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Stochastic Processes and their Applications 93 (2001) 149–180

stochastic processes and their applications

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Stochastic optimization under constraints ☆

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Received 21 June 2000; received in revised form 23 October 2000; accepted 31 October 2000

Abstract

We study a stochastic optimization problem under constraints in a general framework including financial models with constrained portfolios, labor income and large investor models and reinsurance models. We also impose American-type constraint on the state space process. General objective functions including deterministic or random utility functions and shortfall risk loss functions are considered. We first prove existence and uniqueness result to this optimization problem. In a second part, we develop a dual formulation under minimal assumptions on the objective functions, which are the analogue of the asymptotic elasticity condition of Kramkov and Schachermayer (1999). © 2001 Elsevier Science B.V. All rights reserved.

MSC: 93E20; 90A09; 90A10

Keywords: Stochastic optimization; Convex and state constraints; Optional decomposition; Duality theory; Finance and insurance

1. Introduction

A basic optimization problem in mathematical finance, such as optimal portfolio choice or hedging, is formulated as

$$\text{maximize over } X \in \mathscr{X}(x): \quad E[U(X_T)], \tag{1.1}$$

where $\mathscr{X}(x)$ is the set of self-financed wealth processes starting from an initial capital x and U is a concave objective function (eventually state-dependent). The expression of a self-financed wealth process starting from an initial capital x and with an investment

☆ We would like to thank an anonymous referee for helpful comments.

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PII: S0304-4149(00)00089-2

θ in the n securities of price process S is [1]

$$X^{x,\theta} = S^0 \left(x + \int \theta \, \mathrm{d}(S/S^0) \right).$$

Here [2] θ (resp. $\mathrm{diag}(S)\theta$) is the number of shares (resp. the amount) invested in securities S. The process S^0 is a numéraire, i.e. a strictly positive price process of some other asset, and so $(X^{x,\theta} - \theta.S)/S^0$ (resp. $X^{x,\theta} - \theta.S$) is the number of shares (resp. amount) invested in S^0. Usually, S^0 is chosen to be the riskless bond (when it exists). When asset S^0 has (discounted) price equal to 1 at any times, this means actually that one optimizes the objective function from the discounted wealth $\tilde{X} = X/S^0$. In a markovian context, problem (1.1) may be studied by a direct dynamic programming Bellman equation, see e.g. Merton (1971) and Duffie et al. (1997). When dealing with more general processes, it is now well known that the dual martingale approach provides an elegant and powerful tool for solving (1.1). Classical references in the incomplete market model are the papers of Karatzas et al. (1991), He and Pearson (1991) and Kramkov and Schachermayer (1999). The key point in this approach is the duality relation between the set of self-financed wealth processes and the set $\mathscr{P}^0$ of martingale measures Q under which S/S^0 is a Q-local martingale. Extensions to the case of constrained investment are considered by Cvitanić and Karatzas (1992) (constraints on the proportion $\pi = \mathrm{diag}(S)\theta/X^{x,\theta}$) and by Cuoco (1997) (constraint on the amount) in an Itô processes model. Cuoco (1997) and Cvitanić et al. (2000) study problem (1.1) in the presence of labor income, i.e. when x is random. The wealth process in these papers is only required to be nonnegative at the terminal date T. In the case of a complete Itô processes model, He and Pagès (1993) and El Karoui and Jeanblanc (1998) consider the more delicate case of nonnegative wealth constraint over the whole interval $[0,T]$. While such an American-type constraint is guaranteed by the nonnegativity of the terminal wealth in an incomplete market model or when constraints are imposed on proportions, this is no more the case in the presence of constraints on the amount or share and/or with random endowment.

In this paper, we investigate the general structure of such stochastic optimization problems. We prescribe a convex family of semimartingales for the normalized state process $\tilde{X} = X/S^0$ where S^0 is a given strictly positive process. This includes financial models with constrained portfolios, random endowment and large investor, as well as reinsurance models. Given a process (d_t), we impose an American-type constraint on the state process (X_t) in the form $X_t \geqslant d_t$ for all $t \in [0,T]$. For $d \equiv 0$, this is a nonnegativity constraint over the whole interval $[0,T]$, and for general stochastic process d, this may be interpreted as a portfolio insurance problem with American guarantee (see El Karoui et al., 2000). We consider general objective concave functions U defined on $(0,\bar{x})$, where $\bar{x}$ is random and valued in $(0,\infty)$, including deterministic or random utility functions, and loss functions associated to shortfall risk minimization problems, see Föllmer and Leukert (2000), Cvitanić (1998) or Pham (2000a). Using

[1] Given a vector-valued semimartingale $\tilde{S}$ and $\theta \in L(\tilde{S})$, we write $\int \theta \, \mathrm{d}\tilde{S}$ for the vector stochastic integral of θ with respect to S.

[2] Given a vector $S = (S^1,\ldots,S^n)$, we denote by $\mathrm{diag}(S)$ the $n \times n$ diagonal matrix of elements S^i. We also denote by 1_n the vector in $\mathbb{R}^n$ of components 1.

general optional decomposition under constraints of Föllmer and Kramkov (1997), we provide a static characterization of the state process X in terms of a suitable set of probability measures and nondecreasing processes: this set of probability measures is the dual set associated to the convex constraints on the family of state process, and the set of nondecreasing processes is the dual set associated to the American state space constraints. We are then able to prove an existence and uniqueness result to our optimization problem. In a second part, we develop a dual formulation under minimal assumptions on the objective function, which extend the asymptotic elasticity condition of Kramkov and Schachermayer (1999) from deterministic utility functions on $(0,\infty)$ to random objective concave functions on $(0,\bar{x})$. The main point is the study of the additional terms arising from the convex and state space constraints, which lead to a mixed control/singular dual optimization problem.

The outline of the paper is organized as follows. Section 2 describes the general framework and formulates the optimization problem. In Section 3, we show how the conditions of our abstract setting can be applied in several examples motivated by finance and insurance problems. In Section 4, we state an existence and uniqueness result. Section 5 develops a dual formulation while the final Section 6 is devoted to the proof of this duality theorem.

2. The optimization problem

Let $(\Omega,\mathscr{F},P)$ be a probability space equipped with a filtration $\mathbb{F}=(\mathscr{F}_t)_{0\leqslant t\leqslant T}$ satisfying the usual conditions of right-continuity and completeness. $T>0$ is a fixed finite time horizon, and we assume that $\mathscr{F}_0$ is trivial and $\mathscr{F}_T=\mathscr{F}$. We consider a family $\tilde{\mathscr{X}}$ of semimartingales, with initial value 0, and predictably convex in the following sense: for any $\tilde{X}^1$, $\tilde{X}^2\in\tilde{\mathscr{X}}$ and for any predictable process ζ valued in $[0,1]$, we have

$$\int\zeta\,\mathrm{d}\tilde{X}^1+\int(1-\zeta)\,\mathrm{d}\tilde{X}^2\in\tilde{\mathscr{X}}-\mathscr{I}, \tag{2.1}$$

where $\mathscr{I}$ is the set of nondecreasing adapted processes with initial value 0. We shall assume the following standing condition:

(CP) The family $\tilde{\mathscr{X}}$ is closed for the semimartingale topology.

The semimartingale topology is associated to the Emery distance between two semimartingales $\tilde{X}^1$ and $\tilde{X}^2$ defined as

$$D_E(\tilde{X}^1,\tilde{X}^2)=\sum_{n\geqslant1}2^{-n}E\left[\sup_{0\leqslant t\leqslant T\wedge n}|\tilde{X}_t{}^1-\tilde{X}_t{}^2|\wedge 1\right]. \tag{2.2}$$

For this metric, the space of semimartingales is complete. We refer to Mémin (1980) (see also the next section) for other properties of the semimartingale topology.

Given $\tilde{X}^0\in\tilde{\mathscr{X}}$, we set

$$\tilde{\mathscr{X}}^0=\{\tilde{X}-\tilde{X}^0\colon \tilde{X}\in\tilde{\mathscr{X}}\text{ and }\tilde{X}-\tilde{X}^0\text{ is locally bounded from below}\},$$

so that $\tilde{\mathscr{X}}^0$ is a predictably convex family of semimartingales, locally bounded from below, closed for the semimartingale topology, with initial value 0 and containing the constant process 0. We then consider the set $\bar{\mathscr{P}}^0$ of all probability measures $Q \sim P$ with the property: there exists $A \in \mathscr{I}_p$, set of nondecreasing predictable processes with $A_0 = 0$, such that

$$V - A \quad \text{is a } Q\text{-local supermartingale for any } V \in \tilde{\mathscr{X}}^0. \tag{2.3}$$

The upper variation process of $\tilde{\mathscr{X}}^0$ under $Q \in \bar{\mathscr{P}}^0$ is the element $\tilde{A}^0(Q)$ in $\mathscr{I}_p$ satisfying (2.3) and such that $A - \tilde{A}^0(Q) \in \mathscr{I}_p$ for any $A \in \mathscr{I}_p$ satisfying (2.3). We denote by $\mathscr{P}^0$ the subset of elements $Q \in \bar{\mathscr{P}}^0$ such that

$$\tilde{A}^0_T(Q) \quad \text{is bounded a.s.} \tag{2.4}$$

We make the standing assumption that $\tilde{X}^0 \in \tilde{\mathscr{X}}$ can be chosen so that

$$\text{(H0)} \quad \mathscr{P}^0 \neq \emptyset$$

and

$$\text{(H1)} \quad E^Q \sup_t |\tilde{X}^0_t| < \infty, \quad \forall Q \in \mathscr{P}^0.$$

Remark 2.1. The set of probability measures $\mathscr{P}^0$ and the upper variation process are derived by using Lemma 2.1 of Föllmer and Kramkov (1997): This result states that $Q \in \bar{\mathscr{P}}^0$ iff there is an upper bound for all the predictable processes arising in the Doob–Meyer decomposition of the special semimartingale $V \in \tilde{\mathscr{X}}^0$ under Q. In this case, the upper variation process is equal to this upper bound.

We are given a strictly positive price process S^0 and we are interested on the family of state processes

$$\mathscr{X}(x) = \{S^0(x + \tilde{X}) \colon \tilde{X} \in \tilde{\mathscr{X}}\}, \quad x \in \mathbb{R}.$$

The main family of examples we have in mind for applications is described below and will be developed more explicitly in the next section.

Example 2.1. Let $\tilde{S}$ be a semimartingale in $\mathbb{R}^n$ and $L(\tilde{S})$ be the set of predictable processes integrable with respect to $\tilde{S}$. We prescribe a subset Θ of $L(\tilde{S})$ containing the zero element and convex in the following sense: for any predictable process ζ valued in $[0,1]$ and for all $\theta^1, \theta^2 \in \Theta$, we have $\zeta\theta^1 + (1-\zeta)\theta^2 \in \Theta$. We consider a family $\{\tilde{H}^\theta \colon \theta \in \Theta\}$ of adapted processes with finite variation, with initial value 0 satisfying:

$$\tilde{H}^{\zeta\theta^1 + (1-\zeta)\theta^2} - \int \zeta \, \mathrm{d}\tilde{H}^{\theta^1} - \int (1-\zeta) \, \mathrm{d}\tilde{H}^{\theta^2} \in \mathscr{I}. \tag{2.5}$$

Then the family $\tilde{\mathscr{X}} = \{\int \theta \, \mathrm{d}\tilde{S} + \tilde{H}^\theta \colon \theta \in \Theta\}$ is predictably convex in the sense (2.1). The family of state processes considered is

$$\mathscr{X}(x) = \left\{ S^0 \left(x + \int \theta \, \mathrm{d}\tilde{S} + \tilde{H}^\theta \right) : \theta \in \Theta \right\}, \quad x \in \mathbb{R}.$$

The closure property (CP) is stated for a large class of models described in the next section. In all examples below, we shall choose $\tilde{X}^0 = \tilde{H}^0$ corresponding to the element of $\tilde{\mathscr{X}}$ for $\theta = 0$. A sufficient condition ensuring (H1) is the boundedness of $\tilde{H}^0$. For the moment, let us just mention that the incomplete market setting of the introduction is embedded in the family $\mathscr{X}(x)$ by taking $\tilde{S}=S/S^0$ as the discounted securities price, $\Theta = L(\tilde{S})$, $\tilde{H}^\theta \equiv 0$. More generally, the subset Θ models constraints on portfolio θ and the process $\tilde{H}^\theta$ allows to take into account the terms arising from labor income and large investor in financial models. We shall also see how they can be used in reinsurance models.

Given a semimartingale $d = (d_t)_{0\leqslant t\leqslant T}$ and $x\in\mathbb{R}$, we impose an American-type constraint on the family of state processes $X\in\mathscr{X}(x)$:

$$X_t \geqslant d_t, \quad 0\leqslant t\leqslant T. \tag{2.6}$$

By considering the family of processes $\{\tilde{X} - d/S^0\colon \tilde{X}\in\tilde{\mathscr{X}}\}$ which still satisfies the convexity property (2.1) and the closure property (CP) by the invariance of the Emery distance under translation (see (2.2)), we may focus without loss of generality to nonnegativity state constraint

$$X_t \geqslant 0, \quad 0\leqslant t\leqslant T. \tag{2.7}$$

Given $x\in\mathbb{R}$, we then denote by $\mathscr{X}_+(x)$ the set of all processes $X\in\mathscr{X}(x)$ satisfying (2.7). We also denote by $\mathscr{X}_e(x)$ the set of all processes $X\in\mathscr{X}(x)$ satisfying the weaker European constraint: $X_T\geqslant 0$, a.s., and the process X is bounded from below.

The general liquidity constraint (2.6) appears in the problem of portfolio insurance with American guarantee (see El Karoui et al., 2000). In the classical context of incomplete market model, the European constraint in $\mathscr{X}_e(x)$ of nonnegative terminal wealth suffices to ensure that the optimal state never reaches zero before terminal time T so that the American nonnegativity constraint (2.7) is not binding. Notice also that when constraints are imposed on proportions as in Cvitanić and Karatzas (1992), the state process is nonnegative by construction. In our general framework, the American constraint is not satisfied by the optimal state under the European constraint, and we have to take special attention to this state space constraint.

We consider a measurable function $U:[0,\infty)\times\Omega\to\mathbb{R}\cup\{-\infty\}$. To alleviate notations, we omit the dependence in the state $\omega\in\Omega$ and we write $U(x)$. The function U is uppersemicontinuous and concave. We assume that $\inf\{x\geqslant 0\colon U(x) > -\infty\} = 0$ a.s. and we set

$$\bar{x} = \sup\{x\geqslant 0\colon U(x) > -\infty\},$$

so that the convex domain of U, $\operatorname{dom} U := \{x\geqslant 0\colon U(x) > -\infty\}$ satisfies $\operatorname{int}(\operatorname{dom} U) = (0,\bar{x})$, a.s. Notice that by the uppersemicontinuity of U, we have $U(\bar{x}) < \infty$ if $\bar{x} < \infty$. We also assume that U is nondecreasing on dom U. Our interest is on the optimization problem

$$J(x) = \sup_{X\in\mathscr{X}_+(x)} E[U(X_T)], \quad x\in\mathbb{R}. \tag{2.8}$$

Since $U(x) = -\infty$ for $x > \bar{x}$, we clearly have

$$J(x) = \sup_{X\in\mathscr{X}_+(x)} E[U(X_T\wedge\bar{x})], \quad x\in\mathbb{R}. \tag{2.9}$$

Example 2.2. (i) When U is a deterministic function with $\bar{x}=\infty$, problem (2.8) is an utility maximization problem from terminal state.

