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Finite groups with almost distinct character degrees

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Abstract

Finite groups with the nonlinear irreducible characters of distinct degrees, were classified by the authors and Berkovich. These groups are clearly of even order. In groups of odd order, every irreducible character degree occurs at least twice. In this article we classify finite nonperfect groups G, such that $\chi(1) = \theta(1)$ if and only if $\theta = \overline{\chi}$ for any nonlinear $\chi \neq \theta \in \mathrm{Irr}(G)$. We also present a description of finite groups in which $xG' \subseteq \mathrm{class}(x) \cup \mathrm{class}(x^{-1})$ for every $x \in G - G'$. These groups generalize the Frobenius groups with an abelian complement, and their description is needed for the proof of the above mentioned result on characters.

Keywords: Groups of odd order; Character degrees; Extended Camina pairs

1. Introduction

A well-known conjecture states that S_3 is the only nonabelian finite group with conjugacy classes of distinct sizes. For solvable groups, this conjecture was proved by Zhang in [14] and independently by Knörr, Lempken and Thielcke in [12]. It is easy to see that a nonabelian group G has conjugacy classes of distinct sizes if and only if it has noncentral conjugacy classes of distinct sizes (see [8]).

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A similar problem arises concerning degrees of the irreducible characters of finite nonabelian groups. Here one investigates nonabelian groups with nonlinear characters of distinct degrees. These groups were classified by Berkovich, Chillag and Herzog in [2]. It was shown, that a group G satisfies that property if and only if it is of one of the following types: (i) An extraspecial 2-group; (ii) A Frobenius group of order $p^n(p^n-1)$ for some prime power p^n, with an elementary abelian kernel G' of order p^n and a cyclic complement; (iii) The Frobenius group of order 72, with a complement isomorphic to the quaternion group of order 8. All these groups are of even order.

In the case of nonabelian groups of odd order, there are at least two nonidentity conjugacy classes of each size and at least two nonprincipal irreducible characters of each degree. Therefore, the corresponding problems for groups of odd order are:

(1) Characterize nonabelian groups of odd order with exactly two noncentral classes of each size.
(2) Characterize nonabelian groups of odd order with exactly two nonlinear irreducible characters of each degree.

Herzog and Schönheim proved in [8] that the nonabelian group of order 21 is the only group satisfying the conditions of problem (1).

In this paper we consider a more general case than problem (2). The finite groups G considered, which need not be of odd order, are nonperfect and satisfy the property that two distinct irreducible characters of G are of the same degree if and only if they are complex conjugate to each other. Denote by $\mathrm{Irr}(G)$ the set of all irreducible ordinary characters of G, and by $\mathrm{Lin}(G)$ the subset of linear characters of G. Moreover, let $\Phi(G)$ denote the Frattini subgroup of G and let $\Omega(G)$ denote the subgroup of a p-group G, generated by all elements of G of order p. Our main result is:

Theorem 1. *Let G be a nonabelian and nonperfect finite group in which $\chi(1)=\theta(1)$ for distinct $\chi,\theta\in\mathrm{Irr}(G)-\mathrm{Lin}(G)$ if and only if $\theta=\overline{\chi}$. Then one of the following holds*:

(1) *G is an extraspecial 2-group, with $\mathrm{Irr}(G)-\mathrm{Lin}(G)=\{\chi\}$ and $\chi^2(1)=|G|/2$.*
(2) *G is a 2-group, $|G'|=2$, $Z(G)$ is a cyclic group of order 4, with $\mathrm{Irr}(G)-\mathrm{Lin}(G)=\{\chi,\overline{\chi}\}$, and $\chi^2(1)=|G|/4$.*
(3) *G is an extraspecial 3-group, with $\mathrm{Irr}(G)-\mathrm{Lin}(G)=\{\chi,\overline{\chi}\}$ and $\chi^2(1)=|G|/3$.*
(4) *G is a Frobenius group, either of odd order $\frac{p^n-1}{2}p^n$ for some odd prime p, or of order $(p^n-1)p^n$ for some prime p, with an abelian kernel G' of order p^n and a cyclic complement.*
(5) *G is a Frobenius group of order 72, with a complement isomorphic to the quaternion group Q_8 of order 8.*
(6) *G is a Frobenius group with a nonabelian kernel G' and a complement H, where G' is a 2-group and H is cyclic. If $N=[G',G',G']$, then G/N is a Frobenius group with the kernel G'/N of order q^2, where $q=2^r$, r is odd and $r\geqslant 3$ and a complement isomorphic to H of order $q-1$. Moreover, $Z(G'/N)=G''/N=\Phi(G')/N=\Omega(G'/N)$ is an elementary abelian 2-group of order q and of index q in G'/N.*

