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Decomposition of some pointed Hopf algebras given by the canonical Nakayama automorphism

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Abstract

Every finite dimensional Hopf algebra is a Frobenius algebra, with Frobenius homomorphism given by an integral. The Nakayama automorphism determined by it yields a decomposition with degrees in a cyclic group. For a family of pointed Hopf algebras, we determine necessary and sufficient conditions for this decomposition to be strongly graded.

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1. Introduction

Let $\mathbf{k}$ be a field, A a finite dimensional $\mathbf{k}$-algebra and DA the dual space $\mathrm{Hom}_{\mathbf{k}}(A, \mathbf{k})$, endowed with the usual A-bimodule structure. Recall that A is said to be a Frobenius algebra if there exists a linear form $\varphi\colon A \to \mathbf{k}$, such that the map $A \to DA$, defined by $x \mapsto x\varphi$, is a left A-module isomorphism. This linear form $\varphi\colon A \to \mathbf{k}$ is called a Frobenius homomorphism. It is well known that this is equivalent to saying that the map $x \mapsto \varphi x$, from A to DA, is an isomorphism of right A-modules. From this it follows easily that there exists an automorphism ρ of A, called the Nakayama automorphism of A with respect to φ, such that $x\varphi = \varphi\rho(x)$, for all $x \in A$. It is easy to check that a linear form $\widetilde{\varphi}\colon A \to \mathbf{k}$ is another Frobenius homomorphism if and only if there exists an invertible element x in A, such that $\widetilde{\varphi} = x\varphi$. It is also easy to check that the Nakayama automorphism of A with respect to $\widetilde{\varphi}$ is the map given by $a \mapsto \rho(x)^{-1}\rho(a)\rho(x)$.

Let A be a Frobenius $\mathbf{k}$-algebra, $\varphi\colon A \to \mathbf{k}$ a Frobenius homomorphism and $\rho : A \to A$ the Nakayama automorphism of A with respect to φ.

Definition 1.1. We say that ρ has order $m \in \mathbb{N}$ and we write $\mathrm{ord}_\rho = m$, if $\rho^m = \mathrm{id}_A$ and $\rho^r \neq \mathrm{id}_A$, for all $r < m$.

Let G be a group. Recall that a G-graded algebra is a $\mathbf{k}$-algebra A together with a decomposition $A = \bigoplus_{g\in G} A_g$ of A as a direct sum of subvector spaces, such that $A_g A_{g'} \subseteq A_{gg'}$ for all $g, g' \in G$. When $A_g A_{g'} = A_{gg'}$ for all

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$g, g' \in G$, the grading is called *strong*, and the algebra is said to be *strongly graded*. Assume that ρ has finite order and that $\mathbf{k}$ has a primitive ord_ρth root of unity ω. For $n \in \mathbb{N}$, let C_n be the group of nth roots of unity in $\mathbf{k}$. Since the polynomial $X^{\mathrm{ord}_\rho} - 1$ has distinct roots ω^i $(0 \le i < \mathrm{ord}_\rho)$, the algebra A becomes a C_{ord_ρ}-graded algebra

$$A = A_{\omega^0} \oplus \cdots \oplus A_{\omega^{\mathrm{ord}_\rho - 1}}, \quad \text{where } A_z = \{a \in A : \rho(a) = za\}. \tag{1.1}$$

As is well known, every finite dimensional Hopf algebra H is Frobenius, any nonzero right integral $\varphi \in H^*$ being a Frobenius homomorphism. Let t be a nonzero right integral of H. Let $\alpha \in H^*$ be the modular element of H, defined by $at = \alpha(a)t$ (notice that this is the inverse of the modular element α considered in [6]). For $x \in H^*$, let $r_x : H \to H$ be defined by $r_x(b) = bx = x(b_{(1)})b_{(2)}$. Then, as follows from [6, Theorem 3(a)], we have

$$\rho(b) = \alpha^{-1}(b_{(1)})\mathcal{S}^2(b_{(2)}), \quad \text{i.e. } \rho = \mathcal{S}^2 \circ r_{\alpha^{-1}}.$$

Since α is a group-like element, $\mathcal{S}^2 \circ r_{\alpha^{-1}} = r_{\alpha^{-1}} \circ \mathcal{S}^2$ and therefore

$$\rho^l = \mathcal{S}^{2l} \circ r_{\alpha^{-l}}, \quad \text{i.e. } \rho^l(h) = \alpha^{-l}(h_{(1)})\mathcal{S}^{2l}(h_{(2)}) = \alpha^l(\mathcal{S}(h_{(1)}))\mathcal{S}^{2l}(h_{(2)}), \tag{1.2}$$

for all $l \in \mathbb{Z}$. Now, α has finite order and, by the Radford formula for $\mathcal{S}^4$ (see [7] or [9, Theorem 3.8]), the antipode $\mathcal{S}$ has finite order with respect to composition. Thus, the automorphism ρ has finite order, which implies that finite dimensional Hopf algebras are examples of the situation considered above.