(ii) When U is on the form $U(x) = u(x + B)$ where u is a utility function as in (i) and B is a contingent claim, i.e. an $\mathscr{F}_T$-measurable nonnegative random variable, problem (2.8) is an utility-based pricing problem, see Davis (1997) and Karatzas and Kou (1996).

(iii) Take U on the form

$$U(x) = \begin{cases} -l(B-x), & 0 \leqslant x \leqslant B, \\ -\infty, & x > B, \end{cases}$$

where B is a nonnegative $\mathscr{F}_T$-measurable random variable and l is a convex nondecreasing function on $\mathbb{R}_+$. Then, $\bar{x} = B$ and problem (2.9) is written equivalently in [3]

$$-J(x) = \inf_{X \in \mathscr{X}_+(x)} E[l(B - X_T)^+], \quad x \geqslant 0.$$

This is a shortfall risk minimization problem in finance and insurance.

3. Examples

In this section, we provide several examples motivated by finance and insurance problems. We show how one may check the technical conditions (CP) and (H0).

Actually, the set of probability measures and the upper variation process are determined in the examples by using Remark 2.1. On the other hand, to prove the closure property (CP), we shall use the following properties of the semimartingale topology stated by Mémin (1980):

(P1) The space of stochastic integrals $\{\int \theta \, \mathrm{d}\tilde{S} \colon \theta \in L(\tilde{S})\}$, where $\tilde{S}$ is a vector-valued semimartingale, is closed for the semimartingale topology.

(P2) $(V^n)_n$ is a sequence of semimartingales converging to V in the semimartingale topology iff there exists a sequence (also denoted $(V^n)_n$) and a probability measure $Q \sim P$ with bounded density $\mathrm{d}Q/\mathrm{d}P$ such that $(V^n)_n$ is a Cauchy sequence in $\mathscr{M}^2(Q) \oplus \mathscr{A}(Q)$ where $\mathscr{M}^2(Q)$ is the Banach space of Q-square integrable martingales and $\mathscr{A}(Q)$ is the Banach space of predictable processes with finite Q-integrable variation.

3.1. Incomplete market model

This is a particular case of Example 2.1 with $\tilde{S} = S/S^0$ the securities price discounted by a numéraire S^0 (usually a riskless bond), $\Theta = L(\tilde{S})$, $\tilde{H}^\theta = 0$. The family of state processes is the set of self-financed wealth processes: $\mathscr{X}(x) = \{S^0(x + \int \theta \, \mathrm{d}\tilde{S}) \colon \theta \in L(\tilde{S})\}$. We choose $\tilde{X}^0 = \tilde{H}^0 = 0$. In this case, $\tilde{X}^0 = \tilde{\mathscr{X}} = \{\int \theta \, \mathrm{d}\tilde{S} \colon \theta \in L(\tilde{S})\}$, $\mathscr{P}^0$ is the set of all probability measures Q equivalent to P such that $\tilde{S}$ is a Q-local martingale and $\tilde{A}^0(Q) = 0$. The closure property (CP) follows from property (P1).

[3] Given a real number a, we denote $a^+ = \max(a, 0)$ and $a^- = \max(-a, 0)$.

3.2. Constrained portfolios

We consider a standard multivariate Itô processes model for the stock price $S = (S^1, \dots, S^n)'$:

$$dS_t = \operatorname{diag}(S_t)(\mu_t \, dt + \sigma_t \, dW_t).$$

Here W is a n-dimensional Brownian motion and $\mathbb{F}$ is the P-augmentation of the filtration generated by W. μ is a $\mathbb{R}^n$-valued $\mathbb{F}$-adapted process and σ is a $n \times n$ matrix-valued $\mathbb{F}$-adapted process satisfying a nondegeneracy condition:

$$\sigma_t \sigma_t' \quad \text{is definite positive} \tag{3.1}$$

almost surely for all $t \in [0, T]$. There is also a riskless asset of price process

$$S_t^0 = \exp\left(\int_0^t r_s \, ds\right), \tag{3.2}$$

where the interest rate process r is an real-valued $\mathbb{F}$-adapted process, assumed to be bounded uniformly in (t, ω).

The self-financed wealth process starting from initial wealth x and with an amount θ invested in securities S is given by

$$\begin{aligned} X_t^{x,\theta} &= S_t^0 \left(x + \int_0^t \theta_u \operatorname{diag}(S_u)^{-1} d(S_u/S_u^0)\right) \\ &= S_t^0 \left(x + \int_0^t \theta_u \, d\tilde{S}_u\right), \end{aligned}$$

where

$$d\tilde{S}_t = \frac{1}{S_t^0} \sigma_t (\lambda_t \, dt + dW_t) \tag{3.3}$$

and $\lambda_t = \sigma_t^{-1}(\mu_t - r_t 1_n)$. The amount θ is constrained to lie in a closed convex set K in $\mathbb{R}^n$ containing 0. This class of constraints contains short-sale prohibition, buying constraints, rectangular constraints or collateral constraints and was studied by Cvitanić and Karatzas (1992) and Cuoco (1997).

This model is a particular class of Example 2.1 with S^0 given by (3.2), $\tilde{S}$ given by (3.3), $\Theta = \{\theta \in L(\tilde{S}): \ \theta_t \in K, \ 0 \leqslant t \leqslant T\}$ and $\tilde{H}^\theta = 0$. Again, we choose $\tilde{X}^0 = \tilde{H}^0 = 0$ so that $\tilde{\mathscr{X}}^0 = \tilde{\mathscr{X}} = \{\int \theta \, d\tilde{S}: \ \theta \in \Theta\}$. By the martingale representation theorem for Brownian motion (see e.g. Karatzas and Shreve, 1991), any probability measure equivalent to P has a density process in the form

$$Z^\nu = \mathscr{E}\left(-\int (\lambda + \sigma^{-1}\nu)' \, dW\right) \tag{3.4}$$

(here $\mathscr{E}$ is the Doléans–Dade exponential) where ν lies in the set $\mathscr{N}$ of $\mathbb{R}^n$-valued $\mathbb{F}$-adapted process such that $\int_0^T |\sigma_t^{-1}\nu_t|^2 \, dt < \infty$ and $E[Z_T^\nu] = 1$. By Girsanov's theorem, the Doob–Meyer decomposition of $V^\theta = \int \theta \, d\tilde{S} \in \tilde{\mathscr{X}}^0$ under $P^\nu = Z_T^\nu P$, $\nu \in \mathscr{N}$, is

$$V^\theta = \int \frac{1}{S^0} \theta' \sigma \, dW^\nu + A^{\nu,\theta}, \tag{3.5}$$

where W^ν is a P^ν-Brownian motion and $A^{\nu,\theta}$ is the predictable compensator under P^ν:

$$A^{\nu,\theta} = \int \frac{1}{S^0}(-\theta'\nu)\,\mathrm{d}t.$$

Denote by $\delta(\nu) = \sup_{\theta\in K} -\theta'\nu$ the support function of K and let $\tilde{K} = \{\nu \in \mathbb{R}^n: \delta(\nu) < \infty\}$ the effective domain of δ. It follows that there is an upper bound for $\{A^{\nu,\theta}, \theta \in \Theta\}$ iff ν is valued in $\tilde{K}$. Therefore, by Remark 2.1, $\mathscr{P}^0$ consists of all probability measures P^ν, $\nu \in \mathcal{N}(\tilde{K}) = \{\nu \in \mathcal{N}: \nu$ valued in $\tilde{K}$ and $\int_0^T 1/S_T^0 \delta(\nu_t)\,\mathrm{d}t$ is bounded$\}$. Moreover, the upper variation process is given by

$$\tilde{A}^0(P^\nu) = \int \frac{1}{S^0}\delta(\nu)\,\mathrm{d}t.$$

Cvitanić and Karatzas (1992, in the case of proportion) and Cuoco (1997) have obtained same results for the set $\mathscr{P}^0$, called in their papers auxiliary martingale measures or state price densities.

The next result states the closure property (CP) in this model.

Lemma 3.1. *The set $\tilde{\mathscr{X}} = \{\int \theta\,\mathrm{d}\tilde{S}: \theta \in \Theta\}$ is closed for the semimartingale topology.*

Proof. Let $V^n = \int \theta^n\,\mathrm{d}\tilde{S}$, $\theta^n \in \Theta$, be a sequence converging to V in the semimartingale topology. By (P1), (P2) and the martingale representation for Brownian motion, there exist $\theta \in L(\tilde{S})$ and P^ν, $\nu \in \mathcal{N}$, such that V^n converges to $V = \int \theta\,\mathrm{d}\tilde{S}$ in $\mathscr{M}^2(P^\nu) \oplus \mathscr{A}(P^\nu)$. As in (3.5), the canonical decomposition of V^n and V in $\mathscr{M}^2(P^\nu) \oplus \mathscr{A}(P^\nu)$ is

$$V^n = \int \theta^{n\prime}\tilde{\sigma}\,\mathrm{d}W^\nu + A^{\nu,\theta^n},$$

$$V = \int \theta'\tilde{\sigma}\,\mathrm{d}W^\nu + A^{\nu,\theta},$$

where we set $\tilde{\sigma} = \sigma/S^0$. Therefore, $\int \theta^{n\prime}\tilde{\sigma}\,\mathrm{d}W^\nu$ converges to $\int \theta'\tilde{\sigma}\,\mathrm{d}W^\nu$ in $\mathscr{M}^2(P^\nu)$, which means that

$$E^{P^\nu}\left[\int_0^T (\theta_t^n - \theta_t)'\tilde{\sigma}_t\tilde{\sigma}_t'(\theta_t^n - \theta_t)\,\mathrm{d}t\right] \to 0,$$

as n goes to infinity. We deduce that (possibly along a subsequence):

$$(\theta_t^n - \theta_t)'\tilde{\sigma}_t\tilde{\sigma}_t'(\theta_t^n - \theta_t) \to 0 \quad \text{a.s. for all } t \in [0,T]$$

as n goes to infinity. By the nondegeneracy condition (3.1) and since S^0 is strictly positive, this implies:

$$\theta_t^n \to \theta_t \quad \text{a.s. for all } t \in [0,T].$$

Therefore, $\theta_t \in K$, a.s. for all t. We conclude that $\theta \in \Theta$, which ends the proof. $\square$

3.3. Labor income model

We consider the same model as in the previous paragraph 3.2 but we assume further that the agent receives an income with a rate e_t per unit of time. His self-financed wealth process starting from initial wealth x and with an amount θ invested in securities S is given by

$$X_t^{x,\theta} = S_t^0 \left(x + \int_0^t \theta_u \, \mathrm{d}\tilde{S}_u + \int_0^t \frac{1}{S_u^0} e_u \, \mathrm{d}u \right).$$

This model is a particular case of Example 2.1 with S^0 given by (3.2), $\tilde{S}$ given by (3.3), $\Theta = \{\theta \in L(\tilde{S}) \colon \theta_t \in K, \ 0 \leqslant t \leqslant T\}$ and $\tilde{H}^\theta = \int (1/S^0) e \, \mathrm{d}t$. Here $\tilde{H}^\theta$ is independent of θ and obviously satisfies the convexity property (2.5). We choose $\tilde{X}^0 = \tilde{H}^0 = \int (1/S^0) e \, \mathrm{d}t$ so that $\tilde{\mathscr{X}}^0 = \{\int \theta \, \mathrm{d}\tilde{S} \colon \theta \in \Theta\}$ and $\mathscr{P}^0$ are same as in paragraph 3.2. The closure property (CP) for $\tilde{\mathscr{X}} = \{\int \theta \, \mathrm{d}\tilde{S} + \int (1/S^0) e \, \mathrm{d}t \colon \theta \in \Theta\}$ is satisfied from the invariance of the Emery distance in (2.2) under translation.

If we assume that the agent may consume during $[0, T]$, then his self-financed wealth process becomes:

$$X_t^{x,\theta,c} = S_t^0 \left(x + \int_0^t \theta_u \, \mathrm{d}\tilde{S}_u + \int_0^t \frac{1}{S_u^0} (e_u - c_u) \, \mathrm{d}u \right),$$

where $c \in \mathscr{C}_+$, the set of nonnegative $\mathbb{F}$-adapted processes such that $\int_0^T c_t \, \mathrm{d}t < \infty$ a.s. Here c represents the consumption rate process per unit of time. We have then to consider the family $\tilde{\mathscr{X}} = \{\int \theta \, \mathrm{d}\tilde{S} + \int 1/S^0 (e - c) \, \mathrm{d}t \colon \theta \in \Theta, c \in \mathscr{C}_+\}$, which satisfies the convexity property (2.1). We choose $\tilde{X}^0 = \int (1/S^0) e \, \mathrm{d}t$ corresponding to the element of $\tilde{\mathscr{X}}$ for $\theta = 0$ and $c = 0$. In this case, $\tilde{\mathscr{X}}^0 = \{\int \theta \, \mathrm{d}\tilde{S} - \int (1/S^0) c \, \mathrm{d}t \colon \theta \in \Theta, \ c \in \mathscr{C}_+\}$. Since c is nonnegative, $\mathscr{P}^0$ is same as in the case without consumption and so as in Section 3.2. The closure property (CP) for the family $\tilde{\mathscr{X}}$ may also be proved. It requires a little more work than in Lemma 3.1 and is omitted here. Proof may be obtained upon request from the authors, see also Pham (2000b).

