The characteristic of convexity of a Banach space and normal structure

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Received 16 January 2007
Available online 3 April 2007
Submitted by B. Sims

Abstract

We present some sufficient conditions for normal structure of Banach spaces and their dual spaces in terms of the characteristic of convexity, the James constant, and the coefficient of weak orthogonality. Many known results are improved and strengthened. We also show that some of our results are sharp.

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Keywords: Uniform normal structure; Ultrapower; Characteristic of convexity; James constant; Coefficient of weak orthogonality

1. Introduction

A Banach space $X$ is said to have (weak) normal structure (see [2]) if for every (weakly compact) closed bounded convex subset $K$ in $X$ that contains more than one point, there exists a point $x_0 \in K$ such that

$$\sup \{ \| x_0 - y \| : y \in K \} < \sup \{ \| x - y \| : x, y \in K \}.$$

In reflexive spaces, normal structure and weak normal structure are the same. It is well known (see [8]) that if $X$ fails to have weak normal structure, then there exist a weakly compact convex subset $C \subset X$ and a sequence $(x_n) \subset C$ such that $\text{dist}(x_{n+1}, \text{co}\{x_k\}_{k=1}^n) \to \text{diam} C = 1$. A Banach space $X$ is said to have uniform normal structure if there exists $0 < c < 1$ such that for any
closed bounded convex subset $K$ of $X$ that contains more than one point, there exists $x_0 \in K$ such that

$$\sup \{ \| x_0 - y \| : y \in K \} < c \sup \{ \| x - y \| : x, y \in K \}.$$ 

It was proved by W.A. Kirk that every reflexive Banach space with normal structure has the fixed point property (see [14]).

In this paper, we investigate some sufficient conditions for (uniform) normal structure of Banach spaces and their dual spaces in terms of many parameters namely the characteristic of convexity, the James constant, and the coefficient of weak orthogonality. We also show that some results are sharp.

First we recall some basic facts about ultrapowers. Let $\mathcal{F}$ be a filter on $\mathbb{N}$ and let $X$ be a Banach space. A sequence $\{x_n\}$ in $X$ converges to $x$ with respect to $\mathcal{F}$, denoted by $\lim_{\mathcal{F}} x_n = x$, if for each neighborhood $U$ of $x$, $\{ i \in \mathbb{N} : x_i \in U \} \in \mathcal{F}$. A filter $\mathcal{U}$ on $\mathbb{N}$ is called an ultrafilter if it is maximal with respect to set inclusion. An ultrafilter is called trivial if it is of the form $\{ A \subset \mathbb{N} : i_0 \in A \}$ for some fixed $i_0 \in \mathbb{N}$, otherwise, it is called nontrivial.

Let $l_\infty(X)$ denote the subspace of the product space $\prod_{n \in \mathbb{N}} X$ equipped with the norm $\| (x_n) \| := \sup_{n \in \mathbb{N}} \| x_n \| < \infty$.

Let $\mathcal{U}$ be an ultrafilter on $\mathbb{N}$ and let

$$N_\mathcal{U} = \left\{ (x_n) \in l_\infty(X) : \lim_{\mathcal{U}} \| x_n \| = 0 \right\}.$$

The ultrapower of $X$, denoted by $\tilde{X}$, is the quotient space $l_\infty(X)/N_\mathcal{U}$ equipped with the quotient norm. Write $(x_n)_{\mathcal{U}}$ to denote the elements of the ultrapower. It follows from the definition of the quotient norm that

$$\| (x_n)_{\mathcal{U}} \| = \lim_{\mathcal{U}} \| x_n \|.$$

Note that if $\mathcal{U}$ is nontrivial, then $X$ can be embedded into $\tilde{X}$ isometrically. For more details see [18]. We also note that if $X$ is super-reflexive, that is $\tilde{X}^* = (\tilde{X})^*$, then $X$ has uniform normal structure if and only if $\tilde{X}$ has normal structure (see [12]).

In what follows, we let $B_X$, $S_X$ and $X^*$ stand for the closed unit ball, the unit sphere, and the dual space of a Banach space $X$, respectively.