Notice that by (1.2), if $\rho^l = \mathrm{id}$, then $\alpha^l = \varepsilon$ and then $\mathcal{S}^{2l} = \mathrm{id}$. The converse is obvious. So, the order of ρ is the lcm of those of α and $\mathcal{S}^2$. In particular, the number of terms in the decomposition associated with $\mathcal{S}^2$ divides that in the one associated with ρ. Also, from (1.2) we get that $\rho = \mathcal{S}^2$ if and only if H is unimodular.

The main aim of the present work is to determine conditions for decomposition (1.1) to be strongly graded. Besides the fact that the theory for algebras which are strongly graded over a group is well developed (see for instance [4]), our interest in this problem originally came from the homological results in [3].

The decomposition using $\mathcal{S}^2$ instead of ρ was considered in [8]. We show below that if $\mathcal{S}^2 \neq \mathrm{id}$, then this decomposition is not strongly graded. On the other hand, as shown in [8], under suitable assumptions its homogeneous components are equidimensional. It is an interesting problem to know whether a similar thing happens with the decomposition associated with ρ. For instance, all the liftings of quantum linear spaces have equidimensional decompositions, as shown in Remark 4.4.

2. The unimodular case

Let H be a finite dimensional Hopf algebra with antipode $\mathcal{S}$. In this brief section we first show that the decomposition of H associated with $\mathcal{S}^2$ is not strongly graded, unless $\mathcal{S}^2 = \mathrm{id}$ (this applies in particular to decomposition (1.1) when H is unimodular and $\mathrm{ord}_\rho > 1$). We finish by giving a characterization of unimodular Hopf algebras in terms of decomposition (1.1).

Lemma 2.1. *Let H be a finite dimensional Hopf algebra. Suppose $H = \bigoplus_{g \in G} H_g$ is a graduation over a group. Assume there exists $g \in G$ such that $\varepsilon(H_g) = 0$. Then the decomposition is not strongly graded.*

Proof. Suppose the decomposition is strongly graded. Then there are elements $a_i \in H_g$ and $b_i \in H_{g^{-1}}$ such that $1 = \sum_i a_i b_i$. Then $1 = \varepsilon(1) = \sum_i \varepsilon(a_i)\varepsilon(b_i)$, a contradiction. □

Corollary 2.2. *Assume that $\mathcal{S}^2 \neq \mathrm{id}$ and that*

$$H = \bigoplus_{z \in \mathbf{k}^*} H_z, \quad \textit{where } H_z = \{h \in H : \mathcal{S}^2(h) = zh\}.$$

Then this decomposition is not strongly graded.

Proof. Since $\varepsilon \circ \mathcal{S}^2 = \varepsilon$, then $\varepsilon(H_z) = 0$ for all $z \neq 1$. □

Let now $\varphi \in \int_{H^*}^r$ and $\Gamma \in \int_H^l$, such that $\langle \varphi, \Gamma \rangle = 1$, and let $\alpha \colon H \to \mathbf{k}$ be the modular map associated with $t = S(\Gamma)$. Let ρ be the Nakayama automorphism associated with φ. Assume that $\mathbf{k}$ has a root of unity ω of order ord_ρ. We consider the decomposition associated with ρ, as in (1.1)

$$H = H_{\omega^0} \oplus \cdots \oplus H_{\omega^{\mathrm{ord}_\rho - 1}}. \tag{2.1}$$

Corollary 2.3. *If H is unimodular and $\mathcal{S}^2 \neq \mathrm{id}$, then the decomposition (2.1) is not strongly graded.* □

Proposition 2.4. *If $h \in H_{\omega^i}$, then $\alpha(h) = \omega^{-i}\epsilon(h)$.*

Proof. By [6, Proposition 1(e)], or the proof of [9, Proposition 3.6], $\langle \varphi, t\rangle = 1$. Then

$$\epsilon(h) = \epsilon(h)\langle \varphi, t\rangle = \langle \varphi, th\rangle = \langle \varphi, \rho(h)t\rangle = \langle \varphi, \omega^i ht\rangle = \omega^i \alpha(h).$$