3.4. Large investor model

We consider a model as in Section 3.2 but we assume further that the stock price S is influenced through its rate of return by the portfolio strategy θ:

$$\mu = \tilde{\mu} + f(\theta).$$

Here f is some given function which transduces the effect of the portfolio chosen by the investor on the price process. The self-financed wealth process is given by

$$X_t^{x,\theta} = S_t^0 \left(x + \int_0^t \theta_u \, \mathrm{d}\tilde{S}_u + \int_0^t \theta_u' f(\theta_u) \, \mathrm{d}u \right),$$

where

$$\mathrm{d}\tilde{S}_t = \frac{1}{S_t^0} \sigma_t (\tilde{\lambda}_t \, \mathrm{d}t + \mathrm{d}W_t) \tag{3.6}$$

and $\tilde{\lambda}_t = \sigma_t^{-1}(\tilde{\mu}_t - r_t 1_n)$. This is a particular case of Example 2.1 with $\tilde{S}$ given by (3.6), $\Theta = \{\theta \in L(\tilde{S}) \colon \int_0^T |\theta_t' f(\theta_t) \, \mathrm{d}t| < \infty, \theta_t \in K, \ 0 \leqslant t \leqslant T\}$ and $\tilde{H}^\theta = \int \theta' f(\theta) \, \mathrm{d}t$. Convexity of Θ and property (2.5) are satisfied provided that the function $h(\theta) = \theta' f(\theta)$ satisfies the conditions:

h is concave in θ

and there exists $k \geqslant 0$, $\forall \theta^1, \theta^2 \in K$, $\forall \alpha \in [0,1]$, $\forall t \in [0,T]$,

$$|h(t, \alpha\theta^1 1 + (1-\alpha)\theta^2)| \leqslant k(1 + |h(t,\theta^1)| + |h(t,\theta^2)|).$$

This is the setting studied by Cuoco and Cvitanić (1998). We choose $\tilde{X}^0 = \tilde{H}^0 = 0$ so that $\tilde{\mathscr{X}}^0 = \tilde{\mathscr{X}} = \{\int \theta \, \mathrm{d}\tilde{S} + \int h(\theta)\, \mathrm{d}t \colon \theta \in \Theta\}$. As in Section 3.2, any probability measure equivalent to P has a density process in the form:

$$Z^{\nu} = \mathscr{E}\left(-\int (\tilde{\lambda} + \sigma^{-1}\nu)' \, \mathrm{d}W \right),$$

where ν lies in the set $\mathscr{N}$ of $\mathbb{R}^n$-valued $\mathbb{F}$-adapted process such that $\int_0^T |\sigma_t^{-1}\nu_t|^2 \, \mathrm{d}t < \infty$ and $E[Z_T^{\nu}] = 1$. By Girsanov's theorem, the predictable compensator of $V^{\theta} = \int \theta \, \mathrm{d}\tilde{S} + \int h(\theta)\, \mathrm{d}t \in \tilde{\mathscr{X}}^0$ under $P^{\nu} = Z_T^{\nu} P$, $\nu \in \mathscr{N}$, is given by

$$A^{\nu,\theta} = \int \frac{1}{S^0}(h(\theta) - \theta'\nu)\, \mathrm{d}t.$$

Denote by $\tilde{h}(\nu) = \sup_{\theta \in K}(h(\theta) - \theta'\nu)$ the convex conjugate of $-h(-\theta)$ and let $\tilde{\mathscr{K}} = \{\nu \in \mathbb{R}^n \colon \tilde{h}(\nu) < \infty\}$ its effective domain. We deduce from Remark 2.1 that $\mathscr{P}^0$ consists of all probability measures P^{ν}, $\nu \in \mathscr{N}(\tilde{\mathscr{K}}) = \{\nu \in \mathscr{N} \colon \nu$ valued in $\tilde{\mathscr{K}}$ and $\int_0^T 1/S_t^0 \tilde{h}(\nu_t)\, \mathrm{d}t$ is bounded$\}$. Moreover, the upper variation process is given by

$$\tilde{A}^0(P^{\nu}) = \int \frac{1}{S^0}\tilde{h}(\nu)\, \mathrm{d}t.$$

Same results have been obtained by Cuoco and Cvitanić (1998). The closure property (CP) may also be proved in this model under a Lipschitz condition on function h, see Pham (2000b).

3.5. Reinsurance model

We consider an insurance company which reinsures a fraction $1 - \theta_t$ of the incoming claims. The times of arrival of the claims are modelled by a Poisson process (N_t) with constant intensity π and the magnitude of the incoming claims is constant, equal to $\delta \geqslant 0$. The premium rate per unit of time received by the company is a constant $\alpha \geqslant 0$ and the premium rate per unit of time paid by the company to the reinsurer is $\beta \geqslant \alpha$. The risk process of the insurance company is given by:

$$X_t^{x,\theta} = x + \int_0^t (\alpha - \beta(1 - \theta_u))\, \mathrm{d}u - \int_0^t \theta_u \delta \, \mathrm{d}N_u.$$

The reinsurance trading strategy is constrained to remain in $K = [0,1]$. This is a particular case of Example 2.1 with $\tilde{S} = -\delta N$, $S^0 = 1$, $\Theta = \{\theta \in L(\tilde{S}) \colon \theta_t \in K,\ 0 \leqslant t \leqslant T\}$ and $\tilde{H}^{\theta} = \int (\alpha - \beta(1-\theta))\, \mathrm{d}t$. We choose $\tilde{X}^0 = \tilde{H}^0 = \int (\alpha - \beta)\, \mathrm{d}t$ so that $\tilde{\mathscr{X}}^0 = \{\int -\theta\delta \, \mathrm{d}N + \int \beta\theta \, \mathrm{d}t \colon \theta \in \Theta\}$. Assuming that $\mathbb{F}$ is the filtration generated by the Poisson process, it is well-known from the martingale representation theorem for random measures (see e.g. Brémaud, 1981) that all probability measures $Q \sim P$ have a density process in the form

$$Z^{\rho} = \mathscr{E}\left(\int (\rho - 1)\, \mathrm{d}\tilde{N} \right),$$

where $\tilde{N} = N - \int \pi \,\mathrm{d}t$ is the P-compensated martingale of N and $\rho \in \mathscr{D} = \{(\rho_t)_{0 \leqslant t \leqslant T}$ predictable process: $\rho_t > 0$, a.s., $0 \leqslant t \leqslant T$, $\int_0^T |\ln \rho_t| + \rho_t \,\mathrm{d}t < \infty$ and $E[Z_T^\rho] = 1\}$. By Girsanov's theorem, the predictable compensator of an element $V^\theta = \int -\theta\delta \,\mathrm{d}N + \int \beta\theta \,\mathrm{d}t \in \tilde{\mathscr{X}}^0$ under $P^\rho = Z_T^\rho P$ is

$$A^{\rho,\theta} = \int \theta(\beta - \rho\delta\pi) \,\mathrm{d}t.$$

We then deduce from Remark 2.1 that $\mathscr{P}^0 = \{P^\rho\colon \rho \in \mathscr{D}\}$ and the upper variation process of P^ρ is

$$\tilde{A}^0(P^\rho) = \int (\beta - \rho\delta\pi)_+ \,\mathrm{d}t.$$

The closure property (CP) may also be proved in this model by using properties (P1) and (P2), see Pham (2000b).

4. Existence and uniqueness

We denote by $L^0_+(\mathscr{F}_T)$ the set of nonnegative $\mathscr{F}_T$-measurable random variables and we set for $x \in \mathbb{R}$:

$$\mathscr{C}_+(x) = \{H \in L^0_+(\mathscr{F}_T)\colon H \leqslant X_T \quad \text{a.s. for some } X \in \mathscr{X}_+(x)\}.$$

The following lemma formulates dynamic problem (2.8) into an equivalent static one.

Lemma 4.1.

$$J(x) = \sup_{H \in \mathscr{C}_+(x)} E[U(H)] = \sup_{H \in \mathscr{C}_+(x)} E[U(H \wedge \bar{x})], \quad x \in \mathbb{R}. \tag{4.1}$$

(1) *If* $X^* \in \mathscr{X}_+(x)$ *solves* (2.8), *then* $H^* = X_T^* \in \mathscr{C}_+(x)$ *solves* (4.1)
(2) *Conversely, if* $H^* \in \mathscr{C}_+(x)$ *solves* (4.1), *then* $X^* \in \mathscr{C}_+(x)$, *s.t.* $H^* \leqslant X_T^*$, *solves* (2.8).

Proof. Let $X \in \mathscr{X}_+(x)$. Then $X_T \in \mathscr{C}_+(x)$ and so $E[U(X_T)] \leqslant \sup_{H \in \mathscr{C}_+(x)} E[U(H)]$, hence

$$J(x) \leqslant \sup_{H \in \mathscr{C}_+(x)} E[U(H)] = \sup_{H \in \mathscr{C}_+(x)} E[U(H \wedge \bar{x})], \tag{4.2}$$

where the equality in (4.2) is clear since $U(x) = -\infty$ for $x > \bar{x}$.

Conversely, given $H \in \mathscr{C}_+(x)$, there exists $X \in \mathscr{X}_+(x)$ such that $X_T \geqslant H$ a.s. Since U is nondecreasing on its domain, we deduce that $E[U(H \wedge \bar{x})] \leqslant E[U(X_T \wedge \bar{x})]$ and so by (2.9)

$$\sup_{H \in \mathscr{C}_+(x)} E[U(H \wedge \bar{x})] \leqslant J(x), \tag{4.3}$$

which proves (4.1).

(1) Suppose that $X^* \in \mathscr{X}_+(x)$ solves (2.8). Then $H^* = X_T^* \in \mathscr{C}_+(x)$ and we have

$$J(x) = E[U(X_T^*)] = E[U(H^*)],$$

which shows that H^* solves (4.1).

(2) Suppose that $H^* \in \mathscr{C}_+(x)$ solves (4.1). Then there exists $X^* \in \mathscr{X}_+(x)$ such that $H^* \leqslant X_T^*$ a.s. Since U is nondecreasing on its domain, we have

$$J(x) = E[U(H^* \wedge \bar{x})] \leqslant E[U(X_T^* \wedge \bar{x})],$$

which shows that X^* solves (2.9). □

The main purpose of this section is to provide an existence and uniqueness result to the problem (4.1). We first state a dual characterization of the set $\mathscr{C}_+(x)$. We denote by $\mathscr{T}$ the set of all stopping times valued in $[0, T]$.

Proposition 4.1. *Let $H \in L_+^0(\mathscr{F}_T)$ and $x \in \mathbb{R}$. Then*

$$H \in \mathscr{C}_+(x) \quad \textit{iff} \;\; v(H) := \sup_{Q \in \mathscr{P}^0, \tau \in \mathscr{T}} E^Q\left[\frac{H}{S_T^0} 1_{\tau = T} - \tilde{X}_\tau^0 - \tilde{A}_\tau^0(Q)\right] \leqslant x. \tag{4.4}$$

Proof (*Necessary condition*). Let $H \in \mathscr{C}_+(x)$. Then there exists $X = S^0\tilde{X} \in \mathscr{X}_+(x)$ such that $H/S_T^0 1_{t=T} \leqslant \tilde{X}_t$, a.s. $0 \leqslant t \leqslant T$. Let $Q \in \mathscr{P}^0$. By definition of $\mathscr{P}^0$, the process $\tilde{X} - \tilde{X}^0 - \tilde{A}^0(Q)$ is a Q-local supermartingale, bounded from below by a Q-integrable random variable under (2.4) and (H1), and is actually a Q-supermartingale from Fatou's lemma. We deduce that

$$\begin{aligned} E^Q\left[\frac{H}{S_T^0} 1_{\tau = T} - \tilde{X}_\tau^0 - \tilde{A}_\tau^0(Q)\right] &\leqslant E^Q[\tilde{X}_\tau - \tilde{X}_\tau^0 - \tilde{A}_\tau^0(Q)] \\ &\leqslant x \end{aligned}$$

for all $Q \in \mathscr{P}^0$ and $\tau \in \mathscr{T}$. This shows that $v(H) \leqslant x$.

(*Sufficient condition*). Consider the adapted process $g_t = H/S_T^0 1_{t=T} - \tilde{X}_t^0$, $0 \leqslant t \leqslant T$. Since

$$v(H) = \sup_{Q \in \mathscr{P}^0, \tau \in \mathscr{T}} E^Q[g_\tau - \tilde{A}_\tau^0(Q)] \leqslant x < \infty,$$

then by the stochastic control Lemma A.1 of Föllmer and Kramkov (1997), there exists a RCLL version of the process:

$$V_t = \operatorname{ess} \sup_{Q \in \mathscr{P}^0, \tau \in \mathscr{T}_t} E^Q[g_\tau - \tilde{A}_\tau^0(Q) + \tilde{A}_t^0(Q) | \mathscr{F}_t] \quad 0 \leqslant t \leqslant T, \tag{4.5}$$

where $\mathscr{T}_t$ is the set of stopping times valued in $[t, T]$. Moreover, for any $Q \in \mathscr{P}^0$, the process $V - \tilde{A}^0(Q)$ is a Q-local supermartingale. By the optional decomposition under constraints of Föllmer and Kramkov (see their Theorem 3.1), the process V admits a decomposition

$$V = v(H) + U - C,$$

where $U \in \tilde{\mathscr{X}}^0$ and C is an (optional) nondecreasing process with $C_0 = 0$. Hence there exists $\tilde{X} \in \tilde{\mathscr{X}}$ such that

$$V_t \leqslant v(H) + \tilde{X}_t - \tilde{X}_t^0 \quad \text{a.s. } 0 \leqslant t \leqslant T. \tag{4.6}$$

Since $v(H) \leqslant x$ and $V_t \geqslant g_t \geqslant -\tilde{X}_t^0$, by (4.5), inequality (4.6) implies that

$$X_t(x) := S_t^0(x + \tilde{X}_t) \geqslant 0, \quad 0 \leqslant t \leqslant T$$

and so $X(x)\in\mathscr{X}_+(x)$. Moreover, inequality (4.6) for $t=T$ shows that

$$H\leqslant X_T(x), \quad \text{a.s.}$$

Therefore $H\in\mathscr{C}_+(x)$ and the proof is ended. □

Characterization (4.4) in the last proposition means that $v(H)$ is the least initial state value which allows to "dominate" in the almost sure sense the $\mathscr{F}_T$-measurable random variable H by a nonnegative state process. In a financial context, $v(H)$ is usually called the superreplication cost of the American option $(H1_{t=T})_t$. Notice in particular that expression of $v(H)$ does not depend on the choice of $\tilde{X}^0$ and $\mathscr{P}^0$ satisfying (H0) and (H1).

Remark 4.1. Arguments in the proof of Proposition 4.1 show that the process $X\in\mathscr{X}_+(x)$ such that $X_T\geqslant H$ a.s. is given by the optional decomposition of the process V in (4.5).

Remark 4.2. In the case of European constraint $\mathscr{X}_e(x)$ and by setting $\mathscr{C}_e(x)=\{H\in L^0_+(\mathscr{F}_T)$: $H\leqslant X_T$ a.s. for some $X\in\mathscr{X}_e(x)\}$, we can prove similarly (see Pham, 2000b) that

$$H\in\mathscr{C}_e(x) \quad \text{iff} \quad v_e(H):=\sup_{Q\in\mathscr{P}^0} E^Q\left[\frac{H}{S_T^0}-\tilde{X}_T^0-\tilde{A}_T^0(Q)\right]\leqslant x.$$

Remark 4.3. When $\tilde{X}^0$ is nonincreasing and $\tilde{A}^0(Q)=0$, the supremum in $\tau\in\mathscr{T}$ in the expression of $v(H)$ is attained for $\tau=T$ and so $v(H)=v_e(H)$. This means that the optimal state under the European constraint is also the optimal state under the American constraint. In the general case, typically in the presence of random endowment and/or constraints, we have $v_e(H)<v(H)$.

Problem (4.1) is then a convex optimization problem under (infinite) linear constraints characterized by (4.4). As a first consequence of the previous proposition, we can provide a necessary and sufficient condition ensuring that the set of constraints $\mathscr{X}_+(x)$ and $\mathscr{C}_+(x)$ are nonempty.

Corollary 4.1. *$\mathscr{X}_+(x)$ and $\mathscr{C}_+(x)$ are nonempty if and only if*

$$v(0)=\sup_{Q\in\mathscr{P}^0,\tau\in\mathscr{T}} E^Q[-\tilde{X}_\tau^0-\tilde{A}_\tau^0(Q)]\leqslant x. \tag{4.7}$$

Proof. Suppose that $\mathscr{X}_+(x)\neq\emptyset$. Then there exists $X\in\mathscr{X}_+(x)$ such that $X_T\geqslant 0$. Hence $H\equiv 0\in\mathscr{C}_+(x)$ and by Proposition 4.1, we have $v(0)\leqslant x$. Conversely, suppose that $x\geqslant v(0)$. Then by Proposition 4.1, $H=0\in\mathscr{C}_+(x)$, i.e. there exists $X\in\mathscr{X}_+(x)$ such that $X_T\geqslant 0$. In particular $\mathscr{X}_+(x)$ and $\mathscr{C}_+(x)$ are nonempty. □

In the following, we then focus on the optimization problems (2.8) and (4.1) for $x\geqslant v(0)$. The main result of this section proves the existence and uniqueness of a solution to the static problem (4.1). It also provides some qualitative properties of the associated value function. We make the following assumption on the objective function U.

