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Splitting methods for SU(N) loop approximation

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Abstract

The problem of finding the correct asymptotic rate of approximation by polynomial loops in dependence of the smoothness of the elements of a loop group seems not well-understood in general. For matrix Lie groups such as SU(N), it can be viewed as a problem of nonlinearly constrained trigonometric approximation. Motivated by applications to optical FIR filter design and control, we present some initial results for the case of SU(N)-loops, $N \geq 2$. In particular, using representations via the exponential map and first order splitting methods, we prove that the best approximation of an SU(N)-loop belonging to a Hölder–Zygmund class Lip_α, $\alpha > 1/2$, by a polynomial SU(N)-loop of degree $\leq n$ is of the order $\mathrm{O}(n^{-\alpha/(1+\alpha)})$ as $n \to \infty$. Although this approximation rate is not considered final, to our knowledge it is the first general, nontrivial result of this type.
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1. Introduction

The study of classes of periodic functions with values in a Lie group G (so-called loops) is of theoretical importance as a simple example of infinite-dimensional Lie groups [8], but also occurs in a more practical context. For example, polarization mode dispersion in fiber-optical communication systems is studied using SU(2)-valued functions [7], lossless multi-port communication involves para-unitary transfer matrices (in other words, SU(N)-valued functions) [13], the theory of orthogonal wavelet constructions involves polyphase

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symbol representations that are SU(N)-valued Laurent polynomials [1,10]. In all these areas, approximation problems by trigonometric polynomials with values in these matrix Lie groups arise in a natural way—they are, e.g., the key to constructing and operating optical FIR filter architectures for polarization mode dispersion compensation [7].

The classical Jackson–Bernstein theory of approximation by trigonometric polynomials gives a quantitative answer to the connection between approximation rates and smoothness properties of periodic functions: A function f belongs to the Hölder–Zygmund class $\mathrm{Lip}_\alpha(\mathbb{T} \to \mathbb{C})$, $\alpha > 0$, if and only if its best approximations $E_n(f)$ by complex trigonometric polynomials of degree $\le n$ satisfy $E_n(f) = \mathrm{O}(n^{-\alpha})$ as $n \to \infty$. After searching in the literature, we did not find a similar result on quantitative approximation for general loop groups. The only reference is to [8] where the density of the subgroup of polynomial loops in the loop group $C^\infty(\mathbb{T} \to G)$ is proved for any compact semi-simple Lie group G. In particular, the polynomial loops are dense in $C^\infty(\mathbb{T} \to \mathrm{SU}(N))$ for all $N \ge 2$, see also [4,5] for detailed proofs.

In the present note we provide a Jackson-type estimate for SU(N) loops.

Theorem 1. *Let $U(t) \in \mathrm{Lip}_\alpha(\mathbb{T} \to \mathrm{SU}(N))$, where $\alpha > 1/2$ and $N \ge 2$. Then there exists a sequence of polynomial loops $U_n(t) \in \Pi_n(\mathbb{T} \to \mathrm{SU}(N))$ of degree $\le n$ such that*

$$\|U - U_n\|_C \le C_{\alpha,N,U}(n+1)^{-\alpha/(1+\alpha)}, \quad n \ge 0.$$

The approach for establishing this result is straightforward. By suitable factorization, we reduce the problem to studying the special case $N = 2$ and $U(t) = \mathrm{e}^{A(t)}$, where $A(t) \in \mathrm{Lip}_\alpha(\mathbb{T} \to \mathrm{su}(2))$. Next we approximate $A(t)$ componentwise by linear methods, and then use the splitting method for the exponential map to obtain a polynomial SU(2)-valued loop. The proof is carried out in detail in Section 3, and uses preliminary facts collected in a separate Section 2.

We do not consider the approximation rate obtained in Theorem 1 final in any respect. Rather we think that by publishing a result of possibly only temporary value we will attract further attention to a widely open and challenging area of nonlinear approximation: constructive approximation of manifold-valued functions. This field is currently fueled by several communities (symplectic integration of dynamical systems [3,6], manifold subdivision [9,14], etc.) where mostly local approximation schemes are developed. We add another angle by focusing on traditional polynomial approximations (here for periodic functions, i.e., loops). Note that the approximation problem for SU(2)-valued functions is a partial case ($d = 4$) of the problem of approximating loops on the unit sphere $S^{d-1} \subset \mathbb{R}^d$ by trigonometric loops. This, and the closely related question of SO(N)-loop approximation for $N \ge 3$ are other simple test cases of manifold-valued periodic functions that await treatment.

2. Definitions and auxiliary results

Throughout this paper, we consider various loop spaces $X(\mathbb{T} \to \mathcal{M})$, where $\mathcal{M} \subset \mathbb{R}^d$ or $\mathcal{M} \subset \mathbb{C}^d$ is some manifold, and the topology is induced from the coordinate spaces $X(\mathbb{T} \to \mathbb{R}^d)$ via a topology on $\mathcal{M}$. In particular, if $\mathcal{M} = G$ is one of the complex matrix Lie groups of dimension N, we have $G \subset \mathbb{C}^{N^2}$, and will use the topology induced by the Frobenius norm $\|\cdot\|_F$ (i.e., by the Euclidean norm in $\mathbb{C}^{N^2}$) or the spectral norm $\|\cdot\|_2$ (i.e., the operator norm for linear maps induced by the Euclidean norm in $\mathbb{C}^N$). We will drop the subscript in the norm whenever it does not matter which one we use. The Lie algebra associated with a Lie group G will be denoted by g.