So, $\alpha(h) = \omega^{-i}\epsilon(h)$, as we want. □

Corollary 2.5. *H is unimodular if and only if $H_{\omega^i} \subseteq \ker(\epsilon)$, for all $i > 0$*

Proof. ($\Rightarrow$): For $h \in H_{\omega^i}$, we have $\epsilon(h) = \alpha(h) = \omega^{-i}\epsilon(h)$ and so $\epsilon(h) = 0$, since $\omega^i \neq 1$.
($\Leftarrow$): For $h \in H_{\omega^i}$ with $i > 0$, we have $\alpha(h) = \omega^{-i}\epsilon(h) = 0 = \epsilon(h)$ and, for $h \in H_{\omega^0}$, we also have $\alpha(h) = \omega^0\epsilon(h) = \epsilon(h)$. □

3. Bosonizations of Nichols algebras of diagonal type

Let G be a finite abelian group, $\mathbf{g} = g_1, \dots, g_n \in G$ a sequence of elements in G and $\boldsymbol{\chi} = \chi_1, \dots, \chi_n \in \hat{G}$ a sequence of characters of G. Let V be the vector space with basis $\{x_1, \dots, x_n\}$ and let c be the braiding given by $c(x_i \otimes x_j) = q_{ij} x_j \otimes x_i$, where $q_{ij} = \chi_j(g_i)$. We consider the Nichols algebra $R = \mathfrak{B}(V)$ generated by (V, c). We give here one of its possible equivalent definitions. Let $T_c(V)$ be the tensor algebra generated by V, endowed with the unique braided Hopf algebra structure such that the elements x_i are primitive and whose braiding extends c. Then, R is obtained as a colimit of algebras $R = \varinjlim R_i$, where $R_0 = T_c(V)$ and R_{i+1} is the quotient of R_i by the ideal generated by its homogeneous primitive elements with degree ≥ 2. See [1,2] for alternative definitions and main properties of Nichols algebras. Assume that R is finite dimensional and let $t_0 \in R$ be a nonzero homogeneous element of greatest degree. Let $H = H(\mathbf{g}, \boldsymbol{\chi}) = R\#\mathbf{k}G$ be the bosonization of R (this is an alternative presentation for the algebras considered by Nichols in [5]). We have:

$$\begin{aligned} &\Delta(g_i) = g_i \otimes g_i, \qquad \Delta(x_i) = g_i \otimes x_i + x_i \otimes 1,\\ &\mathcal{S}(g) = g^{-1}, \qquad \mathcal{S}(x_i) = -g_i^{-1}x_i,\\ &\mathcal{S}^2(g) = g, \qquad \mathcal{S}^2(x_i) = g_i^{-1}x_i g_i = q_{ii}^{-1}x_i. \end{aligned}$$

The element $t_0 \sum_{g\in G} g$ is a nonzero right integral in H. Let α be the modular element associated with it. Since, for all i, the degree of $x_i t_0 \sum_{g\in G} g$ is bigger than the degree of t_0, we have that $x_i t_0 \sum_{g\in G} g = 0$, and so $\alpha(x_i) = 0$. Moreover, $\alpha_{|G}$ is determined by $gt_0 = \alpha(g)t_0 g$. Thus,

$$\rho(g) = \alpha(g^{-1})g \quad \text{and} \quad \rho(x_i) = \alpha(g_i^{-1})q_{ii}^{-1}x_i.$$

This implies that the nonzero monomials $x_{i_1} \cdots x_{i_\ell} g$ are a set of eigenvectors for ρ (which generate H as a $\mathbf{k}$-vector space). In particular, ρ is diagonalizable, whence $\mathbf{k}$ has a primitive ord_ρth root of unity. Consider the subgroups

$$L_1 = \langle q_{11}, \dots, q_{nn}, \alpha(G)\rangle \quad \text{and} \quad L_2 = \alpha(G),$$

of $\mathbf{k}^*$. Since $\rho(x_i g_i^{-1}) = q_{ii}^{-1} x_i g_i^{-1}$ and $\rho(g) = \alpha(g)$, the group L_1 is the set of eigenvalues of ρ. As in the introduction, we decompose

$$H = \bigoplus_{\omega\in L_1} H_\omega, \quad \text{where } H_\omega = \{h \in H : \rho(h) = \omega h\}.$$

Proposition 3.1. *The following are equivalent:*

(1) *$\bigoplus_{\omega\in L_1} H_\omega$ is strongly graded.*
(2) *$L_1 = L_2$.*
(3) *Each H_ω contains an element in G.*
(4) *H is a crossed product $H_1 \ltimes \mathbf{k}L_1$.*

Proof. It is clear that (2) $\Rightarrow$ (3) and (4) $\Rightarrow$ (1).