Assumption 4.1. We have either

(i) the function U^+ is bounded,

or

(ii) there exist $\gamma \in (0,1)$, $\bar{Q} \in \mathscr{P}^0$ with density $\bar{Z}_T = \mathrm{d}\bar{Q}/\mathrm{d}P$ satisfying

$$(\bar{Z}_T)^{-1} \in L^{\bar{p}}(P) \tag{4.8}$$

for some $\bar{p} > \gamma/(1-\gamma)$, $x_0 \in \mathrm{dom}\, U$, $U(x_0) \in L^p(P)$, where $p = \bar{p}/(\gamma(1+\bar{p}))$, $\Lambda \in L^p(P)$ and $k \in L^\infty(P)$ such that the function U^+ satisfies the growth condition

$$U^+(x) \leqslant kx^\gamma + \Lambda \quad \forall x \in \mathrm{dom}\, U \cap [x_0, \infty). \tag{4.9}$$

Remark 4.4. Similar assumptions have been made by Cuoco (1997) and Bank and Riedel (2000) in the case of deterministic utility functions on $(0,\infty)$. In the case of shortfall risk loss function (Example 2.2(iii)), Assumption 4.1(i) is obviously satisfied.

Theorem 4.1. *Let Assumption* 4.1 *hold.*

(1) *For any* $x \in [v(0), \infty)$, *there exists* $H^*(x) \in \mathscr{C}_+(x)$ *solution of* (4.1) *and* $\bar{x}$ *is solution of* (4.1) *for* $x \in [v(\bar{x}), \infty)$. *Moreover, if* U *is strictly concave on* $\mathrm{dom}\, U$ *a.s., any two such solutions coincide a.s.*

(2) *The function* J *is nondecreasing and concave on* $[v(0), \infty)$, *and equal to* $E[U(\bar{x})]$ *on* $[v(\bar{x}), \infty)$. *Moreover, if* $U(x) < U(\bar{x})$ *whenever* $x < \bar{x}$, *then* J *is strictly increasing on* $[v(0), v(\bar{x})]$, *and* $v(H^*(x)) = x$ *for any* $x \in [v(0), v(\bar{x})]$, *and if* U *is strictly concave on* $\mathrm{dom}\, U$ *a.s., then* J *is strictly concave on* $[v(0), v(\bar{x})]$.

We first need to state the two following lemmas.

Lemma 4.2. *The set* $\mathscr{C}_+(x)$ *is convex and closed for the topology of convergence in measure.*

Proof. The convexity of $\mathscr{C}_+(x)$ is immediate from its characterization (4.4) in Proposition 4.1. Let $(H^n)_{n \in \mathbb{N}}$ be a sequence in $\mathscr{C}_+(x)$ converging to $H \in L^0_+(\mathscr{F}_T)$ a.s.. Take $Q \in \mathscr{P}^0$ and $\tau \in \mathscr{T}$. By Fatou's lemma, we have

$$\begin{aligned} E^Q\left[\frac{H}{S^0_T} 1_{\tau=T} - \tilde{X}^0_\tau - \tilde{A}^0_\tau(Q)\right] &\leqslant \liminf_{n\to\infty} E^Q\left[\frac{H}{S^0_T}{}_n 1_{\tau=T} - \tilde{X}^0_\tau - \tilde{A}^0_\tau(Q)\right] \\ &\leqslant x \end{aligned}$$

and so $v(H) \leqslant x$. This proves the closedness property of $\mathscr{C}_+(x)$. $\square$

Lemma 4.3. *Let Assumption* 4.1 *hold. Then the family* $\{U^+(H), H \in \mathscr{C}_+(x)\}$ *is uniformly integrable under* P.

Proof. If the function U^+ is bounded, then this assertion is trivial. Otherwise, we first show that the family $\{H^\gamma, H \in \mathscr{C}_+(x)\}$ is bounded in $L^p(P)$ with $p = \bar{p}/\gamma(1+\bar{p}) > 1$.

By Hölder's inequality we have

$$E[H^{\gamma p}] \leqslant (E[H\bar{Z}_T])^{\gamma p}(E[\bar{Z}_T^{-\gamma p/(1-\gamma p)}])^{1-\gamma p}$$

$$\leqslant (x + E^{\bar{Q}}[\tilde{X}_T^0 + \tilde{A}_T^0(\bar{Q})])^{\gamma p}(E[\bar{Z}_T^{-\bar{p}}])^{1-\gamma p}. \tag{4.10}$$

The last sum in (4.10) is independent of H and finite by (4.8), (H1) and (2.4). Using the hypothesis of growth condition on U^+, there exists $c > 0$ such that for all $H \in \mathscr{C}_+(x)$:

$$E[U^+(H)^p] = E[U^+(H)^p 1_{\{H \leqslant x_0\}}] + E[U^+(H)^p 1_{\{H \geqslant x_0\}}]$$

$$\leqslant E[U^+(x_0)^p] + c(E[(kH^\gamma)^p] + E[\Lambda^p]).$$

In view of (4.10) and assumptions on k and Λ, this proves the $L^p(P)$-boundedness of the family $\{U^+(H), H \in \mathscr{C}_+(x)\}$ and therefore its uniform integrability under P. □

Proof of Theorem 4.1. (1) Let $x \geqslant v(0)$ and $(H_n)_{n \in \mathbb{N}} \in \mathscr{C}_+(x)$ be a maximizing sequence of the problem (4.1), i.e.

$$\lim_{n \to \infty} E[U(H_n)] = J(x). \tag{4.11}$$

Since $H_n \geqslant 0$ $P-$a.s, then by Lemma A.1.1 of Delbaen and Schachermayer (1994), there exists a sequence of $\mathscr{F}_T$-measurable random variables $\hat{H}_n \in \mathrm{conv}(H_n, H_{n+1}, \ldots)$ such that $\hat{H}_n$ converges almost surely to $H^*(x) \in L^0_+(\mathscr{F}_T)$. By Lemma 4.2, $H^*(x) \in \mathscr{C}_+(x)$. By the concavity and upper-semicontinuity of U we have

$$J(x) \leqslant \limsup_{n \to \infty} E[U(\hat{H}_n)]$$

$$= \limsup_{n \to \infty} E[U^+(\hat{H}_n)] - \liminf_{n \to \infty} E[U^-(\hat{H}_n)]$$

$$\leqslant E[U^+(H^*(x))] - E[U^-(H^*(x))]$$

$$= E[U(H^*(x))],$$

where the second inequality follows from Lemma 4.3 and Fatou's lemma. Therefore $J(x) \leqslant E[U(H^*(x))]$ which proves that $H^*(x)$ solves (4.1).

Now, suppose that $x \geqslant v(\bar{x})$, and so $\bar{x} \in \mathscr{C}_+(x)$. Since U is nondecreasing on $\mathrm{dom}\, U$, we have for all $H \in \mathscr{C}_+(x)$:

$$E[U(H \wedge \bar{x}] \leqslant E[U(\bar{x})].$$

By Lemma 4.1, this shows that $\bar{x}$ is solution of (4.1).

Let H_1 and H_2 be two solutions of (4.1) and $\varepsilon \in (0,1)$. Set $H_\varepsilon = (1-\varepsilon)H_1 + \varepsilon H_2$ which lies in $\mathscr{C}_+(x)$ by Lemma 4.2. By concavity of function U, we have

$$E[U(H_\varepsilon)] \geqslant (1-\varepsilon)E[U(H_1)] + \varepsilon E[U(H_2)] \tag{4.12}$$

$$= J(x). \tag{4.13}$$

Suppose that $P[H_1 \neq H_2] > 0$. Then by the strict concavity of U, we should have strict inequality in (4.12), which is a contradiction with (4.13).

(2) Let $v(0) \leqslant x_1 \leqslant x_2$. Since $\mathscr{C}_+(x_1) \subset \mathscr{C}_+(x_2)$, we deduce that $J(x_1) \leqslant J(x_2)$ and so J is nondecreasing on $[v(0), \infty)$. Notice also that $\frac{1}{2}(H^*(x_1) + H^*(x_2)) \in \mathscr{C}_+((x_1 + x_2)/2)$. Then, by concavity of the function U, we have

$$\begin{aligned} J\left(\frac{x_1 + x_2}{2}\right) &\geqslant E\left[U\left(\frac{H^*(x_1) + H^*(x_2)}{2}\right)\right] \\ &\geqslant \frac{1}{2}(E[U(H^*(x_1))] + E[U(H^*(x_2))]) \\ &= \frac{1}{2}(J(x_1) + J(x_2)), \end{aligned}$$

which proves the concavity of J on $[v(0), \infty)$.

We know from Part (1) of this theorem that $J(x) = E[U(\bar{x})]$ for $x \geqslant v(\bar{x})$. We now consider the case $v(0) < v(\bar{x})$. Let us first check that J is strictly increasing on $[v(0), v(\bar{x})]$ under the condition that $U(x) < U(\bar{x})$ if $x < \bar{x}$. On the contrary, there would exist $v(0) \leqslant x_1 < x_2 \leqslant v(\bar{x})$ such that $J(x_1) = J(x_2)$ and $\alpha \in (0, 1)$ such that $x_2 = \alpha x_1 + (1 - \alpha)v(\bar{x})$. By concavity of J, we should have

$$\begin{aligned} J(x_2) &\geqslant \alpha J(x_1) + (1 - \alpha)J(v(\bar{x})) \\ &= \alpha J(x_2) + (1 - \alpha)E[U(\bar{x})]. \end{aligned} \tag{4.14}$$

Since $J(x) \leqslant E[U(\bar{x})]$, for all $x \geqslant v(0)$, by (4.1) and the nondecreasing property of U on $\operatorname{dom} U$, relation (4.14) shows that $J(x_1) = J(x_2) = E[U(\bar{x})]$. Under the condition that $U(x) < U(\bar{x})$ whenever $x < \bar{x}$, this implies that the solution $X^*(x_1)$ of $J(x_1)$ $(= E[U(X_T^*(x_1) \wedge \bar{x})])$ satisfies $X_T^*(x_1) \geqslant \bar{x}$ a.s. This is in contradiction with the fact that $x_1 < v(\bar{x})$ and Proposition 4.1.

Let us prove that for all $x \in [v(0), v(\bar{x})]$, we have $v(H^*(x)) = x$. On the contrary, we should have $v(0) \leqslant \tilde{x} := v(H^*(x)) < x$. Then $H^*(x) \in \mathscr{C}_+(\tilde{x})$ and so $J(\tilde{x}) \geqslant E[U((H^*(x))]$ $= J(x)$. However, J is strictly increasing on $[v(0), v(\bar{x})]$, so $J(x) > J(\tilde{x})$ which is a contradiction.

Let us finally check the strict concavity of J on $[v(0), v(\bar{x})]$. Let $v(0) \leqslant x_1 < x_2 \leqslant v(\bar{x})$. We have $\frac{1}{2}(H^*(x_1) + H^*(x_2)) \in \mathscr{C}_+((x_1 + x_2)/2)$. Since $J(x_1) < J(x_2)$, then $H^*(x_1) \neq H^*(x_2)$ a.s. By the strict concavity of U, we have

$$\begin{aligned} J\left(\frac{x_1 + x_2}{2}\right) &\geqslant E\left[U\left(\frac{H^*(x_1) + H^*(x_2)}{2}\right)\right] \\ &> \frac{1}{2}(E[U(H^*(x_1))] + E[U(H^*(x_2))]) \\ &= \frac{1}{2}(J(x_1) + J(x_2)), \end{aligned}$$

which proves the strict concavity of J on $[v(0), v(\bar{x})]$. □

5. Dual singular formulation

The aim of this section is to provide a description of the structure of the solution to problem (4.1) and (2.8) by means of a dual formulation, following the line of research of Karatzas et al. (1991) or Kramkov and Schachermayer (1999).

We assume that U is C^1 and strictly concave on $(0,\bar{x})$ a.s. We set $U'(0) := \lim_{x\downarrow 0} U'(x)$ and we assume that:

$$U'(\bar{x}) := \lim_{x\nearrow\bar{x}} U'(x) = 0, \quad \text{a.s.} \tag{5.1}$$

In the case of Example 2.2(i) and (ii) with $\bar{x} = \infty$, (5.1) is the usual Inada condition $U'(\infty) = 0$. In the case of Example 2.2(iii) with $U(x) = -l(B - x)$, $\operatorname{dom} U = [0, B]$, (5.1) means that $l'(0) = 0$.

The strictly decreasing continuous function U' from $(0,\bar{x})$ onto $(0, U'(0))$ has a strictly decreasing continuous inverse $I : (0, U'(0))$ onto $(0,\bar{x})$, extended by continuity on $(0,\infty)$ by setting $I(y) = 0$ for $y \geqslant U'(0)$.

We consider the (state-dependent) conjugate function of U:

$$\tilde{U}(y) = \sup_{x>0} [U(x) - xy], \quad y > 0. \tag{5.2}$$

It is well known (see e.g. Rockafellar, 1970) that $\tilde{U}$ is a nonincreasing convex differentiable function on $(0,\infty)$ with $\tilde{U}(0) = U(\bar{x})$. Moreover, the derivative of $\tilde{U}$ is given by

$$\tilde{U}'(y) = -I(y), \quad y > 0, \text{ a.s.} \tag{5.3}$$

We also know that $I(y)$ attains the supremum in (5.2), i.e.

$$\tilde{U}(y) = U(I(y)) - yI(y), \quad y > 0, \text{ a.s.} \tag{5.4}$$

In the sequel, we identify a probability measure $Q \ll P$ with its density process $Z = (Z_t)_{0\leqslant t\leqslant T}$, $Z_t = E[\mathrm{d}Q/\mathrm{d}P|\mathscr{F}_t]$. We now dualize the optimization primal problem (4.1) as follows: we enlarge $\mathscr{P}^0$ by considering the set $\mathscr{P}^0_{\mathrm{loc}}$ of nonnegative local martingales Z with $Z_0 = 1$ and satisfying the properties:

(1) there exists $A \in \mathscr{I}_p$ such that

$$Z(V - A) \quad \text{is a } P\text{-local supermartingale for any } V \in \tilde{\mathscr{X}}^0. \tag{5.5}$$

(2) The upper variation process of $\mathscr{X}^0$ under $Z \in \mathscr{P}^0_{\mathrm{loc}}$, defined as the element $\tilde{A}^0(Z)$ in $\mathscr{I}_p$ satisfying (5.5) and such that $A - \tilde{A}^0(Z) \in \mathscr{I}_p$ for any $A \in \mathscr{I}_p$ satisfying (5.5), is bounded a.s.