We define the class of continuous loops $U(t) \in C(\mathbb{T} \to G)$ by requiring that $U(t) : \mathbb{T} \to G$ is continuous, and set

$$\mathrm{dist}_C(U_1, U_2) := \|U_1 - U_2\|_C := \max_{t\in\mathbb{T}} \|U_1(t) - U_2(t)\|_2.$$

Similarly, the function classes $C^k(\mathbb{T} \to G)$ are introduced. The Hölder–Zygmund classes $\mathrm{Lip}_\alpha(\mathbb{T} \to G) \subset C(\mathbb{T} \to G)$, $\alpha > 0$, are defined by the finiteness of the semi-norm

$$|U|_{\mathrm{Lip}_\alpha} := \begin{cases} \sup\limits_{h>0} h^{-\alpha}\|U(\cdot + h) - U(\cdot)\|_C, & 0 < \alpha < 1, \\ \sup\limits_{h>0} h^{-1}\|U(\cdot + h) - 2U(\cdot) + U(\cdot - h)\|_C, & \alpha = 1, \end{cases}$$

and by recursion for $\alpha > 1$ by requiring $U \in C^1(\mathbb{T} \to G)$ and $U'(t) \in \mathrm{Lip}_{\alpha-1}(\mathbb{T} \to G)$ and setting

$$|U|_{\mathrm{Lip}_\alpha} := |U'|_{\mathrm{Lip}_{\alpha-1}}.$$

Obviously, $C^k(\mathbb{T} \to G) \subset \mathrm{Lip}_\alpha(\mathbb{T} \to G)$. For further use we introduce the notation

$$\|U\|_{\mathrm{Lip}_\alpha} := \|U\|_C + |U|_{\mathrm{Lip}_\alpha}.$$

We note that for the matrix groups considered in this paper, the above introduced Hölder–Zygmund classes of loops form algebras, i.e., the Lip_α property is preserved under multiplication. This fact will be used below without further mentioning.

We are mostly interested in G-valued trigonometric polynomials (also called polynomial loops in G), and their g-valued cousins. I.e., we specifically consider loops $P_n(t) : \mathbb{T} \to G$ of degree $\le n$ whose entries belong to

$$\Pi_n(\mathbb{T} \to \mathbb{C}) := \left\{ p_n(t) = \sum_{|k|\le n} c_k z^k \right\}, \quad z \equiv \mathrm{e}^{\mathrm{i}t}.$$

The nonlinear set of all these $P_n(t)$ is denoted by $\Pi_n(\mathbb{T} \to G)$. An analogous definition holds for $\Pi_n(\mathbb{T} \to g)$, which is obviously a linear set.

Most of the paper deals with SU(2) and su(2) loops. The following simple result about $\Pi_n(\mathbb{T} \to \mathrm{su}(2))$ will be used below.

Lemma 1. *The space $\Pi_n(\mathbb{T} \to \mathrm{su}(2))$ of polynomial loops of degree $\le n$ in* su(2) *possesses a basis $\mathcal{B}_n$ over $\mathbb{R}$ given by the $3(2n+1)$ elements*

$$B_{1,k}(t) := \begin{pmatrix} 0 & z^k \\ -z^{-k} & 0 \end{pmatrix}, \qquad B_{3,k}(t) := \begin{pmatrix} 0 & \mathrm{i}z^k \\ \mathrm{i}z^{-k} & 0 \end{pmatrix}, \quad k = 0, \ldots, n, \tag{1}$$

$$B_{2,k}(t) := \begin{pmatrix} 0 & z^{-k} \\ -z^k & 0 \end{pmatrix}, \qquad B_{4,k}(t) := \begin{pmatrix} 0 & \mathrm{i}z^{-k} \\ \mathrm{i}z^k & 0 \end{pmatrix}, \quad k = 1, \ldots, n, \tag{2}$$

$$B_{5,k}(t) := \begin{pmatrix} \mathrm{i}\cos kt & -\sin kt \\ \sin kt & -\mathrm{i}\cos kt \end{pmatrix}, \quad k = 0, \ldots, n, \tag{3}$$

and

$$B_{6,k}(t) := \begin{pmatrix} \mathrm{i}\sin kt & \cos kt \\ -\cos kt & -\mathrm{i}\sin kt \end{pmatrix}, \quad k = 1, \ldots, n, \tag{4}$$

with the following properties: $\|B_{l,k}(t)\|_F = \sqrt{2}\|B_{l,k}(t)\|_2 = \sqrt{2}$, *and*

$$e^{cB_{l,k}(t)} = \cos c \cdot I + \sin c \cdot B_{l,k}(t) \in \Pi_k(\mathbb{T} \to \mathrm{SU}(2)) \tag{5}$$

for all $c \in \mathbb{R}$, $l = 1, \ldots, 6$, *and all* k. *Moreover,*

$$\prod_{j=1}^{J} e^{c_j B_{l,k_j}(t)} \in \Pi_n(\mathbb{T} \to \mathrm{SU}(2)), \quad l = 1, \ldots, 6, \tag{6}$$

for any product of this form with $0 \le k_1 < k_2 < \cdots < k_J \le n$.

Proof. Evidently, since $A \in \mathrm{su}(2)$ is equivalent to $A^* = -A$ and $tr(A) = 0$, we can always write

$$A = \begin{pmatrix} \mathrm{i} \cdot \beta & \gamma + \mathrm{i} \cdot \delta \\ -\gamma + \mathrm{i} \cdot \delta & -\mathrm{i} \cdot \beta \end{pmatrix}, \quad \beta, \gamma, \delta \in \mathbb{R}.$$

Moreover, $A^2 = -\phi^2 \cdot I$ and one easily verifies by definition of the matrix exponential that

$$e^A = \cos\phi \cdot I + \frac{\sin\phi}{\phi} \cdot A, \qquad \phi := \sqrt{\beta^2 + \gamma^2 + \delta^2} = \frac{\|A\|_F}{\sqrt{2}}. \tag{7}$$

For all three loop types, we have $\phi(t) \equiv 1$ and (7) implies for real c

$$e^{cB_{l,k}(t)} = \cos c \cdot I + \sin c \cdot B_{l,k}(t).$$

The properties $\|B_{l,k}(t)\|_2 = 1$ resp. $\|B_{l,k}(t)\|_F = \sqrt{2}$ are obvious from (1)–(4).

