithmetical problem solving. Application of the method in practice, in contrast, requires the use of a number of other particular techniques in order for the method to “pass” successfully through all the steps of the procedure of resolution. This observation is well illustrated in the case of Problem II.8, for which Diophantus performs the invention in two ways.

Historians of mathematics have proposed a range of interpretations regarding the particular techniques that Diophantus uses in his practice. In a previous study [Christianidis, 1998] I suggested one such interpretation, arising from the comment of the Byzantine monk Maximus Planudes (end of the 13th century) on Problem II.8. I restate briefly, at this point, that interpretation. In the first resolution of Problem II.8, Diophantus translates the requested *tetragōnoi* into the language of the arithmetical theory in successive steps. He first translates the first *tetragōnoi* by putting $X^2 = x^2$. Then he transforms the problem into a form that can be represented by $16 - x^2 = y^2$. After this, he translates the second *tetragōnos* by putting $Y = 2x - 4$. I believe that a sort of argument intervenes between the translation of the first *tetragōnos* and the translation of the second; that there is a sort of hidden mathematical reasoning in which the first square’s receipt of an “abbreviated designation” plays an essential role. According to the interpretation I have suggested, this reasoning could be reconstructed as follows:

\[
16 - x^2 = y^2 \rightarrow 16 - x^2 = x^2 \rightarrow (4 - Y)(4 + Y) = xx \rightarrow (4 + Y) : x = x : (4 - Y) \rightarrow (4 + Y) : x = 2 : 1 \rightarrow Y = 2x - 4.
\]

Now, in the second solution, Diophantus simultaneously puts $X = x$ and $Y = 2x - 4$. There is no delay between the translation of the first *tetragōnos* and the translation of the second. There is here no “hidden,” sophisticated argument. The two relations are, instead, deduced straightforwardly from simple inspection of the structure of the problem.

6. Diophantus and algebra

I will conclude this paper with some thoughts concerning the relationship between the mathematical practice of Diophantus and algebra. As mentioned above, historians of mathematics have expressed a variety of opinions on this issue. Their theses range from the extreme claim that Diophantus is the “father of algebra” to the claim at the other extreme that the work of Diophantus should not be interpreted as an algebraic work, since it is in fact a work belonging to arithmetic. The latter thesis, as noted before, has been supported by Rashed. The majority of historians, however, agree with the view that the work of Diophantus is a work with algebraic characteristics, these characteristics being usually found in its use of symbolism, its operating with στὶς (“species”), and its algorithmic features. The above analysis allows us to approach the issue from a different point of view.

It is true that if one starts from a conception of algebra that emphasizes the solution of equations, as was generally the case with the Arab mathematicians from al-Khwārizmi onward as well as with the Italian algebraists of the Renaissance, then the work of Diophantus appears indeed very different from the works of those algebraists. The fundamental difference could be described in terms used, though in a different context, by Giovanna Cifoletti: the works of the Arab algebraists and their successors “ont privilégié la théorie des équations,” while Diophantus in the *Arithmetica* “a privilégié la solution des problèmes” [Cifoletti, 1995, 1411]. Diophantus’ intention in the *Arithmetica* is not to present a theory for solving algebraic equations. His goal, as I have taken the opportunity to explain on many occasions, was to elaborate a canon on the basis of which several arithmetical problems could be treated and to demonstrate how this canon should be used in practice. In this sense, the aim of the *Arithmetica*, and therefore the specific character of the work, is much different from the aim and character of, for example, the *Algebra* of al-Khwārizmi. The program of Diophantus in the *Arithmetica* was much different from the program of al-Khwārizmi. If, therefore, we characterize algebra on the basis of the work of al-Khwārizmi, then we will find in the *Arithmetica* nothing but algebraic seeds.
But do we reach the same conclusion if we compare the enterprise of Diophantus not with the enterprise of al-Khwārizmi but with that of Viète? Viète had also elaborated a program for problem solving—in his case, by using algebra. The elements of his program are summarized by Henk Bos as follows [Bos, 1996, 188]:

**Elements of Viète’s program**

[A] Dissociate algebra from numbers; make it general.
[B] Elaborate a canon for problem solving.
   [B1] Translate the problem into algebra.
   [B3] Translate the algebraic result (equation, solution) back and solve the problem.
   [B4] Prove that the solution is correct.

The aim of the second part of the program, that is, of the canon, was, according to Bos, “to control the coming and going between a problem (which might be numerical or geometrical or general) and its algebraic counterpart used in analysis” [Bos, 1996, 188]. It is not my objective to discuss, here, Viète’s answer to the different issues raised by this program, which is thoroughly presented by Bos [1996, 2001]. We observe, however, that Viète’s program and Diophantus’ canon share some important elements of structure. The second part of Viète’s program consists of four steps, as does the canon of Diophantus. A main characteristic of both is the transfer of the problem into another framework, which in the case of Diophantus is the “arithmetical theory.” The aim of the transfer is to handle the problem with the tools and the techniques of this new framework. The result of the transfer is the transformation of the problem into an equation. After the solution of the equation, the program of Viète and also the canon of Diophantus provide for the “exit” from the corresponding framework, the “return” to the framework of the initial problem, and finding the answer to the problem. Finally, they both include a final check, a test proof, to verify that the given answer is correct.

These shared characteristics of structure do not entail the conclusion that the two programs are identical. Viète’s intention, when he was working out his program, was entirely different from that of Diophantus. Viète’s intention was “to leave no problem unsolved” (“nullum non problema solvere”) [Bos, 1996, 187; Klein, 1992, 353]. The problems to which his program was addressed were not only arithmetical, but were problems of all kinds: arithmetical, geometrical, and general problems. In contrast, the problems with which the canon of Diophantus can deal are exclusively arithmetical. In addition, the means by which Viète handles the problems is symbolic algebra. The means by which Diophantus handles the problems is the “arithmetical theory.” The “arithmetical theory” is not symbolic algebra—about this, there can be no doubt. Yet, to deny that the “arithmetical theory” of Diophantus has any relationship with algebra, on the grounds that his theory is about arithmetic, would be inaccurate. Or, to put it in another way, the fact that the “arithmetical theory” is arithmetical, that it belongs to arithmetic, is not the whole truth about the “arithmetical theory.” As Henk Bos explains: “Algebra refers to those mathematical theories and practices that involved unknowns and/or indeterminates, employed the algebraic operations, involved equations, and dealt either with numbers or with geometrical magnitudes or with magnitudes in an abstract more general sense. In as far as it dealt with numbers, algebra was part of arithmetic. Algebra dealing with (geometrical or abstract) magnitudes presupposed (tacitly or explicitly) a redefinition of the algebraic operations so as to apply to such magnitudes” [Bos, 2001, 129; emphasis added]. Bos here refers to algebra as practiced by the algebraists of the early modern period. But from this description, we further infer that the characterization of a work or a practice as arithmetic does not necessarily exclude its involvement with algebra. Taking, finally, into consideration the interpretation of the practice of Diophantus suggested in this paper, the description of algebra proposed by Bos can be extended to characterize Diophantus’ work as well.

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