The proof of (3) $\Rightarrow$ (4) is standard. We sketch it for the readers' convenience. For each $\omega \in L_1$, pick $g_\omega \in H_\omega \cap G$. Define $\rho : L_1 \times H_1 \to H_1$ and $f : L_1 \times L_1 \to H_1$ by

$$\omega \cdot a = \rho(\omega, a) = g_\omega a g_\omega^{-1} \quad \text{and} \quad f(\omega, \omega') = g_\omega g_{\omega'} g_{\omega\omega'}^{-1}.$$

Let $H_1 \ltimes_\rho^f \mathbf{k}L_1$ be the algebra with underlying vector space $H_1 \otimes \mathbf{k}L_1$ and with multiplication $(a \otimes \omega)(b \otimes \omega') = a(\omega \cdot b) f(\omega, \omega') \otimes \omega\omega'$. It is easy to see that the map $\Psi : H_1 \ltimes_\rho^f \mathbf{k}L_1 \to H$ given by $\Psi(a \otimes \omega) = a g_\omega$ is an isomorphism of algebras. In particular, $H_1 \ltimes_\rho^f \mathbf{k}L_1$ is associative with unit $1 \otimes 1$, and so ρ is a weak action and f is a normal cocycle that satisfies the twisted module condition.

We now prove that (1) $\Rightarrow$ (2). Notice that H is also $\mathbb{N}_0$-graded by $\deg(g) = 0$ for all $g \in G$, and $\deg(x_i) = 1$. Call this decomposition $H = \bigoplus_{i \in \mathbb{N}} H^i$. Since each H_ω is spanned by elements which are homogeneous with respect to the previous decomposition, we have:

$$H = \bigoplus_{i \geq 0, \omega \in L_1} H_\omega^i, \quad \text{where } H_\omega^i = H_\omega \cap H^i.$$

So, if $\bigoplus_{\omega \in L_1} H_\omega$ is strongly graded, then each H_ω must contain nonzero elements in H^0. Since $H^0 \subseteq \bigoplus_{\omega \in L_2} H_\omega$, we must have $L_1 = L_2$. $\square$

Quantum linear spaces

If the sequence of characters $\boldsymbol{\chi}$ satisfies

- $\chi_i(g_i) \neq 1$,
- $\chi_i(g_j)\chi_j(g_i) = 1$ for $i \neq j$,

then $H(\mathbf{g}, \boldsymbol{\chi})$ is the quantum linear space with generators G and $x_1, \ldots, x_n$, subject to the following relations:

- $g x_i = \chi_i(g) x_i g$,
- $x_i x_j = q_{ij} x_j x_i$,
- $x_i^{m_i} = 0$,

where $m_i = \operatorname{ord}(q_{ii})$. For such algebras it is possible to give an explicit formula for ρ. In fact, the element $t = x_1^{m_1-1} \cdots x_n^{m_n-1} \sum_{g \in G} g$ is a right integral in H. Using this integral, it is easy to check that $\alpha(g) = \chi_1^{m_1-1}(g) \cdots \chi_n^{m_n-1}(g)$. In particular, $\alpha(g_i) = q_{i1}^{m_1-1} \cdots q_{in}^{m_n-1}$. A straightforward computation, using that $\rho(g) = \alpha(g^{-1})g$ and $\rho(x_i) = \alpha(g_i^{-1}) q_{ii}^{-1} x_i$, shows that

$$\rho(x_1^{r_1} \cdots x_n^{r_n} g) = \prod_{1 \leq i < j \leq n} q_{ij}^{(1-m_j)r_i - (1-m_i)r_j} \alpha(g^{-1}) x_1^{r_1} \cdots x_n^{r_n} g.$$

Proposition 3.1 applies to this family of algebras.