Due to (5), the property (6) follows if we can show that

$$\prod_{j=1}^{J} B_{l,k_j}(t), \quad 0 < k_1 < \cdots < k_J \le n, \quad J > 0,$$

is a matrix polynomial of degree $\le n$ for each $l = 1, \ldots, 6$. Since

$$\begin{pmatrix} 0 & \rho_1 z^{k'} \\ -\rho_1^* z^{-k'} & 0 \end{pmatrix} \begin{pmatrix} 0 & \rho_2 z^{k} \\ -\rho_2^* z^{-k} & 0 \end{pmatrix} = -\begin{pmatrix} \rho_1 \rho_2^* z^{k'-k} & 0 \\ 0 & \rho_1^* \rho_2 z^{k-k'} \end{pmatrix},$$

setting $\rho_1 = \rho_2 = 1$ we obtain for even $J = 2J' \ge 2$

$$\prod_{j=1}^{J} B_{1,k_j}(t) = \prod_{j=1}^{J'} (B_{1,k_{2j-1}}(t) B_{1,k_{2j}}(t)) = (-1)^{J'} \begin{pmatrix} z^{-m} & 0 \\ 0 & z^m \end{pmatrix},$$

where $0 < J' \le m = \sum_{j=1}^{J'} (k_{2j} - k_{2j-1}) \le k_J - k_1 < n$. For odd $J = 2J' + 1$, we obtain a similar result since

$$\prod_{j=1}^{J} B_{1,k_j}(t) = (-1)^{J'} \begin{pmatrix} z^{-m} & 0 \\ 0 & z^m \end{pmatrix} B_{1,k_{2J'+1}}(t) = (-1)^{J'} \begin{pmatrix} 0 & z^{-m+k_J} \\ -z^{m-k_J} & 0 \end{pmatrix}$$

leads to $0 \le |m - k_J| \le n$. I.e., in either case the result is a matrix polynomial of degree $\le n$. This settles the case $l = 1$.

The case $l = 2$ is quite similar (just replace m by $-m$). Repeating the argument with $\rho_1 = \rho_2 = i$ gives the cases $l = 3, 4$. To treat $l = 5, 6$, observe that

$$B_{5,k}(t) = U B_{1,k}(t) U^{-1}, \quad k \ge 0, \qquad B_{6,k}(t) = U B_{3,k}(t) U^{-1}, \quad k > 0,$$

where

$$U = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} \in \mathrm{SU}(2).$$

This, e.g., implies

$$\prod_{j=1}^{J} B_{5,k_j}(t) = U\left(\prod_{j=1}^{J} B_{1,k_j}(t)\right)U^{-1} \quad (k_j > 0),$$

and the reduction to the previous cases is achieved, similarly for products of B_{6,k_j}, $k_j > 0$.

Finally, the basis property follows from observing that an arbitrary loop $A_n(t) \in \Pi_n(\mathbb{T} \to \mathrm{su}(2))$ can be written in the form

$$A_n(t) = \begin{pmatrix} \mathrm{i}\beta_n(t) & \gamma_n(t) + \mathrm{i}\delta_n(t) \\ -\gamma_n(t) + \mathrm{i}\delta_n(t) & -\mathrm{i}\beta_n(t) \end{pmatrix}$$
$$= \begin{pmatrix} \mathrm{i}\beta_n(t) & \tilde{\beta}_n(t) \\ -\tilde{\beta}_n(t) & -\mathrm{i}\beta_n(t) \end{pmatrix} + \begin{pmatrix} 0 & p_n(t) \\ -p_n^*(t) & 0 \end{pmatrix}$$

where

$$\beta_n(t) = \sum_{k=0}^{n} a_k \cos kt + \sum_{k=1}^{n} b_k \sin kt, \qquad \tilde{\beta}_n(t) = \sum_{k=1}^{n} b_k \cos kt - a_k \sin kt$$

are a real-valued trigonometric polynomial $\beta_n(t)$ and its conjugate $\tilde{\beta}_n(t)$ associated with the diagonal entries of $A_n(t)$ (similarly, $\gamma_n(t), \tilde{\gamma}_n(t), \delta_n(t), \tilde{\delta}_n(t)$ are real-valued trigonometric polynomials and their conjugates corresponding to the off-diagonal entries), and

$$p_n(t) = \sum_{|k|\le n} c_k z^k \equiv \sum_{|k|\le n} (a'_k + i \cdot b'_k) z^k, \quad c_k \in \mathbb{C},\ a'_k, b'_k \in \mathbb{R}$$

is some complex-valued trigonometric polynomial of degree $\le n$. Thus, by setting $c_{1,k} = a'_k$, $c_{3,k} = b'_k$, $c_{5,k} = a_k$ for $k = 0, \ldots, n$, and $c_{2,k} = a'_{-k}$, $c_{4,k} = b'_{-k}$, $c_{6,k} = b_k$ for $k = 1, \ldots, n$, we obtain

$$A_n(t) = \sum_{l'=1}^{3}\left(\sum_{k=0}^{n} c_{2l'-1,k} B_{2l'-1,k}(t) + \sum_{k=1}^{n} c_{2l',k} B_{2l',k}(t)\right) \equiv \sum_{l=1}^{6} A_{n,l}(t) \tag{8}$$

with $c_{l,k} \in \mathbb{R}$. Since $A_n(t)$ was arbitrarily chosen from $\Pi_n(\mathbb{T} \to \mathrm{su}(2))$, and the number of elements in $\mathcal{B}_n$ coincides with the dimension of $\Pi_n(\mathbb{T} \to \mathrm{su}(2))$, this proves the basis property and Lemma 1 as a whole.

