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An Integer Analogue of Carathéodory's Theorem

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We prove a theorem on Hilbert bases analogous to Carathéodory's theorem for convex cones. The result is used to give an upper bound on the number of nonzero variables needed in optimal solutions to integer programs associated with totally dual integral systems. For integer programs arising from perfect graphs the general bounds are improved to show that if G is a perfect graph with n nodes and w is a vector of integral node weights, then there exists a minimum w-covering of the nodes that uses at most n distinct cliques.

1. INTRODUCTION

A *rational cone* is a set of the form $\{x: Ax \geqq 0\}$, where A is a rational $m \times n$ matrix and 0 is the m component zero vector. By the theorems of Weyl and Minkowski (see [19]), C is a rational cone if and only if there exists a finite set of rational vectors $\{a_1, ..., a_k\}$ that *generate* C, that is, $C = \{\lambda_1 a_1 + \cdots + \lambda_k a_k: \lambda_i \geqq 0,\ i = 1, ..., k\}$. The *dimension* of a rational cone C, denoted by dim C, is the cardinality of a maximal set of linearly independent vectors in C. A well-known result of Carathéodory is the following.

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CARATHÉODORY'S THEOREM. *Let C be a rational cone and suppose that $\{a_1, ..., a_k\}$ generate C. If $x \in C$, then x can be expressed as a nonnegative linear combination of* dim C *vectors in* $\{a_1, ..., a_k\}$.

We study an integer analogue of this result. A set of integer vectors $\{a_1, ..., a_k\}$ is called a *Hilbert basis* if every integer vector in the convex cone they generate can be expressed as a nonnegative integer combination of $a_1, ..., a_k$. Hilbert [10] proved that every rational cone is generated by a finite Hilbert basis. In fact, Jeroslow [13] and Schrijver [15] have shown that every pointed rational cone is generated by a unique minimal Hilbert basis (a cone C is *pointed* if $x \in C$ and $x \neq 0$ imply $x \notin C$).

Suppose that C is a rational cone of dimension n and that $\{a_1, ..., a_k\}$ is a Hilbert basis for C. A question analogous to the one answered by Carathéodory's theorem is: What is the smallest h such that every integral $x \in C$ can be expressed as a nonnegative integer combination of h vectors in $\{a_1, ..., a_k\}$? We show in the next section that in general there is no upper bound for h in terms of n, but if C is pointed then h is less than or equal to $2n-1$.

Hilbert bases are closely related to total dual integrality. A rational linear system $Ax \leqq b$ is called *totally dual integral* if the minimum in the linear programming duality equation

$$\max\{wx\colon Ax \leqq b\} = \min\{yb\colon yA = w,\ y \geqq 0\} \tag{1}$$

can be achieved by an integer vector for each integral w for which the optima exist. It follows from complementary slackness that $Ax \leqq b$, with A integral, is totally dual integral if and only if for each minimal nonempty face of $\{x\colon Ax \leqq b\}$ the set of active rows of A is a Hilbert basis (an *active* row is one in which the corresponding constraint holds as an equality for every point in the face)—see Giles and Pulleyblank [8]. Hoffman [11, 12] and Edmonds and Giles [6] have connected total dual integrality to combinatorial min–max theorems by showing that if b is integral and $Ax \leqslant b$ is totally dual integral then the maximum in (1) can be achieved by an integer vector for each w for which the optima exist.

The problem on Hilbert bases mentioned above is equivalent to: Given a totally dual integral system $Ax \leqq b$, where A is an integral matrix of rank k, what is the smallest h such that for any integral w for which the minimum in (1) exists the minimum can be achieved by an integral solution with at most h nonzero variables? This problem is a generalization of a problem on matroids posed by Cunningham [3]. General results on this problem given in the next section improve some known results when they are specialized to particular combinatorial problems. For systems arising from perfect graphs the general bounds are improved to show that if G is a perfect graph with n nodes and w is a vector of integral node weights then there exists a minimum w-covering of the nodes that uses at most n distinct cliques.