For any $Z \in \mathscr{P}^0_{\mathrm{loc}}$, we set

$$A^0(Z) = \int S^0 \,\mathrm{d}\tilde{A}^0(Z).$$

Following El Karoui and Jeanblanc (1998), we dualize the space of American constraints by considering the set $\mathscr{D}$ of nonnegative, nonincreasing adapted continuous processes $D = (D_t)_{0\leqslant t\leqslant T}$ with $D_0 = 1$. We then define the dual set of American state space constraints by:

$$\mathscr{Y}^0_{\mathrm{loc}} = \left\{ Y = \frac{ZD}{S^0} : Z \in \mathscr{P}^0_{\mathrm{loc}},\ D \in \mathscr{D} \right\}.$$

In order to formulate our dual problem, we shall strengthen condition (H1) by requiring that:

(H′1) X^0 is a process of finite variation with $E[\int_0^T Z_t \,\mathrm{d}\,|\tilde{X}^0_t|] < \infty$, $\forall Z \in \mathscr{P}^0_{\mathrm{loc}}$.

In the sequel, we denote

$$X^0 = \int S^0 \, \mathrm{d}\tilde{X}^0$$

and for any $Y = ZD/S^0 \in \mathscr{Y}^0_{\mathrm{loc}}$, we set by convention $A^0(Y) = A^0(Z)$.

We now define the value function of our dual problem, for all $y > 0$:

$$\tilde{J}(y) = \inf_{Y \in \mathscr{Y}^0_{\mathrm{loc}}} E\left[\tilde{U}(yY_T) + \int_0^T yY_t \, \mathrm{d}X^0_t + \int_0^T yY_t \, \mathrm{d}A^0_t(Y)\right]. \tag{5.6}$$

With respect to the "classical" dual problem in an incomplete market model, the dual problem (5.6) contains a linear additional term related to X^0 and a nonlinear additional term related to the convex constraints $A^0(Z)$. Moreover, due to the state space constraints, this dual problem appears as a mixed control/singular optimization problem with dynamics $\{Y_t = Z_t D_t / S^0_t, \ 0 \leqslant t \leqslant T\}$ governed by a "classical" control term Z and a singular term D.

We make an assumption on the function U which is analogous to the asymptotic elasticity condition of Kramkov and Schachermayer (1999).

Assumption 5.1. (i) *There exist* $x_0 \in \mathrm{dom}\, U$, *with* $x_0/S^0_T \in L^\infty(P)$ and $U(x_0) \in L^1(P)$, $\gamma \in (0,1)$, $\Lambda \in L^1(P)$, *such that*

$$xU'(x) \leqslant \gamma U(x) + \Lambda, \quad a.s. \ \forall x \in \mathrm{dom}\, U \cap [x_0, \infty).$$

(ii) $U(\bar{x}) \in L^1(P)$ *if* $\bar{x} < \infty$ *a.s.*

(iii) $\forall \varepsilon > 0$, $\exists \delta_\varepsilon$ *real-valued in* $[0, \varepsilon)$, $\delta_\varepsilon S^0_T \in \mathrm{dom}\, U$ *and* $U(\delta_\varepsilon S^0_T) \in L^1(P)$.

Remark 5.1. Assumption 5.1(ii) and (iii) are obviously satisfied in the case of objective functions of Example 2.2 whenever S^0 is bounded.

Remark 5.2. (1) In the case of deterministic utility functions U with $\bar{x} = \infty$, Assumption 5.1(i) is equivalent to the asymptotic elasticity condition of Kramkov and Schachermayer (1999), KS in short (stated for $S^0 = 1$):

$$AE(U) := \limsup_{x \to \infty} \frac{xU'(x)}{U(x)} < 1.$$

Without loss of generality, we can assume that $U(\infty) > 0$. If Assumption 5.1(i) is satisfied, then either $U(\infty) < \infty$ and so $AE(U) = 0$ by Lemma 6.1 of KS, either $U(\infty) = \infty$ and so we have $AE(U) \leqslant \gamma < 1$. Conversely, if $AE(U) < 1$, then by Lemma 6.3 of KS, there exists $x_0 \geqslant 0$ and $\gamma \in (AE(U), 1)$ such that $xU'(x) \leqslant \gamma U(x)$ for $x \geqslant x_0$ and so Assumption 5.1(i) is satisfied.

(2) Consider the Example 2.2(iii) of shortfall risk loss function: $U(x) = -l(B - x)$ and $\mathrm{dom}\, U = [0, B]$ with l a C^1 convex function on $(0, \infty)$. Then by convexity of l, we have

$$xU'(x) \leqslant U(x) + l(B), \quad \forall x \in [0, B].$$

Therefore, whenever $l(B) \in L^1(P)$, Assumption 5.1(i) is satisfied with $x_0 = 0$ and γ arbitrary in $(0, 1)$.

The main result of this section is the following.

Theorem 5.1. *Let Assumption* 5.1 *hold and assume that* $\tilde{J}(y) < \infty$ *for some* $y > 0$.

(1) *For all* $x \in [v(\bar{x}), \infty)$, $\bar{x}$ *is solution to* (4.1).

(2) (a) *For all* $x \in (v(0), v(\bar{x}))$, *there exists* $\hat{y} > 0$ *that attains the infimum in* $\inf_{y>0} [\tilde{J}(y) + xy]$.

(b) *Suppose that there exists a solution* $\hat{Y} = \hat{Z}\hat{D}/S^0 \in \mathscr{Y}^0_{\mathrm{loc}}$ *to problem* $\tilde{J}(\hat{y})$. *Then the unique solution of* (4.1) *is given by*

$$H^*(x) = I(\hat{y}\hat{Y}_T). \tag{5.7}$$

The solution to (2.8) *satisfies*

$$\hat{Y}_t X^*_t(x) = E\left[\hat{Y}_T H^*(x) - \int_t^T \hat{Y}_u \,\mathrm{d}X^0_u - \int_t^T \hat{Y}_u \,\mathrm{d}A^0_u(\hat{Y}) | \mathscr{F}_t\right], \quad 0 \leqslant t \leqslant T. \tag{5.8}$$

If in addition, $\hat{D}_T > 0$ *a.s.*, *then we also have*

$$\frac{\hat{Z}_t X^*_t(x)}{S^0_t} = E\left[\frac{\hat{Z}_T H^*(x)}{S^0_T} - \int_t^T \frac{\hat{Z}_u}{S^0_u} \,\mathrm{d}X^0_u - \int_t^T \frac{\hat{Z}_u}{S^0_u} \,\mathrm{d}A^0_u(\hat{Z}) \middle| \mathscr{F}_t\right], \quad 0 \leqslant t \leqslant T. \tag{5.9}$$

(3) *Suppose that for all* $y > 0$, *there exists a solution to the dual problem* $\tilde{J}(y)$. *Then, we have the conjugate duality relations*

$$J(x) = \min_{y \geqslant 0} [\tilde{J}(y) + xy], \quad \forall x > v(0), \tag{5.10}$$

$$\tilde{J}(y) = \max_{x \geqslant v(0)} [J(x) - xy], \quad \forall y > 0. \tag{5.11}$$

In the complete market model case, the set $\mathscr{P}^0_{\mathrm{loc}}$ is reduced to a singleton and the dual problem is an optimization problem over the set $\mathscr{D}$. In such a context, existence of a solution $\hat{D}(y)$ is proved by El Karoui and Jeanblanc (1998). In an incomplete semimartingale model, Kramkov and Schachermayer (1999) enlarged the set of local martingale measures to a properly defined set of supermartingales, so that the dual problem admits a solution. In an incomplete semimartingale model with random endowment and under European constraints, Cvitanić et al. (2000) extended even further the dual domain to the dual space of L^∞. In our general framework with convex and state constraints, it is an open problem to determine how we should extend appropriately the set $\mathscr{P}^0_{\mathrm{loc}} \times \mathscr{D}$ in order to get existence to the dual problem.

Assertion (2)(b) of Theorem 5.1 may be viewed as a dual verification theorem: it provides a characterization of the solution to the primal problem (2.8) in terms of the solution (when it exists) to the dual problem (5.6). In a markovian context, we have to solve a dual singular control problem which leads by the dynamic programming principle to a free boundary problem, see He and Pagès (1993) and El Karoui and Jeanblanc (1998) in a complete diffusion model. This is an alternative to the usual verification theorem on the primal problem which leads to a nonlinear Bellman PDE with Neumann-type boundary conditions due to the state space constraints.

Remark 5.3. When $U(\bar{x})=\infty$, it is clear from (5.6) and $\tilde{U}(0)=U(\bar{x})$ that the solution $\hat{Y}=\hat{Z}\hat{D}/S^0$ satisfy $\hat{Z}_T>0$ and $\hat{D}_T>0$ a.s.

Remark 5.4. When $\hat{D}_T>0$ a.s, relation (5.9) means that the process $M=(\hat{Z}X^*(x)/S^0)-\int(\hat{Z}/S^0)\,\mathrm{d}X^0-\int(\hat{Z}/S^0)\,\mathrm{d}A^0(\hat{Z})$ is a P-martingale. By Itô's product rule and since D is continuous with finite variation, we have

$$\hat{Y}_T X_T^*(x)-\int_0^T \hat{Y}_t\,\mathrm{d}X_t^0-\int_0^T \hat{Y}_t\,\mathrm{d}A_t^0(\hat{Z})$$

$$=x+\int_0^T \frac{\hat{Z}_{t^-}X_{t^-}^*(x)}{S_{t^-}^0}\,\mathrm{d}\hat{D}_t+\int_0^T \hat{D}_t\,\mathrm{d}M_t, \tag{5.12}$$

where $\hat{Y}=\hat{Z}\hat{D}/S^0$. Taking expectation under P in (5.12) and noting by (5.8) that the expected value of the left-hand side in (5.12) is equal to x, we obtain

$$E\left[\int_0^T \frac{\hat{Z}_{t^-}X_{t^-}^*(x)}{S_{t^-}^0}\,\mathrm{d}\hat{D}_t\right]=0.$$

Since $\hat{Z}$ and $X^*(x)$ are nonnegative, S^0 is strictly positive, and $\hat{D}$ is nonincreasing, this shows that

$$\int_0^T \hat{Z}_{t^-}X_{t^-}^*(x)\,\mathrm{d}\hat{D}_t=0,\quad \text{a.s.}$$

When $\hat{Z}>0$, a.s., this means that $\hat{D}$ stays constant equal to 1 as long as the optimal state $X^*(x)$ is strictly positive, and decreases only when $X^*(x)$ hits zero.

Remark 5.5. If the solution $\hat{Z}$ is strictly positive and is a "true" martingale, then it is the density process of an element $\hat{Q}\in\mathscr{P}^0$. From Bayes formula, expression (5.9) of the optimal state process can then be written as

$$X_t^*(x)=E^{\hat{Q}}\left[\frac{S_t^0}{S_T^0}I(\hat{y}\hat{Y}_T)-\int_t^T \frac{S_t^0}{S_u^0}\,\mathrm{d}X_u^0-\int_t^T \frac{S_t^0}{S_u^0}\,\mathrm{d}A_u^0(\hat{Z})\middle|\mathscr{F}_t\right],\quad 0\leqslant t\leqslant T.$$

This is similar to the expression that we would obtain for the optimal state process under European constraints

$$X_t^{e,*}(x)=E^{\hat{Q}^e}\left[\frac{S_t^0}{S_T^0}I\left(\hat{y}_e\frac{\hat{Z}_T^e}{S_T^0}\right)-\int_t^T \frac{S_t^0}{S_u^0}\,\mathrm{d}X_u^0-\int_t^T \frac{S_t^0}{S_u^0}\,\mathrm{d}A_u^0(\hat{Z}^e)\middle|\mathscr{F}_t\right],\quad 0\leqslant t\leqslant T,$$

where $\hat{Z}^e=\mathrm{d}\hat{Q}^e/\mathrm{d}P|_{\mathscr{F}_t}$ is the solution to the dual problem

$$\tilde{J}_e(\hat{y}_e)=\inf_{Z\in\mathscr{P}^0}E\left[\tilde{U}\left(\hat{y}_e\frac{Z_T}{S_T^0}\right)+\int_0^T \hat{y}_e\frac{Z_t}{S_t^0}\,\mathrm{d}X_t^0+\int_0^T \hat{y}_e\frac{Z_t}{S_t^0}\,\mathrm{d}A_t^0(Z)\right] \tag{5.13}$$

and $\hat{y}_e>0$ solves $\tilde{J}_e'(\hat{y}_e)=-x$. This point has been first observed by El Karoui and Jeanblanc (1998) in a complete Itô processes model with random endowment, by using connections between singular control problems and optimal stopping time problems. By a different approach (see in particular Lemma 6.7), we extend their results in a general setting with constraints.

6. Proof of Theorem 5.1

The proof of Theorem 5.1 is broken into several lemmas.

Lemma 6.1. *For all* $x \geqslant v(0)$, $X \in \mathscr{X}_+(x)$, $Y = ZD/S^0$, $Z \in \mathscr{P}^0_{\rm loc}$, $D \in \mathscr{D}$, *the processes*

$$\frac{ZX}{S^0} - \int \frac{Z}{S^0}\,\mathrm{d}X^0 - \int \frac{Z}{S^0}\,\mathrm{d}A^0(Z)$$

and

$$YX - \int Y\,\mathrm{d}X^0 - \int Y\,\mathrm{d}A^0(Z),$$

are supermartingales under P.

Proof. By definition of $\mathscr{P}^0_{\rm loc}$, the process $Z(X/S^0 - \tilde{X}^0 - \tilde{A}^0(Z))$ is a P-local supermartingale. Since $\tilde{X}^0$ and $\tilde{A}^0(Z)$ have finite variation, this implies from Theorem VII.35 in Dellacherie and Meyer (1982) that the process $M = ZX/S^0 - \int Z\,\mathrm{d}\tilde{X}^0 - \int Z\,\mathrm{d}\tilde{A}^0(Z)$ is a P-local supermartingale. Moreover, M is bounded from below by the random variable $-\int_0^T Z_t\,\mathrm{d}|\tilde{X}^0_t| - \int_0^T Z_t\,\mathrm{d}\tilde{A}^0(Z)$, which is integrable under P by condition (H1′). We deduce by Fatou's lemma that $M = ZX/S^0 - \int Z/S^0\,\mathrm{d}X^0 - \int Z/S^0\,\mathrm{d}A^0(Z)$ is a P-supermartingale.