**Lemma 1.** (See [17, Lemma 2].) If $X$ is a super-reflexive Banach space and fails to have normal structure, then there are $\tilde{x}_1, \tilde{x}_2, \tilde{x}_3 \in S_{\tilde{X}}$ and $\tilde{f}_1, \tilde{f}_2, \tilde{f}_3 \in S(\tilde{X})^*$, such that

(a) $\| \tilde{x}_i - \tilde{x}_j \| = 1$ and $\tilde{f}_i(\tilde{x}_j) = 0$ for all $i \neq j$,
(b) $\tilde{f}_i(\tilde{x}_i) = 1$ for $i = 1, 2, 3$, and
(c) $\| \tilde{x}_3 - (\tilde{x}_2 + \tilde{x}_1) \| \geq \| \tilde{x}_2 + \tilde{x}_1 \|.$

**2. Results**

**2.1. The characteristic of convexity and the James constant**

The modulus of convexity [4] of a Banach space $X$ is the function $\delta_X : [0, 2] \to [0, 1]$ defined by

$$\delta_X(\varepsilon) = \inf \left\{ 1 - \frac{1}{2} \| x + y \| : x, y \in S_X, \| x - y \| = \varepsilon \right\}.$$
The function $\delta$ is continuous on $[0, 2)$, nondecreasing on $[0, \varepsilon_0(X)]$ and strictly increasing on $[\varepsilon_0(X), 2]$. Here $\varepsilon_0(X) = \sup\{\varepsilon: \delta_X(\varepsilon) = 0\}$ is the characteristic of convexity of $X$. We say that $X$ is uniformly convex provided $\varepsilon_0(X) = 0$. Also, $X$ is uniformly nonsquare provided $\varepsilon_0(X) < 2$.

The James constant of $X$ is defined as

$$J(X) = \sup\{\min\{\|x + y\|, \|x - y\|\}: x, y \in S_X\}.$$ 

It follows easily that $X$ is uniformly nonsquare if and only if $J(X) < 2$. Moreover, it is proved in [7] that $J(X) = \sup\{\varepsilon: \varepsilon \leq 2(1 - \delta_X(\varepsilon))\}$.

The search for an upper bound of the James constant for which (uniform) normal structure of a Banach space is guaranteed is widely studied. The best known result is that $X$ has uniform normal structure if $J(X) < \frac{1 + \sqrt{5}}{2}$ [6]. It is also well known that $\varepsilon_0(X) < 1$ implies both $X$ and $X^*$ have uniform normal structure (see also [8, Theorems 6.1 and 7.2]). We now give a new sufficient condition in terms of both constants.

**Theorem 2.** Let $X$ be a Banach space.

1. If $J(X)\varepsilon_0(X^*) < 2$, then $X$ has uniform normal structure.
2. If $J(X^*)\varepsilon_0(X) < 2$, then $X^*$ has uniform normal structure.

**Proof.** To prove (1), we assume that $J(X)\varepsilon_0(X^*) < 2$. Since $J(X) \geq \sqrt{2}$, the condition $J(X)\varepsilon_0(X^*) < 2$ implies $X$ is super-reflexive. Suppose next that $X$ fails weak normal structure. Then there are elements $\tilde{x}_1, \tilde{x}_2, \tilde{x}_3 \in S_X^2$ and $\tilde{f}_1, \tilde{f}_2, \tilde{f}_3 \in S_{\tilde{X}}$ satisfying conditions in Lemma 1. Hence $\|\tilde{f}_2 - \tilde{f}_1\| \geq (\tilde{f}_2 - \tilde{f}_1)(\tilde{x}_2 - \tilde{x}_1) = 2$, so $\|\tilde{f}_2 - \tilde{f}_1\| = 2$. Since the characteristics of convexity of $X^*$ and of $(X)^*$ are the same,

$$\|\tilde{f}_2 + \tilde{f}_1\| = \|\tilde{f}_2 - (-\tilde{f}_1)\| \leq \varepsilon_0(X^*),$$

and similarly, $\|\tilde{f}_3 + \tilde{f}_1\| \leq \varepsilon_0(X^*)$. Now, we obtain a contradiction since

$$J(X) \geq \min\{\|\tilde{x}_3 - \tilde{x}_2 + \tilde{x}_1\|, \|\tilde{x}_3 - \tilde{x}_2 - \tilde{x}_1\|\}$$

$$\geq \min\{\|\tilde{x}_3 - \tilde{x}_2 + \tilde{x}_1\|, \|\tilde{x}_3 - \tilde{x}_2 - \tilde{x}_1\|\}$$

$$\geq \min\left\{\left(\frac{\tilde{f}_3 + \tilde{f}_1}{\varepsilon_0(X^*)}(\tilde{x}_3 - \tilde{x}_2 + \tilde{x}_1), \left(\frac{\tilde{f}_3 + \tilde{f}_1}{\varepsilon_0(X^*)}(\tilde{x}_3 - \tilde{x}_2 - \tilde{x}_1)\right)\right)\right\}$$

$$= \frac{2}{\varepsilon_0(X^*)},$$

We have proved that $X$ has weak normal structure if $J(X)\varepsilon_0(X^*) < 2$. The conclusion that $X$ has uniform normal structure follows since the condition $J(X)\varepsilon_0(X^*) < 2$ is invariant under the taking of ultrapowers.