We note for later use that, by construction, the entries of the terms $A_{n,l}(t)$ in the decomposition (8) can be obtained from the entries of $A_n(t)$ by applying elementary operations and conjugation map $f \to \tilde{f}$. E.g. for $l = 5, 6$ this statement follows from separating even and odd parts of $\beta_n(t)$,

$$\frac{\beta_n(t) + \beta_n(-t)}{2} = \sum_{k=0}^{n} a_k \cos kt, \qquad \frac{\beta_n(t) - \beta_n(-t)}{2} = \sum_{k=1}^{n} b_k \sin kt.$$

Similarly, for $l = 1, \dots, 4$ we observe that as a function of t, $p_n(t) = (\gamma_n(t) - \tilde{\beta}_n(t)) + \mathrm{i}\delta_n(t)$, and use conjugation to split into $k \geq 0$ and $k < 0$ parts,

$$\frac{p_n(t) - \mathrm{i}\tilde{p}_n(t)}{2} = \frac{c_0}{2} + \sum_{k=0}^{n} c_k z^k, \qquad \frac{p_n(t) + \mathrm{i}\tilde{p}_n(t)}{2} = \frac{c_0}{2} + \sum_{|k|\leq n}^{n} c_k z^k,$$

in combination with separating c_k into real and imaginary parts:

$$\frac{p_n(t) + p_n(-t)^*}{2} = \sum_{|k|\leq n} a'_k z^k, \qquad \frac{p_n(t) - p_n(-t)^*}{2\mathrm{i}} = \sum_{|k|\leq n} b'_k z^k.$$

The next two lemmas follow by applying classical results from trigonometric approximation theory for Lip_α classes, see [2] or [15].

Lemma 2. *Let $A(t) \in \mathrm{Lip}_\alpha(\mathbb{T} \to \mathrm{su}(2))$, $\alpha > 0$. Then there exists $A_n(t) \in \Pi_n(\mathbb{T} \to \mathrm{su}(2))$ such that*

$$\|A - A_n\|_C \leq C_\alpha (n+1)^{-\alpha} |A|_{\mathrm{Lip}_\alpha}, \quad n \geq 0, \tag{9}$$

and

$$\|A_n\|_{\mathrm{Lip}_\alpha} \leq C_\alpha |A|_{\mathrm{Lip}_\alpha}, \quad n \geq 0. \tag{10}$$

Lemma 3. *Let $f(t) \sim \sum_{k\in\mathbb{Z}} c_k z^k \in \mathrm{Lip}_\alpha(\mathbb{T} \to \mathbb{C})$, $\alpha > 0$. Then*

$$\|f - S_n f\|_C \leq C_\alpha \frac{\ln(n+2)}{(n+1)^\alpha} |f|_{\mathrm{Lip}_\alpha}, \qquad S_n f(t) = \sum_{|k|\leq n} c_k z^k, \quad n \geq 0, \tag{11}$$

and

$$|c_n| \leq C_\alpha \frac{1}{(n+1)^\alpha} \|f\|_{\mathrm{Lip}_\alpha}, \quad n \geq 0. \tag{12}$$

Our strategy is to use factorization techniques and the exponential map to obtain similar approximation estimates for arbitrary $U(t) \in \mathrm{Lip}_\alpha(\mathbb{T} \to \mathrm{SU}(N))$.

Lemma 4. *For any $\alpha > 1/2$ and any $U(t) \in \mathrm{Lip}_\alpha(\mathbb{T} \to \mathrm{SU}(N))$, there exist constant matrices $U_{0,l} \in \mathrm{SU}(N)$ and loops $A_l(t) \in \mathrm{Lip}_\alpha(\mathbb{T} \to \mathrm{su}(2))$ such that*

$$U(t) = \prod_{l=1}^{L} U_{0,l} \mathrm{e}^{\hat{A}_l(t)}, \quad t \in \mathbb{T},\ L := N(N-1)/2. \tag{13}$$

Here, $\hat{A}_l(t) = T_{ij} A_l(t)$ denotes the canonical extension of $A_l(t)$ to a $\mathrm{su}(N)$ loop by the map

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \longmapsto T_{ij} A = \begin{pmatrix} I_{i-1} & 0 & 0 & 0 & 0 \\ 0 & a_{11} & 0 & a_{21} & 0 \\ 0 & 0 & I_{j-i+1} & 0 & 0 \\ 0 & a_{21} & 0 & a_{22} & 0 \\ 0 & 0 & 0 & 0 & I_{N-j} \end{pmatrix}$$

for some index pair (i, j) with $1 \leq i < j \leq N$ (I_k denotes the $k \times k$ identity matrix). Moreover,

$$\|A_l\|_{\mathrm{Lip}_\alpha} \leq C(\alpha, N, U) \|U\|_{\mathrm{Lip}_\alpha}, \quad l = 1, \dots, L. \tag{14}$$

Proof. In the first step, the QR factorization with complex Givens rotations is adapted to the loop case. The complex Givens rotation $U \in \mathrm{SU}(2)$ of a 2×2 matrix X which annihilates the sub-diagonal entry x_{21} is defined as follows:

$$U^* X \equiv \begin{pmatrix} x_{11}^*/d & x_{21}^*/d \\ -x_{21}/d & x_{11}/d \end{pmatrix} \begin{pmatrix} x_{11} & \cdots \\ x_{21} & \cdots \end{pmatrix} = \begin{pmatrix} d & \cdots \\ 0 & \cdots \end{pmatrix}, \tag{15}$$

where $d := \sqrt{|x_{11}|^2 + |x_{21}|^2}$. This definition of U is subject to the assumption that $d^2 = |x_{11}|^2 + |x_{21}|^2 > 0$, i.e., that the first column of X is non-degenerate.