2. Upper Bounds

(a) *Hilbert Bases*

THEOREM 1. *Let C be a pointed rational cone of dimension n and let $\{a_1,..., a_k\}$ be a Hilbert basis for C. If w is an integer vector in C then w can be expressed as nonnegative integer combination of $2n-1$ vectors in $\{a_1,..., a_k\}$.*

Proof. Suppose $w \in C$ is integral. Let A be the matrix with rows $a_1,..., a_k$ and let $1 = (1,..., 1)^{\mathrm{T}}$. Since C is pointed, the linear program $\max\{y1\colon yA = w, y \geqq 0\}$ has a basic optimal solution $\bar{y}$ (it has a solution since $w \in C$ and it is easy to see that it is bounded, by considering a vector d such that $dz > 0$ for each nonzero vector $z \in C$). Let $\lfloor \bar{y} \rfloor$ denote the vector $(\lfloor \bar{y}_1 \rfloor,..., \lfloor \bar{y}_k \rfloor)$, where $\lfloor \bar{y}_i \rfloor$ is the largest integer less than or equal to $\bar{y}_i$. The vector $w - \lfloor \bar{y} \rfloor A$ is an integer vector in C and so can be expressed as $\lambda_1 a_1 + \cdots + \lambda_k a_k$, where λ_i is a nonnegative integer for $i = 1,..., k$. Since $\bar{y}$ is a basic solution, it has at most n nonzero components. So $(\bar{y} - \lfloor \bar{y} \rfloor)1 < n$. Now since $\bar{y}$ is an optimal solution, $\sum\{\lambda_i\colon i = 1,..., k\} \leqq (\bar{y} - \lfloor \bar{y} \rfloor)1$. So $(\lambda_1 + \lfloor \bar{y}_1 \rfloor)\, a_1 + \cdots + (\lambda_k + \lfloor \bar{y}_k \rfloor)\, a_k$ expresses w as a nonnegative integer combination of $a_1,..., a_k$ with at most $2n-1$ nonzero multipliers. ∎

The bound of $2n-1$ does not hold for general rational cones. Indeed, the following example shows that in general there is no upper bound in terms of n. Let $p_1,..., p_k$ be distinct primes, $k \geqq 2$. For $i = 1,..., k$ let q_i be the product of p_1 through p_k, excluding p_i. The greatest common divisor of $q_1,..., q_k$ is 1, so there exists integer λ_i, $i = 1,..., k$ such that $\lambda_1 q_1 + \cdots + \lambda_k q_k = 1$. For $i = 1,..., k$ let $\delta_i = 1$ if $\lambda_i \geqq 0$ and $\delta_i = -1$ if $\lambda_i < 0$. Now $\{\delta_1 q_1,..., \delta_k q_k\}$ is a Hilbert basis for a cone of dimension 1, the real line. But, since the greatest common divisor of any proper subset of $\{q_1,..., q_k\}$ is greater than 1, any expression of 1 as a nonnegative integer combination of $\{\delta_1 q_1,..., \delta_k q_k\}$ requires all k vectors.

Despite this negative result, it is true that for any linear space L of dimension n there exists a Hilbert basis for this space with at most $2n$ vectors. (This follows from the fact that the set of integer points in L is a lattice.)

We have not found any examples to show that the bound in Theorem 1 cannot be lowered to n (it cannot be less than n). For pointed cones of dimension 2 it is true that the bound can be lowered to 2; this can be proven by showing that if C is a pointed cone of dimension 2 then the unique minimal Hilbert basis for C has the property that every pair of "adjacent" vectors in the basis form a Hilbert basis for the cone they generate.

(b) *Total Dual Integrality*

A system $Ax \leqq 0$, where A is an integral matrix, is totally dual integral if and only if the rows of A form a Hilbert basis (since $\{x: Ax \leqq 0\}$ has a unique minimal face). So, in general, there is no upper bound in terms of $\operatorname{rank}(A)$ on the number of non-zero variables needed in an integral solution that achieves the minimum in (1), where $Ax \leqq b$ is a totally dual integral system with A integral and w is an integer vector such that the minimum exists. However, if $\{x: Ax \leqq b\}$ is of full dimension then the active rows of any minimal face of the polyhedron generate a pointed cone. So Theorem 1 implies the following result.

THEOREM 2. *Let $Ax \leqq b$ be a totally dual integral system, with A an integral matrix of rank n, such that the polyhedron $\{x: Ax \leqq b\}$ is of full dimension. If w is an integer vector such that the minimum in* (1) *exists then the minimum can be achieved by an integral optimal solution with at most $2n-1$ nonzero variables.*

A generalisation of total dual integrality was introduced by Baum and Trotter [1], who defined a linear system $Ax \leqq b$ to have the *integer rounding* property if

$$\min\{yb: yA = w,\ y \geqq 0,\ y \text{ integral}\} = \lceil \min\{yb: yA = w,\ y \geqq 0\} \rceil \quad (2)$$

for each integral w for which the right-hand side exists. (If r is a rational number, then $\lceil r \rceil$ is the least integer greater than or equal to r.) It can be checked that $Ax \leqq b$ has the integer rounding property if and only if $[Ax - bx_0 \leqq 0,\ x_0 \leqq 0]$ is a totally dual integral system (Giles and Orlin [7]). Thus, Theorem 2 can be used to derive a result on integral systems with the integer rounding property. One can also obtain such a result by directly modifying the proof of Theorem 1, the following special case of which can be applied to a problem of Cunningham [3].