Example 3.2. Let $\mathbf{k}$ be a field of characteristic $\neq 2$ and let $G = \{1, g\}$. Set $g_i = g$ and $\chi_i(g) = -1$ for $i \in \{1, \ldots, n\}$. Then, $q_{ij} = -1$ for all i, j, and $\alpha(g) = (-1)^n$. In this case, the algebra H is generated by $g, x_1, \ldots, x_n$ subject to the relations

- $g^2 = 1$,
- $x_i^2 = 0$,
- $x_i x_j = -x_j x_i$,
- $g x_i = -x_i g$.

By Proposition 3.1, we know that H is strongly graded if and only if n is odd.

4. Liftings of quantum linear spaces

In this section we consider a generalization of quantum linear spaces: that of their liftings. As above, G is a finite abelian group, $\mathbf{g} = g_1, \dots, g_n \in G$ is a sequence of elements in G and $\boldsymbol{\chi} = \chi_1, \dots, \chi_n \in \hat{G}$ is a sequence of characters of G, such that

$$\chi_i(g_i) \neq 1, \tag{4.1}$$

$$\chi_i(g_j)\chi_j(g_i) = 1, \quad \text{for } i \neq j. \tag{4.2}$$

Again, let $q_{ij} = \chi_j(g_i)$ and let $m_i = \operatorname{ord}(q_{ii})$. Let now $\lambda_i \in \mathbf{k}$ and $\lambda_{ij} \in \mathbf{k}$ for $i \neq j$ be such that

$$\lambda_i(\chi_i^{m_i} - \varepsilon) = \lambda_{ij}(\chi_i\chi_j - \varepsilon) = 0.$$

Suppose that $\lambda_{ij} + q_{ij}\lambda_{ji} = 0$ whenever $i \neq j$. The *lifting* of the quantum affine space associated with this data is the algebra $H = H(\mathbf{g}, \boldsymbol{\chi}, \boldsymbol{\lambda})$, with generators G and $x_1, \dots, x_n$, subject to the following relations:

$$gx_i = \chi_i(g)x_i g, \tag{4.3}$$

$$x_i x_j = q_{ij}x_j x_i + \lambda_{ij}(1 - g_i g_j), \tag{4.4}$$

$$x_i^{m_i} = \lambda_i(1 - g_i^{m_i}). \tag{4.5}$$

It is well known that the set of monomials $\{x_1^{r_1} \cdots x_n^{r_n} g : 0 \le r_i < m_i,\ g \in G\}$ is a basis of H. It is a Hopf algebra with comultiplication defined by

$$\Delta(g) = g \otimes g, \quad \text{for all } g \in G, \tag{4.6}$$

$$\Delta(x_i) = g_i \otimes x_i + x_i \otimes 1. \tag{4.7}$$

The counit ε satisfies $\varepsilon(g) = 1$, for all $g \in G$, and $\varepsilon(x_i) = 0$. Moreover, the antipode $\mathcal{S}$ is given by $\mathcal{S}(g) = g^{-1}$, for all $g \in G$, and $\mathcal{S}(x_i) = -g_i^{-1}x_i$. We note that $\mathcal{S}^2(g) = g$ and $\mathcal{S}^2(x_i) = q_{ii}^{-1}x_i$.

Let $\mathbb{S}_n$ be the symmetric group on n elements. For $\sigma \in \mathbb{S}_n$ let

$$t_\sigma = x_{\sigma_1}^{m_{\sigma_1}-1} \cdots x_{\sigma_n}^{m_{\sigma_n}-1} \sum_{g \in G} g.$$

Note that $t_\sigma \neq 0$.

Lemma 4.1. *The following hold:*

(1) $\lambda_{ji}g_i g_j$ *lies in the center of* $H(\mathbf{g}, \boldsymbol{\chi}, \boldsymbol{\lambda})$ *for* $i \neq j$.
(2) $\lambda_i g_i^{m_i}$ *lies in the center of* $H(\mathbf{g}, \boldsymbol{\chi}, \boldsymbol{\lambda})$.
(3) $t_\sigma g = t_\sigma$, *for all* $g \in G$.
(4) $t_\sigma x_{\sigma_n} = 0$.