Let us assume for a moment that the loop $U(t) \in \mathrm{Lip}_\alpha(\mathbb{T} \to \mathrm{SU}(N))$ satisfies

$$u_{11}(t) \neq 0, \quad t \in \mathbb{T}. \tag{16}$$

Then $|u_{11}(t)|^2 + |u_{21}(t)|^2 \geq |u_{11}(t)|^2 \geq c_0 > 0$ on $\mathbb{T}$. Let $U_{[1,2]}(t)$ be the complex Givens rotation for the sub-matrix of $U(t)$ corresponding to row/column indices 1 and 2 defined according to (15). Then $\hat{U}_{[1,2]}(t) := T_{12} U_{[1,2]}(t)$ and $U^{[1]}(t) := \hat{U}^*_{[1,2]}(t) U(t)$ belong to $\mathrm{Lip}_\alpha(\mathbb{T} \to \mathrm{SU}(N))$, with norms controlled by a constant depending on α, $U(t)$, and N. Note that due to (15) the leading diagonal element of $U^{[1]}(t)$ is real and positive, and thus automatically satisfies (16). With this in mind, we can recursively construct $\hat{U}_{[1,j]} = T_{1j} U_{[1,j]}(t)$ and $U^{[j-1]}(t)$, $j = 2, \ldots, N$ such that

$$U(t) = \left(\prod_{j=2}^{N} \hat{U}_{[1,j]}(t) \right) U^{[N-1]}(t), \qquad U^{[N-1]}(t) = \begin{pmatrix} 1 & 0 \\ 0 & \tilde{U}^{[N-1]}(t) \end{pmatrix},$$

where all loops have inherited the Lip_α property. In particular, we have $\tilde{U}^{[N-1]}(t) \in \mathrm{Lip}_\alpha(\mathbb{T} \to \mathrm{SU}(N-1))$. Assuming that its leading diagonal entry satisfies again (16), it can be further factorized using the same method. Continuing this way, we would eventually arrive at a factorization of the form

$$U(t) = \prod_{l=1}^{L} U_{l,0} \hat{U}_l(t), \tag{17}$$

where each $\hat{U}_l(t)$ is the lift of a SU(2) loop to a SU(N) loop by some T_{ij}, i.e. some $\hat{U}_{[i,j]}(t)$ for $1 \leq i < j \leq N$, and (under the above assumptions) $U_{0,l} = I_N$.

It remains to remove the assumption (16). We establish the following auxiliary fact:

For any loop $U(t) \in \mathrm{Lip}_\alpha(\mathbb{T} \to \mathrm{SU}(N))$ with $\alpha > 1/2$ and $N \geq 2$, there is a constant matrix $U_0 \in \mathrm{SU}(N)$ such that $U_0^* U(t)$ satisfies (16).

Evidently, by applying this auxiliary result to the original $U(t)$, then to $\tilde{U}^{[N-1]}(t)$, an so on, we establish (17) in full generality (after this step, some of the $U_{0,l}$ are not identity matrices anymore). To see the auxiliary result, consider the first column $u(t)$ of $U(t)$ which is a $\mathrm{Lip}_\alpha(\mathbb{T} \to \Sigma^{N-1})$ loop in the unit sphere Σ^{N-1} of $\mathbb{C}^N$. Since the auxiliary statement is equivalent to proving that the set

$$\Sigma := \{u_0 \in \Sigma^{N-1} : \ u_0^* \cdot u(t) \neq 0 \quad \text{for all } t \in \mathbb{T}\}$$

is non-empty, we are done if we establish that its complement $\Sigma^c = \Sigma^{N-1} \setminus \Sigma$ has zero measure (the measure is induced from the Lebesgue measure on $S^{2N-1} \cong \Sigma^{N-1}$). For this, recall that for any given integer $n \geq 1$ the loop $u(t) \in \mathrm{Lip}_\alpha(\mathbb{T} \to \Sigma^{N-1})$ can be covered by n spherical caps

$$\mathcal{C}(r, u_m) := \left\{ u \in \Sigma^{N-1} : \ \|u - u_m\| \leq r \right\}$$

of radius $r \le Cn^{-\alpha}$ and with centers at $u_m = u(2\pi m/n)$, $m = 1, \ldots, n$. The constant C depends on the Lipschitz constant of the loop. By its definition, the set

$$\Sigma^c = \left\{ u_0 \in \Sigma^{N-1} : \ u_0^* \cdot u(t) = 0 \quad \text{for some } \ t \in \mathbb{T} \right\}$$

is then contained in the union of the associated sets $\Sigma_{\mathcal{C}(r,v_m)}$, where we define

$$\Sigma_{\mathcal{C}(r,u_m)} := \left\{ u_0 \in \Sigma^{N-1} : \ u_0^* \cdot u = 0 \quad \text{for some } \ u \in \mathcal{C}(r, u_m) \right\}, \qquad m = 1, \ldots, n.$$

The measure of these sets $\Sigma_{\mathcal{C}(r,u_m)}$ does not depend on the location of the centers u_m of the associated spherical cap $\mathcal{C}(r, u_m)$ (use unitary transformation), and is bounded by $\le Cr^2$ for $r \to 0$. Indeed, set w.l.o.g. $u_m = e := (1, 0, \ldots)^{\mathrm{T}} \in \Sigma^{N-1}$, and consider any $u_0 := (a_1, a_2, \ldots)^{\mathrm{T}} \in \Sigma_{\mathcal{C}(r,e)}$. Let $u := (b_1, b_2, \ldots)^{\mathrm{T}} \in \mathcal{C}(r, e)$ such that $u_0^* u = 0$. Consequently,

$$|a_1| = \frac{\left|\sum_{k=2}^{N} a_k^* b_k\right|}{|b_1|} \le \frac{\sqrt{\sum_{k=2}^{N} |b_k|^2}}{|b_1|} \le \frac{r}{1-r}$$

since $u \in \mathcal{C}(r, e)$. Thus,

$$\Sigma_{\mathcal{C}(r,e)} \subset \{u_0 := (a_1, a_2, \ldots)^{\mathrm{T}} \in \Sigma^{N-1} : \ |a_1| \le 2r\}, \qquad r < 1/2,$$

but the latter set has obviously measure $\le Cr^2$ as $r \to 0$.