We note that all the groups in conclusions (1)–(5) of Theorem 1 satisfy the assumptions of that theorem and have at most two nonlinear irreducible characters. Also the groups G/N in conclusion (6) of Theorem 1 satisfy the assumptions of Theorem 1 and have three nonlinear irreducible characters of degrees $\{q-1, (q-1)\sqrt{q/2}, (q-1)\sqrt{q/2}\}$. These groups are isomorphic to the normalizers of the Sylow 2-subgroups of the Suzuki simple groups $Sz(q)$. We do not know if such groups exist with $N \neq 1$.

The smallest example of a group of type (6) is a Frobenius group of order $7 \cdot 64 = 448$ with the kernel isomorphic to the Suzuki 2-group of order 64. We are grateful to Professor Malka Schaps for constructing the character table of such a group of order 448, using the Gap system, which confirmed our theoretical arguments.

Since by Feit–Thompson theorem groups of odd order are solvable, Theorem 1 yields the following answer to problem (2).

Corollary. *Let G be a nonabelian group of odd order with exactly two nonlinear irreducible characters of each degree. Then one of the following holds*:

(1) *G is an extraspecial 3-group, with $\mathrm{Irr}(G) - \mathrm{Lin}(G) = \{\chi, \overline{\chi}\}$ and $\chi^2(1) = |G|/3$.*
(2) *G is a Frobenius group of odd order $\frac{p^n-1}{2}p^n$ for some odd prime p, with an abelian kernel G' of order p^n.*

We note that all the groups in the conclusion of the corollary have exactly two nonlinear complex conjugate irreducible characters.

Most of our notation is standard, following Gorenstein's book [7] and Isaacs' book [11]. If G is a Frobenius group, then by its kernel and its complement we mean the Frobenius kernel of G and one of the Frobenius complements of G. The conjugacy class of $x \in G$ in G will be denoted by $\mathrm{class}_G(x)$. Further notation will be introduced as needed.

In Section 2 we prove two preliminary lemmas. Section 3 is devoted to a proof of the following theorem, describing p-groups in which at most two nonlinear irreducible characters share a common degree.

Theorem 2. *Let G be a nonabelian finite p-group and suppose that it has at most two nonlinear irreducible characters of each degree. Then one of the following holds*:

(1) *$p = 2$, $|G'| = |Z(G)| = 2$ (so G is an extraspecial 2-group), with $\mathrm{Irr}(G) - \mathrm{Lin}(G) = \{\chi\}$ and $\chi^2(1) = |G|/2$.*
(2) *$p = 2$, $|G'| = 2$, $Z(G)$ is a cyclic group of order 4 with $\mathrm{Irr}(G) - \mathrm{Lin}(G) = \{\chi, \overline{\chi}\}$ and $\chi^2(1) = |G|/4$.*
(3) *$p = 2$, $|G'| = 2$, $Z(G)$ is an elementary abelian group of order 4, with $\mathrm{Irr}(G) - \mathrm{Lin}(G) = \{\chi_1, \chi_2\}$, where both characters are real-valued and $\chi_1^2(1) = \chi_2^2(1) = |G|/4$.*
(4) *$p = 3$, $|G'| = |Z(G)| = 3$ (so G is an extraspecial 3-group), with $\mathrm{Irr}(G) - \mathrm{Lin}(G) = \{\chi, \overline{\chi}\}$ and $\chi^2(1) = |G|/3$.*
(5) *$p = 2$, $|G'| = 4$, $|Z(G)| = 2$, $Z_2(G)/Z(G)$ is an elementary abelian group of order 4, with $\mathrm{Irr}(G) - \mathrm{Lin}(G) = \{\chi, \chi_1, \chi_2\}$, where the three characters are real-valued and $\chi_1^2(1) = \chi_2^2(1) = |G|/8$, $\chi^2(1) = |G|/2$.*

Moreover, in cases (1)–(4), χ, χ_1, χ_2 vanish on $G - Z(G)$.

The assumption in Theorem 2 is, of course, more general than that of Theorem 1. The results of Theorem 2 will be used in the proof of Theorem 1.