2. By (1), $X^*$ has uniform normal structure whenever $J(X^*)\varepsilon_0(X^{**}) < 2$ which is equivalent to $J(X^*)\varepsilon_0(X) < 2$.

**Corollary 3.** If $\varepsilon_0(X) = 1$, then $X^*$ has uniform normal structure.

**Proof.** It is clear that the condition $\varepsilon_0(X) = 1$ implies $X$ is uniformly nonsquare, and so is $X^*$. Consequently, $\varepsilon_0(X)J(X^*) = J(X^*) < 2$ and the assertion follows.
Remark 4. Corollary 3 gives new information. In particular, if $\varepsilon_0(X) = 1$, then $X^*$ has the fixed point property for mappings of asymptotically nonexpansive type [15] (cf. [13]). The upper bound 1 in Corollary 3 is best possible, that is, for any $0 < \eta < 1$ there is a Banach space $X$ such that $X^*$ fails to have uniform normal structure and $\varepsilon_0(X) = 1 + \eta$. In fact, we consider the Bynum space $\ell_{p,1}$ (see [3]) which is the $\ell_p$ space equipped with the norm
\[
\|x\|_{p,1} = \|x^+\|_p + \|x^-\|_p
\]
where $x^+$ and $x^-$ are positive and negative parts of $x \in \ell_p$, i.e., $(x^+_n) = \max\{x_n, 0\}$ and $(x^-_n) = \max\{-x_n, 0\}$. It is proved that $\varepsilon_0(\ell_{p,1}) = 2^{1/p}$ and $\ell_{q,\infty} = \ell_{p,1}^*$, where $\frac{1}{p} + \frac{1}{q} = 1$, does not have even normal structure.

Since $\frac{1}{2}\varepsilon_0(X^*) = \rho_X'(0) := \lim_{t \to 0} \frac{\rho_X(t)}{t}$, where
\[
\rho_X(t) = \sup \left\{ \frac{\|x + ty\| + \|x - ty\|}{2} - 1 : x, y \in S_X \right\},
\]
Corollary 3 also strengthens Theorem 2.1 of Prus [16] and Proposition 3 of Turett [21]. More precisely, if $\rho_X'(0) = \frac{1}{2}$ or $\varepsilon_0(X^*) = 1$, then $X$ has uniform normal structure.

2.2. The coefficient of weak orthogonality

The WORTH property was introduced by B. Sims in [19] as follows: a Banach space $X$ has the WORTH property if
\[
\lim_{n \to \infty} \|x_n + x\| - \|x_n - x\| = 0
\]
for all $x \in X$ and all weakly null sequences $(x_n)$. In [20], the author defined the coefficient of weak orthogonality, which measures the “degree of WORTH-whileness.” As in [9], we prefer to use its reciprocal, $\mu(X)$, which is defined as the infimum of the set of real numbers $r > 0$ such that
\[
\limsup_{n \to \infty} \|x_n + x\| \leq r \limsup_{n \to \infty} \|x_n - x\|
\]
for all $x \in X$ and for all weakly null sequence $(x_n)$ in $X$. It is known that $X$ has the WORTH property if and only if $\mu(X) = 1$. We also note here that $\mu(X) = \mu(X^*)$ if $X$ is reflexive [10, Theorem 3].

Before going to the result, we start with the following lemma which is a refinement of [7, Lemma 2.3].

Lemma 5. Let $X$ be a Banach space for which $B_{X^*}$ is w*-sequentially compact (for example, $X$ is reflexive or separable, or has an equivalent smooth norm). Suppose that $X$ fails to have weak normal structure. Then, for any $\varepsilon > 0$, there exist $z_1, z_2 \in S_X$ and $g_1, g_2 \in S_{X^*}$ such that the following conditions are satisfied:

(a) $\|z_1 - z_2\| - 1 < \varepsilon$ and $|g_i(z_j)| < \varepsilon$ for all $i \neq j$,
(b) $g_i(z_i) = 1$ for $i = 1, 2$, and
(c) $\|z_2 + z_1\| \leq \mu(X) + \varepsilon$.