Putting things together, the measure of Σ^c is therefore less than $Cnr^2 \le C' n^{1-2\alpha}$ which gives the claim for $n \to \infty$ since $\alpha > 1/2$ was assumed. This establishes the factorization (17) in full generality.

The second step is to represent each of the $\hat{U}_l(t)$ by the exponential map. Because $\hat{U}_l(t) = T_{i,j} U_l(t)$ for some index pair (i, j) and some $U_l(t) \in \mathrm{Lip}_\alpha(\mathbb{T} \to \mathrm{SU}(2))$, $\alpha > 1/2$, it is enough to consider the case $N = 2$. It is well-known that there are continuous SU(2) loops that cannot be represented as the exponential of a continuous su(2) loop. We will therefore prove the desired representation only up to a constant SU(2) factor, i.e., we will show that for any $U(t) \in \mathrm{Lip}_\alpha(\mathbb{T} \to \mathrm{SU}(2))$, $\alpha > 1/3$, there is a $A(t) \in \mathrm{Lip}_\alpha(\mathbb{T} \to \mathrm{su}(2))$, and a $U_0 \in \mathrm{SU}(2)$ such that

$$U(t) = U_0 \mathrm{e}^{A(t)}, \qquad \|A\|_{\mathrm{Lip}_\alpha} \le C(\alpha, U) \|U\|_{\mathrm{Lip}_\alpha}. \tag{18}$$

We again use a covering argument for Lipschitz loops, by identifying SU(2) with the unit ball S^3 in $\mathbb{R}^4$. Since $U(t)$ is Lip_α, we see that the associated loop on S^3 is contained in the union of n closed spherical caps $\mathcal{C}_m$ of radius $\le Cn^{-\alpha}$ on S^3. Let us consider the system of the $2n$ caps $\{\mathcal{C}_m, -\mathcal{C}_m, m = 1, \ldots, n\}$. Evidently, each of these caps has S^3 surface area $\le Cn^{-3\alpha}$. Thus, the union of all these caps covers an area $\le 2Cn^{1-3\alpha}$ and, since $\alpha > 1/3$, we can find a finite n_0 such that the open complement $S^3 \setminus \cup_{m=1}^{n_0} (\mathcal{C}_m \cup (-\mathcal{C}_m))$ is not empty.

By construction, this means that there is a $U_0 \in \mathrm{SU}(2)$ such that $U(t)$ does not intersect with some ball around U_0 and some ball around $-U_0$. Equivalently, the loop $V(t) := U_0^* U(t)$ does not intersect with some balls around $\pm I$. So, if we write

$$V(t) = \begin{pmatrix} a(t) + \mathrm{i} \cdot b(t) & c(t) + \mathrm{i} \cdot d(t) \\ -c(t) + \mathrm{i} \cdot d(t) & a(t) - \mathrm{i} \cdot b(t) \end{pmatrix}, \qquad a^2(t) + b^2(t) + c^2(t) + d^2(t) = 1,$$

then these four real-valued component functions are Lip_α and satisfy

$$|a(t)| \le r_0 < 1, \qquad b^2(t) + c^2(t) + d^2(t) \ge 1 - r_0^2 > 0, \quad t \in \mathbb{T},$$

with some $r_0 < 1$. Thus, $\Delta(t) := \arccos(a(t)) \in [\arccos r_0, \pi - \arccos r_0]$ belongs to $\mathrm{Lip}_\alpha(\mathbb{T} \to \mathbb{R})$, and so do the three functions

$$\beta(t) := \frac{\Delta(t)b(t)}{\sin \Delta(t)}, \qquad \gamma(t) := \frac{\Delta(t)c(t)}{\sin \Delta(t)}, \qquad \delta(t) := \frac{\Delta(t)d(t)}{\sin \Delta(t)}.$$

It remains to verify that the element $A(t) \in \mathrm{Lip}_\alpha(\mathbb{T} \to \mathrm{su}(2))$ given by

$$A(t) := \begin{pmatrix} \mathrm{i} \cdot \beta(t) & \gamma(t) + \mathrm{i} \cdot \delta(t) \\ -\gamma(t) + \mathrm{i} \cdot \delta(t) & -\mathrm{i} \cdot \beta(t) \end{pmatrix},$$

satisfies $V(t) = \mathrm{e}^{A(t)}$ as desired. Since by construction

$$\begin{aligned} \beta^2(t) + \gamma^2(t) + \delta^2(t) &= \frac{\Delta^2(t)}{\sin^2(\Delta(t))}(b^2(t) + c^2(t) + d^2(t)) = \frac{\Delta^2(t)}{\sin^2(\Delta(t))}(1 - a^2(t)) \\ &= \frac{\Delta^2(t)}{\sin^2(\Delta(t))}(1 - \cos^2(\Delta(t))) = \Delta(t)^2, \end{aligned}$$

this follows from the elementary formula (7) which proves the formula in (18). The proof of the Lip_α estimate is obvious because the transformations from $U(t)$ to $A(t)$ are all diffeomorphisms preserving the Lip_α property (since the transformation is nonlinear, the constant depends in general on U). Together with (17), this proves the statement of Lemma 4.

It should be noted that it is impossible to use the argument in the second step of the proof with $\alpha < 1/3$, due to the fact that there exist space-filling curves on S^3 that are Lip_α-continuous for any such α. This is analogous to the corresponding statements for Peano and Hilbert curves for the unit cube in $\mathbb{R}^d$, where the critical Hölder exponent is $1/d$. Similarly, we cannot drop the assumption $\alpha > 1/2$ in the first step of the proof. However, we believe that statements of the type "*any* Lip_α *loop in* $\mathrm{SU}(2)$ *is, up to a simple correction, representable as the exponential of a* Lip_α *loop in* $\mathrm{su}(2)$ *(or by a product of a few such exponentials)*" can be deduced from the available factorization theorems for loop groups, see [8,12], and hold for more general Lie groups. Unfortunately, we could not yet locate these statements in the literature.