THEOREM 3. *Let A be a 0–1 matrix of rank n, such that $Ax \leqq 1$ has the integer rounding property. If w is an integer vector such that $\min\{y1: yA = w,\ y \geqq 0\}$ exists, then there exists an optimal solution to $\min\{y1: yA = w,\ y \geqq 0,\ y \text{ integral}\}$ with at most $2n-1$ nonzero variables.*

Proof. We may assume n is at least 2. Suppose w is an integer vector and $\bar{y}$ is a basic optimal solution to $\min\{y1: yA = w,\ y \geqq 0\}$. We have that $\bar{y} - \lfloor \bar{y} \rfloor$ is an optimal solution to

$$\min\{y1: yA = w - \lfloor \bar{y} \rfloor A,\ y \geqq 0\}. \quad (3)$$

So there exists an integral solution, y', to (3) with objective value $\lceil(\bar{y}-\lfloor\bar{y}\rfloor)1\rceil$. We claim that $(\bar{y}-\lfloor\bar{y}\rfloor)1<n-1$. Once this is shown the result follows, since $y^*=\lfloor\bar{y}\rfloor+y'$ is an optimal solution to $\min\{y1\colon yA=w, y\geqq 0, y \text{ integral}\}$ (because $y^*1=\lceil\bar{y}1\rceil$). Suppose that the claim is not true. Let $\bar{A}$ be the submatrix of A consisting of the rows corresponding to the nonzero variables, $\bar{y}_{i_1},\dots,\bar{y}_{i_n}$, in the solution $\bar{y}$. The only possible 0–1 vectors whose inner product with $(\bar{y}_{i_1}-\lfloor\bar{y}_{i_1}\rfloor,\dots,\bar{y}_{i_n}-\lfloor\bar{y}_{i_n}\rfloor)$ is an integer are the 0 vector and the vector of all 1's. But $(\bar{y}_{i_1}-\lfloor\bar{y}_{i_1}\rfloor,\dots,\bar{y}_{i_n}-\lfloor\bar{y}_{i_n}\rfloor)\bar{A}=w-(\lfloor\bar{y}_{i_1}\rfloor,\dots,\lfloor\bar{y}_{i_n}\rfloor)\bar{A}$, which is integral. So all columns of $\bar{A}$ are either all 0's or all 1's, which implies that the rank of $\bar{A}$ is 1. But this implies that n is 1, a contradiction. ▮

Cunningham [3] posed the following problem. Let M be a matroid and w a nonnegative integer vector. What is the least integer k such that of all minimum cardinality families of independent sets of M having w as the sum of the incidence vectors of the sets in the family, there exists a family with at most k distinct members? That is, if A is a matrix whose rows are the incidence vectors of independent sets of M, what is the minimum number of non-zero variables needed in an optimal solution to $\min\{y1\colon yA=w, y\geqq 0, y \text{ integral}\}$? Cunningham [3] showed that his algorithm for testing membership in matroid polyhedra gives an upper bound of n^4+1 for k, where n is the number of elements in M. He also gives an improvement of this result due to Schrijver which gives $2n$ as an upper bound. The matroid partition theorem of Edmonds [5] shows that $Ax\leqq 1$ has the integer rounding property, so Theorem 3 lowers the upper bound to $2n-1$.

(c) *Perfect Graphs*

A graph G is called *perfect* if for every induced subgraph H of G, the minimum number of cliques needed to cover the nodes of H is equal to the cardinality of a largest independent set in H.

Let G be a graph and let A be the clique-node incidence matrix of G. A well known result of Lovász [14] is that G is perfect if and only if for every nonnegative integral w both sides of the equation

$$\max\{wx\colon Ax\leqq 1, x\geqq 0\}=\min\{y1\colon yA\geqq w, y\geqq 0\} \tag{4}$$

can be achieved by integral solutions (that is, if and only if $[Ax\leqq 1, x\geqq 0]$ is totally dual integral). Another way of stating this is that G is perfect if and only if for each nonnegative integral w, the maximum weight of an independent set of G is equal to the size of a minimum w-covering of G (a *minimum* w*-covering* is a minimum cardinality family of cliques such that each node i of G is in at least w_i cliques in the family).