There exist groups of each type mentioned in Theorem 2. The extraspecial groups of types (1) and (4) certainly exist and there are exactly two such groups for each possible order. The central product of an extraspecial 2-group with a cyclic group of order 4 is an example of a group of type (2) and there is just one such group for each possible order. The direct product of an extraspecial 2-group with a cyclic group of order 2 is an example for a group of type (3). There are exactly two possibilities for each order. By checking the character tables from the Gap system, prepared by Professor Malka Schaps, we found groups of type (5) of order 32. We do not know if there exist larger examples of groups of type (5).

We also need a description of Frobenius groups which satisfy the assumptions of Theorem 1. Propositions 5 and 6 in Section 4 supply the required information.

In the study [2] of nonabelian groups with nonlinear characters of distinct degrees, Camina groups (which are generalizations of Frobenius groups) were used. In this paper we use extended Camina pairs, which are generalizations of Camina pairs. These two notions will be defined and discussed in the next two paragraphs.

A pair (G, K), where G is a finite group and $1 < K < G$ is a normal subgroup of G, is called a *Camina pair* if $xK \subseteq \mathrm{class}_G(x)$ for each $x \in G - K$. This is also equivalent to each of the following two statements: (i) $|C_G(x)| = |C_{G/K}(xK)|$ for each $x \in G - K$ and (ii) all irreducible characters of G not containing K in their kernel, vanish on $G - K$. A. Camina showed in [4] that if (G, K) is a Camina pair, then either G is a Frobenius group with kernel K, or one of K and G/K is a p-group for some prime p. Camina pairs (G, G') were described in [6]. It was shown that in this case one of the following three statements holds: (i) G is a Frobenius group with kernel G'; (ii) G is a p-group; (iii) G is a Frobenius group with a complement isomorphic to Q_8, the quaternion group of order 8. This result was used in [2].

A pair (G, K), where G is a finite group and $1 < K < G$ is a normal subgroup of G, is called an *extended Camina pair* if $xK \subseteq \mathrm{class}_G(x) \cup \mathrm{class}_G(x^{-1})$ for each $x \in G - K$. Camina pairs are clearly also extended Camina pairs. Extended Camina pairs were introduced in [1], where such pairs with K being a maximal normal subgroup of G were studied.

In this paper we study Camina pairs of type (G, G'), which generalize the Frobenius groups with abelian complements. We show in Section 5 (Theorem 8) that if (G, G') is an extended Camina pair, then one of the following three statements holds: (i) G is a Frobenius group with kernel G'; (ii) G is a p-group for some prime p; (iii) G/G' is a 2-group.

Our final Section 6 is devoted to a proof of Theorem 1.

2. Preliminary lemmas

In these section we prove two lemmas about nonabelian and nonperfect finite groups.

Lemma 3. *Let G be a nonabelian and nonperfect finite group and let $\chi \in \mathrm{Irr}(G) - \mathrm{Lin}(G)$. Suppose that $\chi(x) \neq 0$ for some $x \in G - G'$ and G has at most two irreducible characters of degree $\chi(1)$. Then the following statements hold*:

(1) *All linear characters of G take the values ± 1 on x.*
(2) *$G' \cup xG' = \langle x\rangle G'$ and $x^2 \in G'$.*
(3) *χ vanishes on $G - \{G' \cup xG'\}$.*

Proof. As χ does not vanish on $G - G'$, χ does not vanish on $G - \ker(\mu)$ for some $\mu \in \mathrm{Lin}(G) - \{1_G\}$. Let $\theta = \mu\chi$. Then $\theta \neq \chi$ and by our assumptions χ and θ are the only irreducible characters of G of degree $\chi(1)$. It follows that $\mu^2\chi = \chi$. Let $A = \{\alpha \in \mathrm{Lin}(G) \mid \alpha\chi = \chi\}$. We claim that $\mathrm{Lin}(G) = A \cup \mu A$. To see that, let $\beta \in \mathrm{Lin}(G) - A$. Then $\beta\chi \neq \chi$, so $\beta\chi = \theta = \mu\chi$. Hence $\mu^{-1}\beta\chi = \chi$ and $\mu^{-1}\beta \in A$, which is the same as $\beta \in \mu A$ and our claim follows. Clearly $\mu^2 \in A$.