Proof. For (1) it is sufficient to see that $\lambda_{ji}g_i g_j$ commutes with x_l. If $\lambda_{ji} = 0$ the result is clear. Assume that $\lambda_{ji} \neq 0$. Then, $\chi_i = \chi_j^{-1}$, and thus

$$\begin{aligned}
\lambda_{ji}g_i g_j x_l &= \lambda_{ji}\chi_l(g_i)\chi_l(g_j)x_l g_i g_j \\
&= \lambda_{ji}\chi_i(g_l)^{-1}\chi_j(g_l)^{-1}x_l g_i g_j \\
&= \lambda_{ji}x_l g_i g_j.
\end{aligned}$$

The proof of (2) is similar to that of (1) and (3) is immediate. For (4), we have:

$$\begin{aligned}
t_\sigma x_{\sigma_n} &= x_{\sigma_1}^{m_{\sigma_1}-1} \cdots x_{\sigma_n}^{m_{\sigma_n}-1} \sum_{g \in G} g x_{\sigma_n} \\
&= x_{\sigma_1}^{m_{\sigma_1}-1} \cdots x_{\sigma_n}^{m_{\sigma_n}-1} x_{\sigma_n} \sum_{g \in G} \chi_{\sigma_n}(g) g \\
&= \lambda_{\sigma_n} x_{\sigma_1}^{m_{\sigma_1}-1} \cdots x_{\sigma_{n-1}}^{m_{\sigma_{n-1}}-1} (1 - g_{\sigma_n}^{m_{\sigma_n}}) \sum_{g \in G} \chi_{\sigma_n}(g) g,
\end{aligned}$$

and the result follows on noticing that

$$\begin{aligned}(1-g_{\sigma_n}^{m_{\sigma_n}})\sum_{g\in G}\chi_{\sigma_n}(g)g &= \sum_{g\in G}\chi_{\sigma_n}(g)g - \sum_{g\in G} g_{\sigma_n}^{m_{\sigma_n}}\chi_{\sigma_n}(g)g \\ &= \sum_{g\in G}\chi_{\sigma_n}(g_{\sigma_n}^{m_{\sigma_n}}g)g_{\sigma_n}^{m_{\sigma_n}}g - \sum_{g\in G}\chi_{\sigma_n}(g)g_{\sigma_n}^{m_{\sigma_n}}g = 0,\end{aligned}$$

since $\chi_{\sigma_n}(g_{\sigma_n}^{m_{\sigma_n}}) = q_{\sigma_n\sigma_n}^{m_{\sigma_n}} = 1$. □

Proposition 4.2. *t_σ is a right integral.*

Proof. Let $M = (m_1 - 1) + \cdots + (m_n - 1)$. Let

$$\mathcal{A} = \{f : \{1, \dots, M\} \to \{1, \dots, n\} : \# f^{-1}(i) = m_i - 1 \text{ for all } i\}.$$

For $f \in \mathcal{A}$, let $x_f = x_{f(1)}x_{f(2)}\cdots x_{f(M)}$. We claim that if $f, h \in \mathcal{A}$, then $x_f \sum_{g\in G} g = \beta x_h \sum_{g\in G} g$ for some $\beta \in \mathbf{k}^*$. To prove this claim, it is sufficient to check it when f and h differ only in $i, i+1$ for some $1 \le i < M$, that is, when $h = f \circ \tau_i$, where $\tau_i \in \mathbb{S}_M$ is the elementary transposition $(i, i+1)$. But, in this case, we have:

$$\begin{aligned}x_f\sum_{g\in G} g &= x_{h\circ\tau_i}\sum_{g\in G} g \\ &= q_{h(i+1)h(i)}x_h\sum_{g\in G} g + \lambda_{h(i+1)h(i)}x_{h,\widehat{i},\widehat{i+1}}(1 - g_{h(i)}g_{h(i+1)})\sum_{g\in G} g \\ &= q_{h(i+1)h(i)}x_h\sum_{g\in G} g\end{aligned}$$

where $x_{h,\widehat{i},\widehat{i+1}} = x_{h(1)}\cdots x_{h(i-1)}x_{h(i+2)}\cdots x_{h(M)}$. The second equality follows from relation (4.4) and item (1) in the previous lemma. The proposition follows now using items (3) and (4) in the lemma. □

Now we see that

- $\alpha(x_i) = 0$,
- $\alpha(g) = \chi_1^{m_1-1}(g)\cdots\chi_n^{m_n-1}(g)$.