Proof. By the assumptions, there exist sequences $(x_n) \subset X$ and $(f_n) \subset S_{X^*}$ such that
(1) \( x_n \xrightarrow{w} 0 \),
(2) \( \text{diam}\{x_n\}_{n=1}^{\infty} = 1 = \lim_{n \to \infty} \|x_n - x\| \) for all \( x \in \co\{x_n\}_{n=1}^{\infty} \),
(3) \( f_n(x_n) = \|x_n\| \) for all \( n \in \mathbb{N} \), and
(4) \( f_n \xrightarrow{w^*} f \) for some \( f \in B_{X^*} \).

Observe that 0 is in the weakly closed convex hull of \( \{x_n\}_{n=1}^{\infty} \) which equals the norm closed convex hull \( \co\{x_n\}_{n=1}^{\infty} \). This implies that \( \lim_{n \to \infty} \|x_n\| = 1 \).

Let \( \varepsilon \in (0, 1) \) be given. Pick \( \eta = \frac{\varepsilon}{3} \). We first choose a natural number \( n_1 \) so that
\[

\|f_n(x_{n_1})\| < \frac{\eta}{2} \quad \text{and} \quad 1 - \eta \leq \|x_{n_1}\| \leq 1.
\]

By the definition of \( \mu(X) \), we have
\[

\limsup_{n \to \infty} \|x_n + x_{n_1}\| \leq \mu(X) \lim_{n \to \infty} \|x_n - x_{n_1}\| = \mu(X).
\]

Next, we choose \( n_2 > n_1 \) so that
\[

\|x_{n_2} + x_{n_1}\| \leq \mu(X) + \eta,
\]
\[

1 - \eta \leq \|x_{n_2}\| \leq 1, \quad 1 - \eta \leq \|x_{n_2} - x_{n_1}\| \leq 1,
\]
\[

\|f_{n_1}(x_{n_2})\| < \eta, \quad \text{and} \quad \|(f_{n_2} - f)(x_{n_1})\| < \frac{\eta}{2}.
\]

This implies that
\[

\|f_{n_2}(x_{n_1})\| \leq \|(f_{n_2} - f)(x_{n_1})\| + \|f(x_{n_1})\| < \eta.
\]

Set
\[

z_1 := \frac{x_{n_1}}{\|x_{n_1}\|}, \quad z_2 := \frac{x_{n_2}}{\|x_{n_2}\|}, \quad g_1 := f_{n_1}, \quad g_2 := f_{n_2}.
\]

We now prove that (a), (b) and (c) are satisfied. Clearly, (b) holds. Moreover, for \( i \neq j \),
\[

\|g_i(z_j)\| = \frac{\|f_{n_1}(x_{n_j})\|}{\|x_{n_j}\|} < \frac{\eta}{1 - \eta} < 2\eta < \varepsilon.
\]

Next, we observe that
\[

\|z_1 - z_2\| = \left\| \frac{x_{n_1}}{\|x_{n_1}\|} - \frac{x_{n_2}}{\|x_{n_2}\|} \right\|
\]
\[

\leq \left\| \frac{x_{n_1}}{\|x_{n_1}\|} - x_{n_1} \right\| + \|x_{n_1} - x_{n_2}\| + \left\| \frac{x_{n_2}}{\|x_{n_2}\|} - x_{n_2} \right\|
\]
\[

= \left| 1 - \|x_{n_1}\| \right| + \|x_{n_1} - x_{n_2}\| + \left| 1 - \|x_{n_2}\| \right|
\]
\[

< 1 + 2\eta < 1 + \varepsilon,
\]
and
\[

\|z_1 - z_2\| \geq g_1(z_1) - g_1(z_2) = g_1(z_1) - g_1(z_2) \geq 1 - \eta > 1 - \varepsilon,
\]
that is (a) is satisfied. Moreover, (c) is satisfied, since
Theorem 7. characteristic of convexity and the coefficient of weak orthogonality.

\[ J(X) < \text{or} \]

This completes the proof. \( \square \)

Lemma 5 can be rewritten in ultrapower language as:

**Lemma 6.** If a super-reflexive Banach space \( X \) fails normal structure, then there are \( \tilde{x}_1, \tilde{x}_2 \in S_X \) and \( \tilde{f}_1, \tilde{f}_2 \in S(X^*)_X \) such that

(a) \( \|\tilde{x}_1 - \tilde{x}_2\| = 1 \) and \( \tilde{f}_1(\tilde{x}_j) = 0 \) for all \( i \neq j \).