Let G be a perfect graph. Grötschel, Lovász, and Schrijver [9] gave a polynomial algorithm to find a minimum w-covering of G for any nonnegative integral w. A by-product of this algorithm is that for any nonnegative integral w, there always exists a minimum w-covering of G with at most n^2+n distinct cliques, where n is the number of nodes of G. Now since the polyhedron $\{x\colon Ax \leqq 1, x \geqq 0\}$ is of full dimension (A is the clique–node incidence matrix of G), Theorem 2 improves this bound to $2n-1$. It is obvious that the upper bound can be made no lower than n. The following result shows that, in fact, n is an upper bound.

THEOREM 4. *Let G be a perfect graph with n nodes. For each nonnegative integral w, there exists a minimum w-covering of G with at most n distinct members.*

Proof. Suppose that w is a nonnegative integer vector. It must be shown that the minimum in (4) can be achieved by an integral solution with at most n nonzero variables.

Let I_1 be the set of rows of A that are active in every solution that achieves the maximum in (4). The proof is by induction on the rank of I_1 (the number of linearly independent rows in I_1). If $\operatorname{rank}(I_1)=0$, then $y=0$ achieves the minimum in (4). Let k be greater than zero and assume that if $\operatorname{rank}(I_1)<k$ then the minimum in (4) can be achieved by an integral solution with at most $\operatorname{rank}(I_1)$ nonzero variables. Now suppose that $\operatorname{rank}(I_1)=k$. Let a_j be a row in I_1 and let y_j be the corresponding variable in the right-hand side of (4). Let t be the greatest value of y_j in an integral solution that achieves the minimum in (4) (t is at least 1 since $j \in I_1$). Consider the linear program

$$\max\{(w-ta_j)\,x\colon Ax \leqq 1, x \geqq 0\}. \tag{5}$$

Let I_2 be the set of rows of A that are active in every solution that achieves the maximum in (5). By the choice of t, $j \notin I_2$. Also, if x^* achieves the maximum in (4) then x^* achieves the maximum in (5) as well (since the dual linear program of (5) has a solution with objective value wx^*-t). So $I_2 \subseteq I_1$ and $\operatorname{rank}(I_2) \leqq \operatorname{rank}(I_1)$. Suppose that $\operatorname{rank}(I_2)=\operatorname{rank}(I_1)$. Since $j \in I_1$, $a_j=\sum\{\lambda_i a_i\colon i \in I_2\}$ for some scalars λ_i, $i \in I_2$. Let x^* achieve the maximum in (4). Since $j \in I_1$, $(a_j, x^*)=1$ ((a_j, x^*) denotes the inner product of a_j and x^*). If $\bar{x}$ achieves the maximum in (5) then $(a_j, \bar{x})=\sum\{\lambda_i(a_i, \bar{x})\colon i \in I_2\}=\sum\{\lambda_i\colon i \in I_2\}$. But $1=(a_j, x^*)=\sum\{\lambda_i(a_i, x^*)\colon i \in I_2\}=\sum\{\lambda_i\colon i \in I_2\}$. So $j \in I_2$, a contradiction. So $\operatorname{rank}(I_2)$ is less than $\operatorname{rank}(I_1)$. By assumption, there exists an integral optimal solution y^* to the dual linear program of (5) with at most $\operatorname{rank}(I_2)$ nonzero variables. Let $\bar{y}_j=y_j^*+t$ and $\bar{y}_i=y_i^*$ for all other i. The solution $\bar{y}$ achieves the minimum in (4) and has at most $\operatorname{rank}(I_1)$ nonzero variables. Since $\operatorname{rank}(I_1) \leqq n$, the theorem is proven. ▌

There are other totally dual integral systems $Ax \leqq b$, $x \geqq 0$ such that $\min\{yb\colon yA \geqq w,\ y \geqq 0\}$ can be achieved by a solution with at most n nonzero variables, where n is the number of variables in the totally dual integral system and w is an integer vector such that the minimum exists. Cunningham pointed out that the algorithm for optimum matchings given in Cunningham and Marsh [4] shows that the totally dual integral matching system has this property (this result also follows from the proof of total dual integrality of matching systems given by Schrijver and Seymour [17]). The systems associated with cross-free families (Schrijver [16]) also have this property since the nonzero dual variables can always be chosen to induce a totally unimodular matrix—these systems include matroid intersection systems and many others.

Remark. Since the writing of this paper, several new results on this topic have come to the authors' attention. Two of these new results are due to Sebö [18], who showed that, by sharpening the technique used in the proof of Theorem 1, the bound given in that result can be lowered to $2n-2$ (when n is at least 2) and that for $n=3$ the bound can be lowered to 3. Also, Chandrasekaran and Tamir [2] independently found an alternative proof of Theorem 4, replacing the linear independence argument by one on lexicographic orderings.

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