On the other hand, by Itô's product rule and since D is continuous with finite variation, we get:

$$YX - \int Y\,\mathrm{d}X^0 - \int Y\,\mathrm{d}A^0(Z) = x + \int D\,\mathrm{d}M + \int \frac{Z_- X_-}{S^0_-}\,\mathrm{d}D. \tag{6.1}$$

Since D is nonnegative and nonincreasing, and S^0, Z, X are nonnegative, this shows that the process

$$YX - \int Y\,\mathrm{d}X^0 - \int Y\,\mathrm{d}A^0(Z)$$

is a P-local supermartingale, bounded from below by an $L^1(P)$ random variable, and hence a P-supermartingale. □

Lemma 6.2. *For all* $H \in L^0_+(\mathscr{F}_T)$, *we have*

$$v(H) = \sup_{Y \in \mathscr{Y}^0_{\rm loc}} E\left[Y_T H - \int_0^T Y_t\,\mathrm{d}X^0_t - \int_0^T Y_t\,\mathrm{d}A^0_t(Y)\right]. \tag{6.2}$$

Proof. Fix some $H \in L^0_+(\mathscr{F}_T)$. Given an arbitrary $\tau \in \mathscr{T}$, we define a sequence $(D^n_t)_n$ of elements in $\mathscr{D}$ by

$$D^n_t = \exp\left(-\int_0^t n 1_{\tau \leqslant u}\,\mathrm{d}u\right), \quad 0 \leqslant t \leqslant T, \ n \in \mathbb{N}.$$

We then have for all $Z \in \mathscr{P}^0$:

$$E\left[\frac{Z_T D_T^n}{S_T^0} H - \int_0^T \frac{Z_t D_t^n}{S_t^0}\, \mathrm{d}X_t^0 - \int_0^T \frac{Z_t D_t^n}{S_t^0}\, \mathrm{d}A_t^0(Z)\right]$$

$$\leqslant \sup_{Z \in \mathscr{P}^0, D \in \mathscr{D}} E\left[\frac{Z_T D_T}{S_T^0} H - \int_0^T \frac{Z_t D_t}{S_t^0}\, \mathrm{d}X_t^0 - \int_0^T \frac{Z_t D_t}{S_t^0}\, \mathrm{d}A_t^0(Z)\right].$$

Since $D_t^n \to 1_{t \leqslant \tau}$ a.s., for all $0 \leqslant t \leqslant T$, we have by Fatou's lemma

$$E\left[\frac{Z_T H}{S_T^0} 1_{\tau = T} - \int_0^\tau \frac{Z_t}{S_t^0}\, \mathrm{d}X_t^0 - \int_0^\tau \frac{Z_t}{S_t^0}\, \mathrm{d}A_t^0(Z)\right]$$

$$\leqslant \sup_{Z \in \mathscr{P}^0, D \in \mathscr{D}} E\left[\frac{Z_T D_T}{S_T^0} H - \int_0^T \frac{Z_t D_t}{S_t^0}\, \mathrm{d}X_t^0 - \int_0^T \frac{Z_t D_t}{S_t^0}\, \mathrm{d}A_t^0(Z)\right].$$

Identifying a probability measure $Q \in \mathscr{P}^0$ with its density process Z, we then obtain from Bayes formula

$$v(H) = \sup_{Q \in \mathscr{P}^0, \tau \in \mathscr{T}} E^Q\left[\frac{H}{S_T^0} 1_{\tau = T} - \tilde{X}_\tau^0 - \tilde{A}_\tau^0(Q)\right]$$

$$\leqslant \sup_{Z \in \mathscr{P}^0, D \in \mathscr{D}} E^Q\left[\frac{Z_T D_T}{S_T^0} H - \int_0^T \frac{Z_t D_t}{S_t^0}\, \mathrm{d}X_t^0 - \int_0^T \frac{Z_t D_t}{S_t^0}\, \mathrm{d}A_t^0(Z)\right]$$

$$\leqslant \sup_{Z \in \mathscr{P}^0_{\mathrm{loc}}, D \in \mathscr{D}} E^Q\left[Y_T H - \int_0^T Y_t\, \mathrm{d}X_t^0 - \int_0^T Y_t\, \mathrm{d}A_t^0(Y)\right],$$

where the last inequality follows from the inclusion $\mathscr{P}^0 \subset \mathscr{P}^0_{\mathrm{loc}}$.

Conversely, by the supermartingale property of $YX - \int Y\, \mathrm{d}X^0 - \int Y\, \mathrm{d}A^0(Z)$ for any $Y \in \mathscr{Y}^0_{\mathrm{loc}}$, see Lemma 6.1, we have

$$\sup_{Y \in \mathscr{Y}^0_{loc}} E\left[Y_T H - \int_0^T Y_t\, \mathrm{d}X_t^0 - \int_0^T Y_t\, \mathrm{d}A_t^0(Y)\right] \leqslant x, \quad \forall H \in \mathscr{C}_+(x). \tag{6.3}$$

Now, by Proposition 4.1, any $H \in L^0_+(\mathscr{F}_T)$ lies in $\mathscr{C}_+(v(H))$. We then deduce from (6.3) for $x = v(H)$:

$$\sup_{Y \in \mathscr{Y}^0_{\mathrm{loc}}} E\left[Y_T H - \int_0^T Y_t\, \mathrm{d}X_t^0 - \int_0^T Y_t\, \mathrm{d}A_t^0(Y)\right] \leqslant v(H),$$

which proves the required equality (6.2). □

Remark 6.1. It is easily checked that the supremum in (6.2) can be taken over $\mathscr{P}^0_{\mathrm{loc},+}$, the subset of elements $Z \in \mathscr{P}^0_{\mathrm{loc}}$ such that $Z_t > 0$, for all t, a.s., and $\mathscr{D}_+$, the subset of elements D in $\mathscr{D}$ such that $D_T > 0$ a.s.

$$v(H) = \sup_{Y \in \mathscr{Y}^0_{\mathrm{loc},+}} E\left[Y_T H - \int_0^T Y_t\, \mathrm{d}X_t^0 - \int_0^T Y_t\, \mathrm{d}A_t^0(Z)\right], \tag{6.4}$$

where $\mathscr{Y}^0_{\mathrm{loc},+} = \{Y = ZD/S^0\colon Z \in \mathscr{P}^0_{\mathrm{loc},+}, D \in \mathscr{D}_+\}$.

Lemma 6.3. *For all* $Z^1 \in \mathscr{P}^0_{\rm loc}$, $Z^2 \in \mathscr{P}^0_{{\rm loc},+}$, *for all* $D^1 \in \mathscr{D}$, $D^2 \in \mathscr{D}_+$, *for all* $\varepsilon \in (0,1)$, *there exist* $Z^\varepsilon \in \mathscr{P}^0_{\rm loc}$ *and* $D^\varepsilon \in \mathscr{D}$ *such that*

$$(1-\varepsilon)Z^1D^1 + \varepsilon Z^2D^2 = Z^\varepsilon D^\varepsilon.$$

Moreover, we have

$$E\left[\int_0^T Z_t^\varepsilon D_t^\varepsilon \, \mathrm{d}\tilde{A}_t^0(Z^\varepsilon)\right] \leqslant (1-\varepsilon)E\left[\int_0^T Z_t^1 D_t^1 \, \mathrm{d}\tilde{A}_t^0(Z^1)\right] + \varepsilon E\left[\int_0^T Z_t^2 D_t^2 \, \mathrm{d}\tilde{A}_t^0(Z^2)\right].$$

Proof. We set $\bar{Y}^1 = Z^1D^1, \bar{Y}^2 = Z^2D^2$ and $\bar{Y}^\varepsilon = (1-\varepsilon)\bar{Y}^1 + \varepsilon\bar{Y}^2$. Notice that $\bar{Y}^\varepsilon$ is a strictly positive process. By Itô's product rule and since D is continuous with finite variation, we have $\bar{Y}^\varepsilon = Z^\varepsilon D^\varepsilon$ where

$$\mathrm{d}Z_t^\varepsilon = Z_{t-}^\varepsilon \left[(1-\varepsilon)\frac{D_t^1}{\bar{Y}_{t-}^\varepsilon}\,\mathrm{d}Z_t^1 + \varepsilon\frac{D_t^2}{\bar{Y}_{t-}^\varepsilon}\,\mathrm{d}Z_t^2\right], \quad Z_0^\varepsilon = 1, \tag{6.5}$$

$$\mathrm{d}D_t^\varepsilon = D_t^\varepsilon \left[(1-\varepsilon)\frac{Z_{t-}^1}{\bar{Y}_{t-}^\varepsilon}\,\mathrm{d}D_t^1 + \varepsilon\frac{Z_{t-}^2}{\bar{Y}_{t-}^\varepsilon}\,\mathrm{d}D_t^2\right], \quad D_0^\varepsilon = 1. \tag{6.6}$$

Since D^i, $i=1,2 \in \mathscr{D}$, it is clear that the process D^ε defined in (6.6) lies also in $\mathscr{D}$. The process Z^ε defined in (6.5) is a local martingale and is nonnegative by the Doléans–Dade exponential formula. Let us check that Z^ε lies in $\mathscr{P}^0_{\rm loc}$. Fix some $V \in \tilde{\mathscr{X}}^0$ and consider the process $\tilde{A} \in \mathscr{I}_p$ defined by

$$\mathrm{d}\tilde{A}_t = (1-\varepsilon)\,\frac{\bar{Y}_{t-}^1}{\bar{Y}_{t-}^\varepsilon}\,\mathrm{d}\tilde{A}_t^0(Z^1) + \varepsilon\frac{\bar{Y}_{t-}^2}{\bar{Y}_{t-}^\varepsilon}\,\mathrm{d}\tilde{A}_t^0(Z^2), \quad \tilde{A}_0 = 0. \tag{6.7}$$

From Itô's formula, we have

$$\mathrm{d}(Z_t^\varepsilon(V_t - \tilde{A}_t)) = (V_{t-} - \tilde{A}_t)\,\mathrm{d}\,Z_t^\varepsilon + Z_{t-}^\varepsilon\,\mathrm{d}(V_t - \tilde{A}_t) + \mathrm{d}[Z^\varepsilon, V - \tilde{A}\,]_t. \tag{6.8}$$

From (6.7), we have

$$\mathrm{d}(V_t - \tilde{A}_t) = (1-\varepsilon)\frac{\bar{Y}_{t-}^1}{\bar{Y}_{t-}^\varepsilon}\,\mathrm{d}(V_t - \tilde{A}_t^0(Z^1)) + \varepsilon\frac{\bar{Y}_{t-}^2}{\bar{Y}_{t-}^\varepsilon}\,\mathrm{d}(V_t - \tilde{A}_t^0(Z^2)). \tag{6.9}$$

From (6.5), we have

$$\begin{aligned}\mathrm{d}[Z^\varepsilon, V - \tilde{A}]_t &= (1-\varepsilon)\frac{D_t^1}{\bar{Y}_{t-}^\varepsilon}\,Z_{t-}^\varepsilon\,\mathrm{d}[Z^1, V - \tilde{A}^0(Z^1)]_t + \varepsilon\frac{D_t^2}{\bar{Y}_{t-}^\varepsilon}\,Z_{t-}^\varepsilon\,\mathrm{d}[Z^2, V - \tilde{A}^0(Z^2)]_t\\ &\quad + (1-\varepsilon)\frac{D_t^1}{\bar{Y}_{t-}^\varepsilon}\,Z_{t-}^\varepsilon\,\mathrm{d}[Z^1, \tilde{A}^0(Z^1)]_t + \varepsilon\frac{D_t^2}{\bar{Y}_{t-}^\varepsilon}\,Z_{t-}^\varepsilon\,\mathrm{d}[Z^2, \tilde{A}^0(Z^2)]_t\\ &\quad - \mathrm{d}[Z^\varepsilon, \tilde{A}]_t.\end{aligned} \tag{6.10}$$

Plugging (6.9) and (6.10) into (6.8) and noting by Itô's formula that for $i = 1,2$:

$$\begin{aligned}\mathrm{d}(Z_t^i(V_t - \tilde{A}_t^0(Z^i))) &= (V_{t-} - \tilde{A}_t^0(Z^i))\,\mathrm{d}Z_t^i + Z_{t-}^i\,\mathrm{d}(V_t - \tilde{A}_t^0(Z^i))\\ &\quad + \mathrm{d}[Z^i, V - \tilde{A}^0(Z^i)]_t,\end{aligned}$$

we obtain

$$
\begin{aligned}
&\mathrm{d}(Z_t^\varepsilon(V_t - \tilde{A}_t)) \\
&\quad = (1-\varepsilon)\frac{D_t^1}{\bar{Y}_{t-}^\varepsilon} Z_{t-}^\varepsilon \, \mathrm{d}(Z_t^1(V_t - \tilde{A}_t^0(Z^1))) + \varepsilon \frac{D_t^2}{\bar{Y}_{t-}^\varepsilon} Z_{t-}^\varepsilon \, \mathrm{d}(Z_t^2(V_t - \tilde{A}_t^0(Z^2))) \\
&\qquad - (1-\varepsilon)\frac{D_t^1}{\bar{Y}_{t-}^\varepsilon} Z_{t-}^\varepsilon (V_{t-} - \tilde{A}_t^0(Z^1))\, \mathrm{d}Z_t^1 - \varepsilon \frac{D_t^2}{\bar{Y}_{t-}^\varepsilon} Z_{t-}^\varepsilon (V_{t-} - \tilde{A}_t^0(Z^2))\, \mathrm{d}Z_t^2 \\
&\qquad + (1-\varepsilon)\frac{D_t^1}{\bar{Y}_{t-}^\varepsilon} Z_{t-}^\varepsilon \, \mathrm{d}[Z^1, \tilde{A}^0(Z^1)]_t + \varepsilon \frac{D_{t-}^2}{\bar{Y}_{t-}^\varepsilon} Z_{t-}^\varepsilon \, \mathrm{d}[Z^2, \tilde{A}^0(Z^2)]_t \\
&\qquad + (V_{t-} - \tilde{A}_t)\, \mathrm{d}Z_t^\varepsilon - \mathrm{d}[Z^\varepsilon, \tilde{A}]_t.
\end{aligned} \tag{6.11}
$$

Now by definition of $\mathscr{P}_{\mathrm{loc}}^0$ and the upper variation process, the processes $Z^i(V - \tilde{A}^0(Z^i))$, $i = 1, 2$, are P-local supermartingales. Moreover, since Z^ε and Z^i are P-local martingales and $\tilde{A}^0(Z^i)$, $\tilde{A}$ are predictable processes with finite variation, then $[Z^i, \tilde{A}^0(Z^i)]$, $i = 1, 2$ and $[Z^\varepsilon, A]$ are P-local martingales (see Theorem VII.36 in Dellacherie–Meyer, 1982). Therefore, relation (6.11) implies that $Z^\varepsilon(V - \tilde{A})$ is a P-local supermartingale. The upper variation process of Z^ε satisfies then

$$
\tilde{A}_t^0(Z^\varepsilon) \leqslant \tilde{A}_t, \quad 0 \leqslant t \leqslant T, \quad \text{a.s.} \tag{6.12}
$$

Notice from (6.7) that $\tilde{A}_T$ is bounded a.s. since $\tilde{A}_T^0(Z^i)$, $i = 1, 2$, are bounded. Hence $\tilde{A}_T^0(Z^\varepsilon)$ is bounded and therefore $Z^\varepsilon \in \mathscr{P}_{\mathrm{loc}}^0$.

From (6.12) and (6.7), we now obtain

$$
\begin{aligned}
E\left[\int_0^T \bar{Y}_t^\varepsilon \, \mathrm{d}\tilde{A}_t^0(Z^\varepsilon)\right] &\leqslant E\left[\int_0^T \bar{Y}_t^\varepsilon \, \mathrm{d}\tilde{A}_t\right] \\
&= E\left[\int_0^T (1-\varepsilon)\bar{Y}_t^1 \, \mathrm{d}\tilde{A}_t^0(Z^1) + \varepsilon \bar{Y}_t^2 \, \mathrm{d}\tilde{A}_t^0(Z^2)\right],
\end{aligned}
$$

which ends proof of Lemma 6.3. □

Remark 6.2. The last lemma shows in particular that the set $\mathscr{Y}_{\mathrm{loc},+}^0 = \{Y = ZD/S^0 \colon Z \in \mathscr{P}_{\mathrm{loc},+}^0, D \in \mathscr{D}_+\}$ is convex and the function

$$
\begin{aligned}
&\mathscr{Y}_{\mathrm{loc},+}^0 \longrightarrow \mathbb{R}_+ \\
&Y \mapsto E\left[\int_0^T Y_t \, \mathrm{d}A_t^0(Y)\right]
\end{aligned}
$$

is convex.