For the first assertion, by Proposition 4.2, we can take σ such that $\sigma_1 = i$. Then,

$$\begin{aligned}x_i t_\sigma &= x_i^{m_i}x_{\sigma_2}^{m_{\sigma_2}-1}\cdots x_{\sigma_n}^{m_{\sigma_n}-1}\sum_{g\in G} g \\ &= x_{\sigma_2}^{m_{\sigma_2}-1}\cdots x_{\sigma_n}^{m_{\sigma_n}-1}\lambda_i(1 - g_i^{m_i})\sum_{g\in G} g = 0,\end{aligned}$$

where the second equality follows from item (2) of Lemma 4.1. In particular, $\alpha(g_i) = q_{i1}^{m_1-1}\cdots q_{in}^{m_n-1}$. Since $\rho(h) = \alpha(\mathcal{S}(h_{(1)}))\mathcal{S}^2(h_{(2)})$, we have:

- $\rho(g) = \alpha(g^{-1})g$,
- $\rho(x_i) = \alpha(g_i^{-1})q_{ii}^{-1}x_i = \prod_{\substack{1\le j\le n\\ j\ne i}} q_{ij}^{1-m_j}x_i$.

Thus, as ρ is an algebra map,

$$\begin{aligned}\rho(x_1^{r_1}\cdots x_n^{r_n}g) &= q_{11}^{-r_1}\cdots q_{nn}^{-r_n}\alpha(g_1^{-r_1}\cdots g_n^{-r_n}g^{-1})x_1^{r_1}\cdots x_n^{r_n}g \\ &= \prod_{1\le i<j\le n} q_{ij}^{(1-m_j)r_i-(1-m_i)r_j}\alpha(g^{-1})x_1^{r_1}\cdots x_n^{r_n}g.\end{aligned}$$

So, the basis $\{x_1^{j_1}\cdots x_n^{j_n}g\}$ is made up of eigenvectors of ρ. Consider the groups

$$\mathbf{k}^* \supseteq L_1 = \langle q_{11}, \dots, q_{nn}, \alpha(G)\rangle \supseteq L_2 = \alpha(G).$$

Using that $\rho(x_i g_i^{-1}) = q_{ii}^{-1} x_i g_i^{-1}$ and $\rho(g) = \alpha(g^{-1})g$, it is easy to see that L_1 is the set of eigenvalues of ρ and that the order of ρ is the l.c.m. of the numbers $m_1, \ldots, m_n$ and the order of the character $\alpha_{|G} \in \hat{G}$ (in particular, $\mathbf{k}$ has a primitive ord_ρth root of unity). As before, we decompose $H = H(\mathbf{g}, \boldsymbol{\chi}, \boldsymbol{\lambda})$ as

$$H = \bigoplus_{\omega \in L_1} H_\omega, \quad \text{where } H_\omega = \{h \in H : \rho(h) = \omega h\}.$$

The following result is the version of Proposition 3.1 for the present context.

Theorem 4.3. *The following are equivalent:*

(1) $\bigoplus_{\omega \in L_1} H_\omega$ *is strongly graded.*
(2) $L_1 = L_2$.
(3) *Each component* H_ω *contains an element in* G.
(4) H *is a crossed product* $H_1 \ltimes \mathbf{k}L_1$.

Proof. Clearly (2) and (3) are equivalent and (4) ⇒ (1). The proof of (3) ⇒ (4) is the same as for Proposition 3.1. Next we prove that (1) ⇒ (3). Let $\omega \in L_1$. By Lemma 2.1, we know that $\varepsilon(H_\omega) \neq 0$. Since H_ω has a basis consisting of monomials $x_1^{r_1} \cdots x_n^{r_n} g$ and $\varepsilon(x_i) = 0$, there must be an element $g \in G$ inside H_ω. □

Remark 4.4. We next show that for liftings of quantum linear spaces, the components in the decomposition $H = \bigoplus_{\omega \in L_1} H_\omega$ are equidimensional. In fact, in this case we can take the basis of H given by

$$\{(x_1 g_1^{-1})^{r_1} \cdots (x_n g_n^{-1})^{r_n} g : 0 \leq r_i < m_i, g \in G\}.$$

Since $\rho(x_i g_i^{-1}) = q_{ii}^{-1} x_i g_i^{-1}$, the map

$$\theta : \mathbb{Z}_{m_1} \times \cdots \times \mathbb{Z}_{m_n} \times G \to \mathbf{k}^*,$$

taking $(r_1, \ldots, r_n, g)$ to the eigenvalue of $(x_1 g_i^{-1})^{r_1} \cdots (x_n g_n^{-1})^{r_n} g$ with respect to ρ, is a well defined group homomorphism. From this it follows immediately that all the eigenspaces of ρ are equidimensional.