(b) \( \tilde{f}_1(\tilde{x}_i) = 1 \) for \( i = 1, 2 \).

(c) \( \|\tilde{x}_1 + \tilde{x}_2\| \leq \mu(X) \).

We now give sufficient conditions for normal structure of a space and its dual in terms of the characteristic of convexity and the coefficient of weak orthogonality.

**Theorem 7.**

(1) If \( \varepsilon_0(X^*)\mu(X) < 2 \), then \( X \) has normal structure.

(2) If \( \varepsilon_0(X)\mu(X) < 2 \), then \( X^* \) has normal structure.

**Proof.** (1) Since \( \mu(X) \geq 1 \) always, the condition \( \varepsilon_0(X^*)\mu(X) < 2 \) implies that \( X \) is super-reflexive. If \( X \) does not have normal structure, then there are vectors \( \tilde{x}_1, \tilde{x}_2 \in S_X \) and \( \tilde{f}_1, \tilde{f}_2 \in S(X^*)_X \) satisfying properties in Lemma 6. Clearly, \( \|\tilde{f}_2 - \tilde{f}_1\| = 2 \). Hence

\[ \varepsilon_0(X^*) \geq \|\tilde{f}_2 + \tilde{f}_1\| \geq \left(\tilde{f}_2 + \tilde{f}_1\right)\left(\frac{\tilde{x}_2 + \tilde{x}_1}{\|\tilde{x}_2 + \tilde{x}_1\|}\right) = \frac{2}{\|\tilde{x}_2 + \tilde{x}_1\|} \geq \frac{2}{\mu(X)}, \]

or \( \varepsilon_0(X^*) \mu(X) \geq 2 \) which is a contradiction.

(2) We assume that \( \varepsilon_0(X)\mu(X) < 2 \). This implies that \( X \) is super-reflexive. In particular, \( \mu(X^*) = \mu(X) \) and \( \varepsilon_0(X^{**}) = \varepsilon_0(X) \). It then follows from \( \varepsilon_0(X^{**})\mu(X^*) = \varepsilon_0(X)\mu(X) < 2 \) and (1) that \( X^* \) has normal structure. \( \square \)

**Remark 8.** (1) Recently, it is proved in [10, Theorems 1 and 2] that if \( C_{NJ}(X) < 1 + \frac{1}{\mu(X)} \), then \( X \) has normal structure. Here \( C_{NJ}(X) = \sup\{\frac{\|x+y\|^2 + \|x-y\|^2}{2\|x\|^2 + \|y\|^2}: x, y \in X \text{ and } (x, y) \neq (0, 0)\} \). Since \( C_{NJ}(X) = C_{NJ}(X^*) \geq 1 + \frac{\varepsilon_0(X^*)^2}{4} \), the class of spaces with \( C_{NJ}(X) < 1 + \frac{1}{\mu(X)} \) is included in the wider class of spaces satisfying \( \varepsilon_0(X^*) \mu(X) < 2 \). Furthermore, let \( X_p \) be the \( p \)-direct sum \( \ell_2 \oplus_p \ell_2 \) where \( 1 < p \leq 2 \). It follows that \( \varepsilon_0(X_p^*) = 1 \) (see [5, Theorem 9]). \( C_{NJ}(X_p) \geq 2^{2/p-1} \), \( J(X_p) \geq 2^{1/p} \) and \( \mu(X_p) = \sqrt{2} \) (see [11, Theorem 7]). If \( p \) is sufficiently close to 1, then neither \( C_{NJ}(X_p) < 1 + \frac{1}{\mu(X_p)} \) nor \( J(X_p) < 1 + \frac{1}{\mu(X_p)} \) holds.
However, the condition $\varepsilon_0((X_p)^*)\mu(X_p) < 2$ still holds true. We also note here that sometimes calculating the characteristic of convexity of the dual space is much easier than calculating the James or von Neumann–Jordan constants of the space (see [1, Example IV.7] and [10, Theorem 4]).

(2) It is easy to see that $\varepsilon_0(\ell_2,1) = \mu(\ell_2,\infty) = \sqrt{2}$ and $\ell_2,\infty$ fails normal structure. Hence we conclude from Theorem 7 that our results are sharp.

Acknowledgment

The author would like to acknowledge suggestions by the referee(s) which led to improvements in the manuscript.

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