Lemma 6.4. *Let Assumption* 5.1(iii) *hold and assume that* $\tilde{J}(y) < \infty$ *for some* $y > 0$. *Then, for all* $x \in (v(0), v(\bar{x}))$, *there exists* $\hat{y} > 0$ *that attains the infimum in* $\inf_{y>0} [\tilde{J}(y) + xy]$.

Proof. Fix $x \in (v(0), v(\bar{x}))$. Under Assumption 5.1(iii), there exists $\tilde{\delta}$ real-valued in $[0, x - v(0))$ such that $\tilde{\delta} S_T^0 \in \operatorname{dom} U$ a.s. By definition of $\tilde{U}$, we have

$$\tilde{U}(yY_T) \geqslant U(\tilde{\delta} S_T^0) - \tilde{\delta} y Z_T D_T, \quad \forall y > 0,\ \forall Y = ZD/S^0 \in \mathscr{Y}_{\mathrm{loc}}^0.$$

Since $E[Z_T D_T] \leqslant E[Z_T] \leqslant 1$, we get

$$E[\tilde{U}(yY_T)] \geqslant U(\tilde{\delta} S_T^0) - \tilde{\delta} y, \quad \forall y > 0,\ \forall Y = ZD/S^0 \in \mathscr{Y}_{\mathrm{loc}}^0.$$

Taking infimum in this last inequality over $Y \in \mathscr{Y}_{\mathrm{loc}}^0$, this implies by definition (5.6) of $\tilde{J}(y)$ and relation (6.2) of $v(H)$ for $H = 0$:

$$\tilde{J}(y) + xy \geqslant U(\tilde{\delta} S_T^0) + y(x - v(0) - \tilde{\delta}), \quad \forall y > 0. \tag{6.13}$$

Since $U(\tilde{\delta} S_T^0) > -\infty$ and $x - v(0) - \tilde{\delta} > 0$, we deduce that $\tilde{J}(y) + xy \to \infty$ as $y \to \infty$. This shows that the proper convex function $y \to \tilde{J}(y) + xy$ attains its infimum in $\hat{y} \in \mathbb{R}_+$.

Let us check that $\hat{y} > 0$. On the contrary, we should have:

$$\tilde{U}(0) \leqslant E\left[\tilde{U}(yY_T) + \int_0^T yY_t \, \mathrm{d}X_t^0 + \int_0^T yY_t \, \mathrm{d}A_t^0(Y)\right] + xy \tag{6.14}$$

for all $y > 0$, $Y \in \mathscr{Y}_{\mathrm{loc}}^0$. By convexity of $\tilde{U}$ and recalling that $\tilde{U}' = -I$, we have

$$\tilde{U}(yY_T) + I(yY_T) yY_T \leqslant \tilde{U}(0). \tag{6.15}$$

Plugging (6.15) into (6.14) and dividing by $y > 0$, we obtain for all $y > 0$, $Y \in \mathscr{Y}_{\mathrm{loc}}^0$:

$$-x \leqslant E\left[-Y_T I(yY_T) + \int_0^T Y_t \, \mathrm{d}X_t^0 + \int_0^T Y_t \, \mathrm{d}A_t^0(Y)\right].$$

By Hypothesis (H′1) and since $I(yY_T) \geqslant 0$ and $I(yY_T) \to \bar{x}$ as $y \to 0$, we get by Fatou's lemma

$$E\left[Y_T \bar{x} - \int_0^T Y_t \, \mathrm{d}X_t^0 - \int_0^T Y_t \, \mathrm{d}A_t^0(Y)\right] \leqslant x, \quad \forall Y \in \mathscr{Y}_{\mathrm{loc}}^0.$$

By Lemma 6.2, this shows that $v(\bar{x}) \leqslant x$, a contradiction.

Lemma 6.5. *Under Assumption* 5.1(i), *there exist* $\alpha_0 \in (0, 1)$, $c > 0$ *and* $\Gamma \in L^1(P)$, *such that*

$$yI(\alpha_0 y) \leqslant c[\tilde{U}(y) 1_{y \leqslant U'(x_0)} + U(x_0) 1_{y > U'(x_0)}] + \Gamma, \quad \textit{a.s.},\ \forall y > 0.$$

Proof. Take $\alpha_0 \in (\gamma, 1)$. Consider first the case $0 < y \leqslant U'(x_0)$. Since I is nonincreasing, we have: $I(\alpha_0 y) \geqslant I(\alpha_0 U'(x_0)) \geqslant I(U'(x_0)) = x_0$. By Assumption 5.1(i), we then obtain

$$\begin{aligned}
\alpha_0 y I(\alpha_0 y) &= I(\alpha_0 y) U'(I(\alpha_0 y)) \\
&\leqslant \gamma U(I(\alpha_0 y)) + \Lambda \\
&\leqslant \gamma[\tilde{U}(y) + yI(\alpha_0 y)] + \Lambda,
\end{aligned}$$

where the last inequality follows from definition (5.2) of $\tilde{U}$. Therefore, $(\alpha_0 - \gamma) yI(\alpha_0 y) \leqslant \gamma \tilde{U}(y) + \Lambda$. By setting $c = \gamma/(\alpha_0 - \gamma)$ and $\Gamma = \Lambda/(\alpha_0 - \gamma)$, we get

$$yI(\alpha_0 y) \leqslant c\tilde{U}(y) + \Gamma, \quad \text{a.s. } \forall 0 < y \leqslant U'(x_0). \tag{6.16}$$

Consider the other case $y > U'(x_0)$. Since I is nonincreasing, we have

$$yI(\alpha_0 y) \leqslant yI(\alpha_0 U'(x_0)) \\ \leqslant c\tilde{U}(U'(x_0)) + \Gamma, \tag{6.17}$$

where the second inequality follows from (6.16). Now, by (5.4) and (5.2), we have

$$\tilde{U}(U'(x_0)) = U(I(U'(x_0))) - U'(x_0)I(U'(x_0)) \\ \leqslant U(I(U'(x_0))) = U(x_0).$$

Substituting into (6.17), we finally get

$$yI(\alpha_0 y) \leqslant cU(x_0) + \Gamma, \quad \forall y > U'(x_0). \qquad \square \tag{6.18}$$

Lemma 6.6. *Let Assumption* 5.1(i) *hold and suppose that there exists a solution* $\hat{Y}(y) \in \mathscr{Y}^0_{\mathrm{loc}}$ *to problem* (5.6) *for some* $y > 0$. *Then* $\tilde{J}$ *is differentiable in* y *and we have*

$$\tilde{J}'(y) = -E\left[\hat{Y}_T(y)I(y\hat{Y}_T(y)) - \int_0^T \hat{Y}_t(y)\,\mathrm{d}X^0_t - \int_0^T \hat{Y}_t(y)\,\mathrm{d}A^0_t(\hat{Y}(y))\right].$$

Moreover, $v(I(y\hat{Y}_T(y))) = -\tilde{J}'(y)$ *and in particular*

$$I(y\hat{Y}_T(y)) \in \mathscr{C}_+(-\tilde{J}'(y)).$$

Proof. Let $\delta > 0$. Then we have by definition (5.6) of $\tilde{J}$,

$$\frac{\tilde{J}(y+\delta) - \tilde{J}(y)}{\delta} \leqslant E\left[\frac{\tilde{U}((y+\delta)\hat{Y}_T(y)) - \tilde{U}(y\hat{Y}_T(y))}{\delta} \right. \\ \left. + \int_0^T \hat{Y}_t(y)\,\mathrm{d}X^0_t + \int_0^T \hat{Y}_t(y)\,\mathrm{d}A^0_t(\hat{Z}(y))\right]. \tag{6.19}$$

By convexity of $\tilde{U}$ and recalling that $\tilde{U}' = -I$, we have

$$\frac{\tilde{U}((y+\delta)\hat{Y}_T(y)) - \tilde{U}(y\hat{Y}_T(y))}{\delta} \leqslant -\hat{Y}_T(y)I((y+\delta)\hat{Y}_T(y)). \tag{6.20}$$

Substituting (6.20) into (6.19) and sending δ to zero, we deduce by Fatou's lemma

$$\limsup_{\delta\downarrow 0} \frac{\tilde{J}(y+\delta) - \tilde{J}(y)}{\delta} \leqslant -E\left[\hat{Y}_T(y)I(y\hat{Y}_T(y)) \right. \\ \left. - \int_0^T \hat{Y}_t(y)\,\mathrm{d}X^0_t - \int_0^T \hat{Y}_t(y)\,\mathrm{d}A^0_t(\hat{Y}(y))\right]. \tag{6.21}$$

Let $\delta < 0$, with $y + \delta > 0$. Then by same arguments as above (using definition of $\tilde{J}$ and convexity of $\tilde{U}$)

$$\frac{\tilde{J}(y+\delta) - \tilde{J}(y)}{\delta} \geqslant -E\left[\hat{Y}_T(y)I((y+\delta)\hat{Y}_T(y)) - \int_0^T \hat{Y}_t(y)\,\mathrm{d}X_t^0 - \int_0^T \hat{Y}_t(y)\,\mathrm{d}A_t^0(\hat{Z}(y))\right]. \tag{6.22}$$

Under Assumption 5.1(i) and by Lemma 6.5, we have for $\delta < 0$ sufficiently small:

$$\begin{aligned} &- \int_0^T \hat{Y}_t(y)\,\mathrm{d}X_t^0 - \int_0^T \hat{Y}_t(y)\,\mathrm{d}A_t^0(\hat{Y}(y)) \\ &\leqslant \hat{Y}_T(y)I((y+\delta)\hat{Y}_T(y)) - \int_0^T \hat{Y}_t(y)\,\mathrm{d}X_t^0 - \int_0^T \hat{Y}_t(y)\,\mathrm{d}A_t^0(\hat{Y}(y)) \\ &\leqslant c[\tilde{U}(y\hat{Y}_T(y))1_{\{y\hat{Y}_T(y)\leqslant U'(x_0)\}} + |U(x_0)|] + \Gamma \\ &\quad - \int_0^T \hat{Y}_t(y)\,\mathrm{d}X_t^0 - \int_0^T \hat{Y}_t(y)\,\mathrm{d}A_t^0(\hat{Z}(y)) \end{aligned} \tag{6.23}$$

for some $c > 0$ and $\Gamma \in L^1(P)$. The left-hand-side of (6.23) is integrable under (H1′). On the other hand, since $\tilde{J}(y) < \infty$, we have

$$E\left[\tilde{U}(y\hat{Y}_T(y)) - \int_0^T \hat{Y}_t(y)\,\mathrm{d}X_t^0 - \int_0^T \hat{Y}_t(y)\,\mathrm{d}A_t^0(\hat{Z}(y))\right] < \infty. \tag{6.24}$$

Moreover, by definition of $\tilde{U}$, we have

$$\begin{aligned} &\tilde{U}(y\hat{Y}_T(y)) - \int_0^T \hat{Y}_t(y)\,\mathrm{d}X_t^0 - \int_0^T \hat{Y}_t(y)\,\mathrm{d}A_t^0(\hat{Z}(y)) \\ &\geqslant U(x_0) - \frac{x_0}{S_T^0} y\hat{Z}_T(y)\hat{D}_T(y) - \int_0^T \hat{Y}_t(y)\,\mathrm{d}X_t^0 - \int_0^T \hat{Y}_t(y)\,\mathrm{d}A_t^0(\hat{Y}(y)). \end{aligned} \tag{6.25}$$

The right-hand side of (6.25) is integrable (with respect to P) by (H′1), Assumption 5.1(i) and since $|\hat{D}(y)| \leqslant 1$. Therefore, the R.H.S. of (6.23) is integrable (with respect to P) and one can apply dominated convergence theorem to (6.22):

$$\liminf_{\delta \nearrow 0} \frac{\tilde{J}(y+\delta) - \tilde{J}(y)}{\delta} \geqslant -E\left[\hat{Y}_T(y)I(y\hat{Y}_T(y)) - \int_0^T \hat{Y}_t(y)\,\mathrm{d}X_t^0 - \int_0^T \hat{Y}_t(y)\,\mathrm{d}A_t^0(\hat{Z}(y))\right].$$

This last inequality combined with (6.21) and convexity of $\tilde{J}$ proves the first assertion of Lemma 6.6.

From (6.2), we already have: $-\tilde{J}'(y) \leqslant v(I(y\hat{Y}_T(y)))$. To state the converse inequality, we proceed as follows. Take an arbitrary element $(Z, D) \in \mathscr{P}^0_{\mathrm{loc},+} \times \mathscr{D}_+$ and let $\varepsilon \in (0, 1)$. By Lemma 6.3, we have

$$(1-\varepsilon)\hat{Z}(y)\hat{D}(y) + \varepsilon ZD = Z^\varepsilon D^\varepsilon$$

with $(Z^\varepsilon, D^\varepsilon) \in \mathscr{P}^0_{\mathrm{loc}} \times \mathscr{D}$. Since $\hat{Y}(y) = \hat{Z}(y)\hat{D}(y)/S^0$ solves $\tilde{J}(y)$, we have

$$E\left[\tilde{U}(y\hat{Y}_T(y)) - \int_0^T \hat{Y}_t(y)\,\mathrm{d}X_t^0 - \int_0^T \hat{Y}_t(y)\,\mathrm{d}A_t^0(\hat{Y}(y))\right]$$

$$\leqslant E\left[\tilde{U}(yY_T^\varepsilon) - \int_0^T Y_t^\varepsilon(y)\,\mathrm{d}X_t^0 - \int_0^T Y_t^\varepsilon(y)\,\mathrm{d}A_t^0(\hat{Y}(y))\right], \tag{6.26}$$

where we set $Y = ZD/S^0$ and $Y^\varepsilon = Z^\varepsilon D^\varepsilon/S^0$. By convexity of $\tilde{U}$ and noting that $Y^\varepsilon - \hat{Y}(y) = \varepsilon(Y - \hat{Y}(y))$, we have

$$\tilde{U}(yY_T^\varepsilon) + \varepsilon y I(yY_T^\varepsilon)(Y - \hat{Y}(y)) \leqslant \tilde{U}(y\hat{Y}_T(y)). \tag{6.27}$$

By Lemma 6.3, we have

$$E\left[\int_0^T Y_t^\varepsilon\,\mathrm{d}A_t^0(Y^\varepsilon)\right] \leqslant (1-\varepsilon)E\left[\int_0^T \hat{Y}_t\,\mathrm{d}A_t^0(\hat{Y}(y))\right] + \varepsilon E\left[\int_0^T Y_t\,\mathrm{d}A_t^0(Y)\right]. \tag{6.28}$$

Plugging (6.27)–(6.28) into (6.26) and dividing by ε, we obtain

$$E\left[Y_T I(yY_T^\varepsilon) - \int_0^T Y_t\,\mathrm{d}X_t^0 - \int_0^T Y_t\,\mathrm{d}A_t^0(Y)\right]$$

$$\leqslant E\left[\hat{Y}_T(y) I(yY_T^\varepsilon) - \int_0^T \hat{Y}_t(y)\,\mathrm{d}X_t^0 - \int_0^T \hat{Y}_t(y)\,\mathrm{d}A_t^0(\hat{Y}(y))\right]. \tag{6.29}$$