As $\chi(x) \neq 0$, we get $\alpha(x) = 1$ for all $\alpha \in A$. As $\mu^2 \in A$, we get $\mu^2(x) = 1$. We claim that $\mu(x) = -1$. If not, then $\mu(x) = 1$ and $(\mu\alpha)(x) = \mu(x)\alpha(x) = 1$ for all $\alpha \in A$, yielding $x \in \bigcap_{\lambda \in \mathrm{Lin}(G)} \ker(\lambda) = G'$, a contradiction. Thus $\lambda(x) = 1$ for all $\lambda \in A$ and $\lambda(x) = -1$ for all $\lambda \in \mathrm{Lin}(G) - A$, proving (1). This implies that $x^2 \in G'$ and $[\langle x\rangle G' : G'] = 2$, yielding (2).

Let $y \in G - G'$ be another element with $\chi(y) \neq 0$. By repeating the arguments of the last paragraph, we get $\lambda(x) = \lambda(y) = \pm 1$ for all $\lambda \in \mathrm{Lin}(G)$. It follows that $\lambda(xy) = 1$ for all $\lambda \in \mathrm{Lin}(G)$, implying that $xy \in G'$. So every element of $G - G'$ on which χ does not vanish lies in the coset $x^{-1}G' = xG'$. This proves (3). $\square$

Lemma 4. *Let G be a nonabelian and nonperfect finite group and let $\chi \in \mathrm{Irr}(G) - \mathrm{Lin}(G)$. Suppose that G has at most two irreducible characters of degree $\chi(1)$. Then $\chi^2(1) \geqslant \frac{[G:G']}{2}$ and if $|G|$ is odd, then $\chi^2(1) \geqslant [G : G']$.*

Proof. By Lemma 3, χ vanishes either on $G - G'$ or on $G - \langle x\rangle G'$, for some $x \in G - G'$ satisfying $x^2 \in G'$. Moreover, in the latter case, $[\langle x\rangle G' : G'] = 2$. Therefore, in any case, χ vanishes on $G - X$, where X is a subgroup of G satisfying $G' \leqslant X$ and $[X : G'] \leqslant 2$. By Lemma 2.29 in [11], $[\chi_X, \chi_X] = [G : X]$. Now a standard argument implies that $[G : X] \leqslant \chi^2(1)$ (see [11, p. 200]). The lemma follows. $\square$

3. On p-groups

In this section we prove Theorem 2.

Proof. We show first that $|G'| \geqslant p^3$ is impossible under our assumptions. If $|G'| \geqslant p^3$, then $[G : G'] \leqslant \frac{|G|}{p^3}$ and the maximal possible degree in $\mathrm{Irr}(G)$ is $(\frac{|G|}{p})^{1/2}$. For $p = 2$, there could be at most one irreducible character of degree $(\frac{|G|}{2})^{1/2}$, so

$$|G| \leqslant \frac{|G|}{8} + \frac{|G|}{2} + 2|G|\left(\frac{1}{8} + \frac{1}{32} + \cdots\right) = |G|\left(\frac{1}{8} + \frac{1}{2} + 2\frac{1/8}{3/4}\right) = |G|\left(\frac{23}{24}\right),$$

a contradiction. For $p \geqslant 3$ we get

$$|G| \leqslant \frac{|G|}{p^3} + 2|G|\left(\frac{1}{p} + \frac{1}{p^3} + \frac{1}{p^5} + \cdots\right) = |G|\left(\frac{1}{p^3} + \frac{2p}{p^2 - 1}\right) \leqslant |G|\left(\frac{1}{27} + \frac{3}{4}\right),$$

a contradiction.

Suppose, next, that $|G'| = p$. Then, by [10, Theorem 7.5, p. 82, and Example 7.6(a), p. 84], $\mathrm{Irr}(G) - \mathrm{Lin}(G)$ consists of $|Z(G)| - |Z(G)/G'|$ characters of degree $|G/Z(G)|^{1/2}$ and consequently $p \leqslant 3$. Moreover, if $p = 3$, then we must have $|Z(G)| = 3$ and G is of type (4). So suppose that $p = 2$. If $|Z(G)| = 2$, then G is of type (1). Otherwise, we must have $|Z(G)| = 4$ and by Theorem 7.5 in [10], the two nonlinear characters of G are real-valued if $Z(G)$ is elementary abelian and they are complex conjugate to each other if $Z(G)$ is cyclic. Thus G is either of type (3) or of type (2). Moreover, it follows again by Theorem 7.5 in [10] that in cases (1)–(4), the characters χ, χ_1, χ_2 vanish on $G - Z(G)$. This completes the analysis of the case $|G'| = p$.