To work with factorizations such as in Lemma 4, we will often use the following simple estimate.

Lemma 5. *For any* $U_k(t), \tilde{U}_k(t) \in C(\mathbb{T} \to \mathrm{SU}(N))$, $k = 1, \ldots, K$, *we have*

$$\left\| \prod_{k=1}^K \tilde{U}_k - \prod_{k=1}^K U_k \right\|_C \le \sum_{k=1}^K \|\tilde{U}_k - U_k\|_C. \tag{19}$$

Proof. By definition of $\|\cdot\|_C$, we have $\|U\|_C = 1$ for arbitrary $U(t) \in C(\mathbb{T} \to \mathrm{SU}(N))$, and $\|\prod_k X_k\|_C \le \prod_k \|X_k\|_C$ for arbitrary $X_k(t) \in C(\mathbb{T} \to GL(N))$. Thus,

$$\begin{aligned} \left\| \prod_{k=1}^K \tilde{U}_k - \prod_{k=1}^K U_k \right\|_C &\le \left\| (\tilde{U}_1 - U_1) \prod_{k=2}^K \tilde{U}_k \right\|_C + \left\| U_1 \left(\prod_{k=2}^K \tilde{U}_k - \prod_{k=2}^K U_k \right) \right\|_C \\ &\le \|\tilde{U}_1 - U_1\|_C + \left\| \prod_{k=2}^K \tilde{U}_k - \prod_{k=2}^K U_k \right\|_C \\ &\cdots \\ &\le \sum_{k=1}^K \|\tilde{U}_k - U_k\|_C. \end{aligned}$$

Finally, we need some technical estimates for exponentials of the form $e^{\sum A_j}$ which are well-known in the theory of splitting methods. Although they hold in much more generality, we formulate them only for su(2).

Lemma 6. (a) *Let* $A, \tilde{A} \in \mathrm{su}(2)$. *Then*

$$\|e^A - e^{\tilde{A}}\| \le C(\|A\|, \|\tilde{A}\|)\|A - \tilde{A}\|. \tag{20}$$

(b) *If* $A_j \in \mathrm{su}(2)$, $j = 1, \ldots, m$, *and* $M \ge 1$. *Then*

$$\left\| e^{\lambda \sum_j A_j} - \prod_{j=1}^{m} e^{\lambda A_j} \right\|_2 \le \frac{\lambda^2}{2} \sum_{j=1}^{m-1} \left\| \left[A_j, \sum_{l=j+1}^{m} A_l \right] \right\|_2, \tag{21}$$

where $[A, \tilde{A}] := A\tilde{A} - \tilde{A}A$ *and* $\lambda \in \mathbb{R}$. *In particular,*

$$\left\| e^{\sum_j A_j} - \left(\prod_{j=1}^{m} e^{\frac{1}{M} A_j} \right)^M \right\|_2 \le \frac{1}{2M} \sum_{j=1}^{m-1} \left\| \left[A_j, \sum_{l=j+1}^{m} A_l \right] \right\|_2. \tag{22}$$

Inequality (20) follows from applying formula (7) while (21) and (22) can be found in [11].

3. Proof of Theorem 1

We will not follow the dependencies of the constants occuring in the estimates below, in general, they will depend on α, N, and U. They are independent of other parameters, and in particular, of the final degree n of the polynomial loop $U_n(t)$ to be constructed.

By Lemmas 4 and 5, it is enough to concentrate on the case $U(t) = e^{A(t)}$, where $A(t) \in \mathrm{Lip}_\alpha(\mathbb{T} \to \mathrm{su}(2))$. For such $A(t)$, we use linear approximation methods (see Lemma 2) and construct polynomial loops

$$A_m(t) := \begin{pmatrix} i \cdot \beta_m(t) & \gamma_m(t) + i \cdot \delta_m(t) \\ -\gamma_m(t) + i \cdot \delta_m(t) & -i \cdot \beta_m(t) \end{pmatrix} \in \Pi_m(\mathbb{T} \to \mathrm{su}(2))$$

with the optimal approximation rate

$$\|A(t) - A_m(t)\| \le Cm^{-\alpha}, \quad m \to \infty.$$

By Lemma 6 (a) this gives

$$\|U - \tilde{U}_m\|_C = \|e^{A(t)} - e^{A_m(t)}\|_C \le Cm^{-\alpha}, \quad m \to \infty, \tag{23}$$

where $\tilde{U}_m(t) := e^{A_m(t)}$.

The remaining effort goes into approximating the loop $\tilde{U}_m(t) := e^{A_m(t)}$ (which is typically not an element of the polynomial loop group) by a suitable element $U_n(t) \in \Pi_n(\mathbb{T} \to \mathrm{SU}(2))$. We proceed in several steps. Using Lemma 1, we write $A_m(t) = \sum_{l=1}^{6} A_{m,l}$, and apply Lemma 6 (b):

$$\left\| e^{A_m(t)} - \left(\prod_{l=1}^{6} e^{A_{m,l}(t)/M} \right)^M \right\|_2 \le \frac{1}{2M} \sum_{l=1}^{5} \left\| \left[A_{m,l}(t), \sum_{j=l+1}^{6} A_{m,j}(t) \right] \right\|_2.$$

Now, by (10) and the remarks following the proof of Lemma 1, we have

$$\|A_{m,l}\|_{\mathrm{Lip}_\alpha} \le C\|A_m\|_{\mathrm{Lip}_\alpha} \le C\|A\|_{\mathrm{Lip}_\alpha} \le C\|U\|_{\mathrm{Lip}_\alpha} \tag{24}$$

because the algebraic operations and trigonometric conjugation leading from $A_m(t)$ to representations for $A_{m,l}(t)$ all preserve the Lip_α property. Estimating the commutators in the above estimate leads to

$$\left\| \tilde{U}_m(t) - \left(\prod_{l=1}^{6} \mathrm{e}^{A_{m,l}(t)/M}\right)^M \right\| \le \frac{C}{M}(\max_l \|A_{m,l}\|_C)^2 \le \frac{C}{M}. \tag{25}$$

The integer M will be chosen later.