Since $Y^\varepsilon \geqslant (1-\varepsilon)\hat{Y}(y)$ and I is nonincreasing, we have by Lemma 6.5:

$$-\int_0^T \hat{Y}_t(y)\,\mathrm{d}X_t^0 - \int_0^T \hat{Y}_t(y)\,\mathrm{d}A_t^0(\hat{Y}(y))$$

$$0 \leqslant \hat{Y}_T(y) I(yY_T^\varepsilon) - \int_0^T \hat{Y}_t(y)\,\mathrm{d}X_t^0 - \int_0^T \hat{Y}_t(y)\,\mathrm{d}A_t^0(\hat{Y}(y))$$

$$\leqslant c[\tilde{U}(y\hat{Y}_T(y))1_{\{y\hat{Y}_T(y) \leqslant U'(x_0)\}} + |U(x_0)|] + \Gamma$$

$$-\int_0^T \hat{Y}_t(y)\,\mathrm{d}X_t^0 - \int_0^T \hat{Y}_t(y)\,\mathrm{d}A_t^0(\hat{Y}(y))$$

for some $c > 0$ and $\Gamma \in L^1(P)$. By same arguments as in (6.23), one can apply the dominated convergence theorem to the R.H.S of (6.29), and Fatou's lemma to the L.H.S of (6.29), and we obtain

$$E\left[Y_T I(y\hat{Y}_T(y)) - \int_0^T Y_t\,\mathrm{d}X_t^0 - \int_0^T Y_t\,\mathrm{d}A_t^0(Y)\right]$$

$$\leqslant E\left[\hat{Y}_T(y) I(y\hat{Y}_T(y)) - \int_0^T \hat{Y}_t(y)\,\mathrm{d}X_t^0 - \int_0^T \hat{Y}_t(y)\,\mathrm{d}A_t^0(\hat{Y}(y))\right] = -\tilde{J}'(y). \tag{6.30}$$

From the arbitrariness of $Y \in \mathscr{Y}^0_{\mathrm{loc},+}$, this proves by (6.4) that $v(I(y\hat{Y}_T(y))) \leqslant -\tilde{J}'(y)$ and finally the required equality. The last assertion of Lemma 6.6 follows from Proposition 4.1. □

Remark 6.3. The last lemma shows in particular that $\mathscr{C}_+(\tilde{J}'(y)) \neq \emptyset$ whenever there exists a solution to problem (5.6) for $y > 0$. Therefore, by Corollary 4.1, $-\tilde{J}'(y) \geqslant v(0)$.

Lemma 6.7. *Let Assumption* 5.1(i) *hold and suppose that there exists a solution* $\hat{Y}(y) = \hat{Z}(y)\hat{D}(y)/S^0 \in \mathscr{Y}^0_{\mathrm{loc}}$ *to problem* 5.6 *with* $\hat{D}_T(y) > 0$ *a.s., for some* $y > 0$. *Then, we have*

$$\tilde{J}'(y) = -E\left[\frac{\hat{Z}_T(y)}{S^0_T} I(y\hat{Y}_T(y)) - \int_0^T \frac{\hat{Z}_t(y)}{S^0_t}\,\mathrm{d}X^0_t - \int_0^T \frac{\hat{Z}_t(y)}{S^0_t}\,\mathrm{d}A^0_t(\hat{Z}(y))\right].$$

Proof. Since $v(I(y\hat{Y}_T(y))) = -\tilde{J}'(y)$ (see Lemma 6.6), we already have by (6.2)

$$-\tilde{J}'(y) \geqslant E\left[\frac{\hat{Z}_T(y)}{S^0_T} I(y\hat{Y}_T(y)) - \int_0^T \frac{\hat{Z}_t(y)}{S^0_t}\,\mathrm{d}X^0_t - \int_0^T \frac{\hat{Z}_t(y)}{S^0_t}\,\mathrm{d}A^0_t(\hat{Z}(y))\right]. \tag{6.31}$$

The converse inequality is proved as follows. Let $0 < \delta < y$, and consider the process D^δ in $\mathscr{D}$ defined by

$$D^\delta_t = \left(\hat{D}_t(y) - \frac{\delta}{y}(1 - \hat{D}_t(y))\right) 1_{\{\hat{D}_t(y) \geqslant \delta/(y+\delta)\}}, \quad 0 \leqslant t \leqslant T,$$

which tends to $\hat{D}_t(y)$ a.s. for all t, as δ goes to zero. We then have by definition of $\tilde{J}$:

$$\begin{aligned}\frac{\tilde{J}(y) - \tilde{J}(y-\delta)}{\delta} \geqslant E\Bigg[&\frac{\tilde{U}(y(\hat{Z}_T(y)\hat{D}_T(y)/S^0_T)) - \tilde{U}((y-\delta)(\hat{Z}_T(y)D^\delta_T/S^0_T))}{\delta} \\ &+ \int_0^T \frac{y\hat{D}_t(y) - (y-\delta)D^\delta_t}{\delta}\,\frac{\hat{Z}_t(y)}{S^0_t}\,\mathrm{d}X^0_t \\ &+ \int_0^T \frac{y\hat{D}_t(y) - (y-\delta)D^\delta_t}{\delta}\,\frac{\hat{Z}_t(y)}{S^0_t}\,\mathrm{d}A^0_t(\hat{Z}(y))\Bigg].\end{aligned} \tag{6.32}$$

By convexity of $\tilde{U}$, we have

$$\begin{aligned}&\frac{\tilde{U}(y\hat{Z}_T(y)\hat{D}_T(y)/S^0_T) - \tilde{U}((y-\delta)\hat{Z}_T(y)D^\delta_T/S^0_T)}{\delta} \\ &\quad \geqslant -\frac{y\hat{D}_T(y) - (y-\delta)D^\delta_T}{\delta}\,\frac{\hat{Z}_T(y)}{S^0_T} I\left((y-\delta)\frac{\hat{Z}_T(y)D^\delta_T}{S^0_T}\right).\end{aligned} \tag{6.33}$$

By definition of D^δ, we have

$$\frac{y\hat{D}_t(y) - (y-\delta)D^\delta_t}{\delta} = 1 + \frac{\delta}{y}(\hat{D}_t(y) - 1) + I_t(\delta), \quad 0 \leqslant t \leqslant T, \tag{6.34}$$

where $I_t(\delta) = (y-\delta)/y[1 - (y+\delta)/\delta\hat{D}_t(y)]1_{\{\hat{D}_t(y) < \delta/(y+\delta)\}}$, $0 \leqslant t \leqslant T$. Notice that $0 \leqslant I_t(\delta) \leqslant (y-\delta)/\delta 1_{\{\hat{D}_t(y) < \delta/(y+\delta)\}}$, $0 \leqslant t \leqslant T$. Therefore, under the assumption that $\hat{D}_T(y) > 0$ a.s

$$I_t(\delta) \to 0 \quad \text{when } \delta \to 0, \quad 0 \leqslant t \leqslant T, \text{ a.s.}$$

We deduce that

$$\frac{y\hat{D}_t(y) - (y-\delta)D_t^\delta}{\delta} \to 1 \quad \text{when } \delta \to 0, \quad 0 \leqslant t \leqslant T, \text{ a.s.} \tag{6.35}$$

Plugging (6.33)–(6.34) into (6.32) and sending δ to zero, we obtain by (6.35) and Fatou's lemma:

$$\tilde{J}'(y) \geqslant -E\left[\frac{\hat{Z}_T(y)}{S_T^0} I(y\hat{Y}_T(y)) - \int_0^T \frac{\hat{Z}_t(y)}{S_t^0}\,\mathrm{d}X_t^0 - \int_0^T \frac{\hat{Z}_t(y)}{S_t^0}\,\mathrm{d}A_t^0(\hat{Z}(y))\right].$$

This last inequality combined with (6.31) ends the proof of Lemma 6.7. □

Proof of Theorem 5.1. (1) Let $v(\bar{x}) \leqslant x < \infty$. Then, we clearly have $\bar{x} < \infty$ a.s. and so $U(\bar{x}) \in L^1(P)$ by Assumption 5.1(ii). Moreover, by Proposition 4.1, $\bar{x} \in \mathscr{C}_+(x)$. Since U is nondecreasing on $\operatorname{dom} U$, we get by (4.1)

$$J(x) \leqslant E[U(\bar{x})].$$

This shows that $\bar{x} \in \mathscr{C}_+(x)$ is solution to (4.1) with $J(x) = E[U(\bar{x})]$.

(2) Let $x \in (v(0), v(\bar{x}))$. The existence of $\hat{y}$ in assertion (a) follows from Lemma 6.4. By Lemma 6.6, $\tilde{J}$ is differentiable in $\hat{y}$ and we have $\tilde{J}'(\hat{y}) = -x$. We also deduce from the second part of Lemma 6.6 that $H^*(x)$ given in (5.7) lies in $\mathscr{C}_+(x)$. Moreover, from definition of $\tilde{U}$ and (5.4), we have for all $H \in \mathscr{C}_+(x)$:

$$\begin{aligned} U(H) &\leqslant \tilde{U}(\hat{y}\hat{Y}_T) + \hat{y}\hat{Y}_T H \\ &= U(H^*(x)) - \hat{y}\hat{Y}_T H^*(x) + \hat{y}\hat{Y}_T H. \end{aligned}$$

Hence, by taking expectation, we obtain

$$\begin{aligned} E[U(H)] &\leqslant E[U(H^*(x))] + \hat{y}(v(H) + \tilde{J}'(\hat{y})) \\ &\leqslant E[U(H^*(x))], \end{aligned}$$

where we used expression of $\tilde{J}'(\hat{y})$ in Lemma 6.6, expression of $v(H)$ in Lemma 6.2, and the fact that $v(H) \leqslant x = -\tilde{J}'(\hat{y})$. This shows that $H^*(x)$ solves (4.1).

From Proposition 4.1, there exists $X^*(x) \in \mathscr{X}_+(x)$ such that

$$H^*(x) \leqslant X_T^*(x), \quad \text{a.s.} \tag{6.36}$$

Since $\hat{Y}X^*(x) - \int \hat{Y}\,\mathrm{d}X^0 - \int \hat{Y}\,\mathrm{d}A^0(\hat{Z})$ is a P-supermartingale by Lemma 6.1, we have

$$E\left[\hat{Y}_T X_T^*(x) - \int_0^T \hat{Y}_t\,\mathrm{d}X_t^0 - \int_0^T \hat{Y}_t\,\mathrm{d}A_t^0(\hat{Z})\right] \leqslant x. \tag{6.37}$$

From the expression of $-\tilde{J}'(\hat{y})$ $(=x)$ in Lemma 6.6 and by (6.36), this shows that we actually have $\hat{Y}_T X_T^*(x) = \hat{Y}_T H^*(x)$ a.s., and equality in (6.37). Therefore $\hat{Y}X^*(x) - \int \hat{Y}\,\mathrm{d}X^0 - \int \hat{Y}\,\mathrm{d}A^0(\hat{Z})$ is a martingale under P, and so relation (5.8) is proved.

If in addition $\hat{D}_T > 0$ a.s. then $\hat{Z}_T X_T^*(x) = \hat{Z}_T H^*(x)$ a.s. From the expression of $-\tilde{J}'(\hat{y})$ $(=x)$ in Lemma 6.7, we deduce that

$$E\left[\frac{\hat{Z}_T}{S_T^0} X_T^*(x) - \int_0^T \frac{\hat{Z}_t}{S_t^0}\,\mathrm{d}X_t^0 - \int_0^T \frac{\hat{Z}_t}{S_t^0}\,\mathrm{d}A_t^0(\hat{Z})\right] = x.$$

This implies that the P-supermartingale $\hat{Z}X^*(x)/S^0 - \int \hat{Z}/S^0\,\mathrm{d}X^0 - \int \hat{Z}/S^0\,\mathrm{d}A^0(\hat{Z})$ is actually a P-martingale, which ends the proof of assertion (2)(b).

(3) By definition of $\tilde{U}$, we have for all $x \geqslant v(0)$, $y > 0$, $H \in \mathscr{C}_+(x)$, $Y \in \mathscr{Y}^0_{\mathrm{loc}}$:

$$U(H) - y\left(Y_T H - \int_0^T Y_t\,\mathrm{d}X^0_t - \int_0^T Y_t\,\mathrm{d}A^0_t(Y)\right)$$

$$\leqslant \tilde{U}(yY_T) + \int_0^T yY_t\,\mathrm{d}X^0_t + \int_0^T yY_t\,\mathrm{d}A^0_t(Y). \tag{6.38}$$

By using the expression of $v(H)$ in (6.2) and since $v(H) \leqslant x$, we obtain by taking expectation in (6.38) and infimum over $Y \in \mathscr{Y}^0_{\mathrm{loc}}$:

$$J(x) \leqslant \tilde{J}(y) + xy, \quad \forall x \geqslant v(0),\ \forall y > 0. \tag{6.39}$$

For $v(\bar{x}) \leqslant x < \infty$, we have seen that $\bar{x}$ is solution to (4.1) and $J(x) = E[U(\bar{x})]$. Since $\tilde{U}$ is nonincreasing with $\tilde{U}(0) = U(\bar{x})$ and recalling that $U(\bar{x}) \in L^1(P)$ by Assumption 5.1(ii), we deduce by the dominated convergence theorem that

$$\tilde{J}(0) = \lim_{y \downarrow 0}\,[\tilde{J}(y) + xy] \leqslant E[U(\bar{x})] = J(x).$$

This last inequality combined with (6.39) proves (5.10). For $v(0) < x < v(\bar{x})$, we get from (5.4)

$$\begin{aligned}\tilde{J}(\hat{y}) &= E[U(H^*(x))] - \hat{y}E\left[\hat{Y}_T H^*(x) - \int_0^T \hat{Y}_t\,\mathrm{d}X^0_t - \int_0^T \hat{Y}_t\,\mathrm{d}A^0_t(\hat{Z})\right]\\ &= J(x) + \hat{y}\tilde{J}'(\hat{y})\\ &= J(x) - x\hat{y},\end{aligned}$$

where we used expression of $\tilde{J}'$ in Lemma 6.6 and the fact that $\tilde{J}'(\hat{y}) = -x$. This last inequality combined with (6.39) proves (5.10).

Fix now $y > 0$ and take $x^* = -\tilde{J}'(y)$ which is larger than $v(0)$ by Remark 6.3. By Lemma 6.6, we have $I(y\hat{Y}_T(y)) \in \mathscr{C}_+(x^*)$. This implies that

$$\begin{aligned}J(x^*) &\geqslant E[U(I(y\hat{Y}_T(y)))]\\ &= E[\tilde{U}(y\hat{Y}_T(y))] + y\hat{Y}_T(y)I(y\hat{Y}_T(y))\\ &= \tilde{J}(y) - y\tilde{J}'(y)\\ &= \tilde{J}(y) + x^* y,\end{aligned}$$

where we used (5.4) and expression of $\tilde{J}'$ in Lemma 6.6. This last inequality combined with (6.39) proves (5.11). Proof of Theorem 5.1 is complete.

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