5. Computing H_1

Assume we are in the setting of the liftings of QLS. Suppose H is a crossed product or, equivalently, that $L_1 = L_2$. Then, there exist elements $\gamma_1, \ldots, \gamma_n \in G$, such that $\alpha(\gamma_i) = q_{ii}$. Set $\tilde{\gamma}_i = g_i^{-1}\gamma_i^{-1}$ and let $y_i = x_i \tilde{\gamma}_i$. It is immediate that $y_i \in H_1$. Let $N = \ker(\alpha_{|G}) \subseteq G$. It is easy to see that H_1 has a basis given by $\{y_1^{r_1} \cdots y_n^{r_n} g : g \in N\}$. Furthermore, H_1 can be presented by generators $N, y_1, \ldots, y_n$ and relations

- $g y_i = \chi_i(g) y_i g$,
- $y_i y_j = q_{ij} \chi_j(\tilde{\gamma}_i) \chi_i^{-1}(\tilde{\gamma}_j) y_j y_i + \chi_j(\tilde{\gamma}_i) \lambda_{ij} (\tilde{\gamma}_i \tilde{\gamma}_j - \gamma_i^{-1} \gamma_j^{-1})$,
- $y_i^{m_i} = \lambda_i \chi_i^{m_i(m_i-1)/2}(\tilde{\gamma}_i)(\tilde{\gamma}_i^{m_i} - \gamma_i^{-m_i})$.

Notice that if $\lambda_i \neq 0$, then $\chi_i^{m_i(m_i-1)/2}(\tilde{\gamma}_i) = \pm 1$. We claim that

$$\lambda_{ij} \tilde{\gamma}_i \tilde{\gamma}_j, \qquad \lambda_{ij} \gamma_i \gamma_j, \qquad \lambda_i \tilde{\gamma}^{m_i} \quad \text{and} \quad \lambda_i \gamma^{m_i}$$

belong to $\mathbf{k}N$. It is clear that $\gamma^{m_i} \in N$, since $\alpha(\gamma^{m_i}) = q_{ii}^{m_i} = 1$. We now prove the remaining part of the claim. Assume that $\lambda_{ij} \neq 0$. Then $\chi_i \chi_j = \varepsilon$. Hence,

- If $l \neq i, j$, then $\chi_l(g_i g_j) = q_{il} q_{jl} = q_{li}^{-1} q_{lj}^{-1} = \chi_i \chi_j (g_l^{-1}) = 1$.
- $q_{ii} = \chi_i(g_i) = \chi_j(g_i^{-1}) = q_{ij}^{-1} = q_{ji} = q_{jj}^{-1}$.

Thus, $m_i = \mathrm{ord}(q_{ii}) = \mathrm{ord}(q_{jj}) = m_j$, and then

$$\chi_i^{m_i-1}(g_i g_j) \chi_j^{m_j-1}(g_i g_j) = (q_{ii} q_{ij} q_{ji} q_{jj})^{m_i-1} = 1 \quad \text{and} \quad \alpha(\gamma_i \gamma_j) = q_{ii} q_{jj} = 1.$$

It is now immediate that $\alpha(g_i g_j) = \chi_1^{m_1-1}(g_i g_j) \cdots \chi_n^{m_n-1}(g_i g_j) = 1$, and so

$$\alpha(\tilde{\gamma}_i \tilde{\gamma}_j) = \alpha(g_i^{-1}\gamma_i^{-1}g_j^{-1}\gamma_j^{-1}) = \alpha(g_j g_i)^{-1}\alpha(\gamma_j\gamma_i)^{-1} = 1.$$

It remains to check that $\lambda_i \tilde{\gamma}^{m_i} \in \mathbf{k}N$. Assume now that $\lambda_i \neq 0$. Then $\chi_i^{m_i} = \varepsilon$. Thus,

- If $l \neq i$, then $\chi_l^{m_l-1}(g_i^{m_i}) = q_{il}^{(m_l-1)m_i} = q_{li}^{(1-m_l)m_i} = \chi_i^{m_i}(g_l^{1-m_l}) = 1$.

Since $\chi_i^{m_i-1}(g_i^{m_i}) = q_{ii}^{m_i(m_i-1)} = 1$, this implies that

$$\alpha(g_i^{m_i}) = \chi_1^{m_1-1}(g_i^{m_i}) \cdots \chi_n^{m_n-1}(g_i^{m_i}) = 1,$$

and so

$$\alpha(\tilde{\gamma}_i^{m_i}) = \alpha(\gamma_i g_i)^{-1} = \alpha(\gamma_i)^{-1}\alpha(g_i)^{-1} = 1.$$

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