The next step is to deal with approximating each of the factors

$$\tilde{U}_{m,l} := \mathrm{e}^{A_{m,l}(t)/M} = \exp\left(\left(\sum_{k=0}^{m} c_{k,l}B_{k,l}(t)\right)\Big/M\right),$$

appearing in the left-hand side of (25) by the polynomial loop

$$U_{m,l}(t) := \prod_{k=0}^{m} \mathrm{e}^{(c_{k,l}/M)B_{k,l}(t)}$$

of degree $\le m$, where $c_{0,l} = 0$ is silently assumed for even l. The fact that $U_{m,l} \in \Pi_m(\mathbb{T} \to \mathrm{SU}(2))$ follows from Lemma 1. For estimating the error we rely on (21) with $\lambda = 1/M$:

$$\|\tilde{U}_{m,l}(t) - U_{m,l}(t)\|_C \le \frac{1}{2M^2}\sum_{k=0}^{m-1}|c_{k,l}| \left\| \left[B_{k,l}(t), \sum_{j=k+1}^{m} c_{j,l}B_{j,l}(t)\right]\right\|_C.$$

Now Lemma 3 will be used in conjunction with (24): On the one hand, $|c_{k,l}| \le C(k+1)^{-\alpha}$, on the other

$$\left\|\left[B_{k,l}(t), \sum_{j=k+1}^{m} c_{j,l}B_{j,l}(t)\right]\right\|_C \le 2\left\|\sum_{j=k+1}^{m} c_{j,l}B_{j,l}(t)\right\|_C \le C\frac{\ln(k+2)}{(k+1)^\alpha},$$

because $\sum_{j=k+1}^{m} c_{j,l}B_{j,l}(t)$ is the difference between $A_{m,l}$ and its k-th partial sum. Since $\alpha > 1/2$, we obtain after substitution

$$\|\tilde{U}_{m,l}(t) - U_{m,l}(t)\|_C \le \frac{C}{M^2}\sum_{k=0}^{m-1}\frac{\ln(k+2)}{(k+1)^{2\alpha}} \le \frac{C}{M^2}, \quad l = 1, \ldots, 6. \tag{26}$$

It remains to use (26) in conjunction with Lemma 5:

$$\left\|\left(\prod_{l=1}^{6}\tilde{U}_{m,l}\right)^M - U_n\right\|_C \le M\sum_{l=1}^{6}\|(\tilde{U}_{m,l} - U_{m,l})\|_C \le \frac{C}{M},$$

where

$$U_n := \left(\prod_{l=1}^{6} U_{m,l}\right)^M \in \Pi_n(\mathbb{T} \to \mathrm{SU}(2)),$$

is a polynomial loop of degree $n \le 6mM$.

Together with (23) and (25), this shows that we have constructed a polynomial loop U_n of degree at most $6mM$ such that

$$\|U - U_n\|_C \le C\left(\frac{1}{m^\alpha} + \frac{1}{M}\right).$$

Choosing m as the integer part of $n^{1/(\alpha+1)}$ and M as the integer part of $\frac{1}{6}n^{\alpha/(\alpha+1)}$, then the degree of $U_n(t)$ is indeed $\le 6mM \le n$, and we arrive at an error estimate of the form $\|U - U_n\|_C \le Cn^{-\alpha/(1+\alpha)}$ for large enough n. This proves Theorem 1.

Further Remarks.

- For $0 < \alpha \le 1/2$, some weaker results are possible. Then in the proof of (26) the estimate

  $$\sum_{k\le m} \frac{\ln(k+2)}{(k+1)^{2\alpha}} \le C < \infty$$

  is not valid anymore, and needs to be replaced by an upper bound that depends on m. This leads to other choices for m and M, and an overall weaker approximation rate, compared to the result stated in Theorem 1. Such results are subject to the availability of factorizations such as proved in Lemma 4 for $\alpha > 1/2$.
- If $U(t) \in \mathrm{Lip}_\alpha(\mathbb{T} \to \mathrm{SU}(2))$ is diagonal and $0 < \alpha \le 1$ then we can establish the optimal bound

  $$\|U - U_n\|_C \le \frac{C}{n^\alpha}, \qquad n \to \infty. \tag{27}$$

  Here is a sketch of the elementary argument. The diagonal element $u(t)$ (of modulus 1) of the matrix $U(t)$ can be approximated by an appropriate polynomial $p_n(t)$ of degree n such that

  $$\|u - p_n\|_C \le \frac{C}{n^\alpha},$$

  and

  $$0 \le 1 - |p_n(t)|^2 \le Cn^{-2\alpha}.$$

  The crucial fact is the upper bound for the deviation of $|p_n(t)|$ from $|u(t)| = 1$, which is stronger than the trivial $\mathrm{O}(n^{-\alpha})$ estimate following from $1 - |p_n(t)|^2 \le 2(1 - |p_n(t)|) \le 2\|u - p_n\|_C$. With this at hand, and $q_n(t)$ determined from $|q_n(t)|^2 = 1 - |p_n(t)|^2$ via spectral factorization, we see that

  $$U_n = \begin{pmatrix} p_n(t) & q_n(t) \\ -q_n^*(t) & p_n^*(t) \end{pmatrix} \in \Pi_n(\mathbb{T} \to \mathrm{SU}(2)),$$

  satisfies (27).

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