It remains to show that if $|G'| = p^2$, then only case (5) is possible. Let m be the largest squared irreducible character degree of G. Clearly $m \leqslant |G|/p = p[G : G']$ and the next smaller possible squared irreducible degree is $m/p^2 \leqslant [G : G']/p$. By Lemma 4, an irreducible character of squared degree m/p^2 can exist only if $m = p[G : G']$ and $p = 2$. Moreover, there cannot exist any smaller nonlinear squared degree in G. We recall that $(|G'| - 1)[G : G'] = (p^2 - 1)[G : G']$ is the sum of the squares of the degrees of the nonlinear irreducible characters of G. If $p > 2$, then we get:

$$\left(p^2 - 1\right)[G : G'] \leqslant 2m \leqslant 2p[G : G'],$$

a contradiction. Hence $p = 2$ and either $m = |G|/2$ and this squared degree appears exactly once or $m \leqslant |G|/4$ and no smaller squared degree exists. In the latter case, the previous argument yields $3[G : G'] = (3/4)|G| \leqslant 2m \leqslant |G|/2$, a contradiction. So $m = |G|/2$ and we must have $3/4|G| = 2 \cdot |G|/8 + |G|/2$. Thus $|G'| = 4$, $|Z(G)| = 2$ and $\mathrm{Irr}(G) - \mathrm{Lin}(G) = \{\chi, \chi_1, \chi_2\}$ with $\chi_1^2(1) = \chi_2^2(1) = |G|/8$ and $\chi^2(1) = |G|/2$. Now, by [9, Satz III, 2.13(a), p. 266], the exponent of $Z(G/Z(G))$ divides the exponent 2 of $Z(G)$, which implies that $G/Z(G)$ is of type (3). Thus $Z_2(G)/Z(G)$ is an elementary abelian group of order 4 and the three nonlinear irreducible characters of G are real-valued, as claimed. □

4. Frobenius groups

The aim of this section is to investigate Frobenius groups satisfying the assumptions of Theorem 1. First we deal with Frobenius groups with an abelian kernel.

Proposition 5. *Let G be a Frobenius group with an abelian kernel K. Assume that $\chi(1) = \theta(1)$ for distinct characters $\chi, \theta \in \mathrm{Irr}(G) - \mathrm{Lin}(G)$ if and only if $\theta = \overline{\chi}$. Then K is an elementary abelian p-group for some prime p and either $|G| = (|K| - 1)|K|$ or G is of odd order $\frac{|K|-1}{2}|K|$.*

Proof. Let H be a complement in G. Then, by Theorem 18.7 in [10, p. 239], $\mathrm{Irr}(G) - \mathrm{Lin}(G)$ contains $\frac{|K|-1}{|H|}$ characters of degree $[G : K]$ and, by our assumptions, either $|H| = \frac{|K|-1}{2}$ or $|H| = |K| - 1$. Hence K is an elementary abelian p-group for some prime p. If $|H| = \frac{|K|-1}{2}$ is even, then each $a \in K$ is inverted by the involution in H and the two characters in $\mathrm{Irr}(G) - \mathrm{Lin}(G)$ are real and of equal degree, in contradiction to our assumptions. Hence, if $|H| = \frac{|K|-1}{2}$, then $|H|$ must be odd and clearly $|K|$ must be odd. Thus either $|G| = (|K| - 1)|K|$ or G is of odd order $\frac{|K|-1}{2}|K|$, as claimed. □

Conversely, the Frobenius groups mentioned in Proposition 5 indeed satisfy the assumptions of that proposition. Next, we deal with Frobenius groups with a nonabelian kernel.

Proposition 6. *Let G be a Frobenius group with a nonabelian kernel K and a complement H. Assume that $\chi(1) = \theta(1)$ for distinct $\chi, \theta \in \mathrm{Irr}(G) - \mathrm{Lin}(G)$ if and only if $\theta = \overline{\chi}$. Then K is a 2-group. If $N = [K, K, K]$, then K/N is of order q^2, where $q = 2^r$, r is odd and $r \geqslant 3$. Moreover, $Z(K/N) = K'/N = \Phi(K)/N = \Omega(K/N)$ is an elementary abelian 2-group of order q and index q in K/N, and H is of order $q - 1$. Finally, if $K = G'$, then H is cyclic.*

Proof. Since K is nonabelian, $|H|$ is odd and by Thompson's theorem K is nilpotent. Since G/K' is a Frobenius group with an abelian kernel K/K' satisfying our assumptions, it follows by Proposition 5 that K/K' is an elementary abelian p-group for some prime p of order p^r, say. Hence K is a p-group satisfying $K' = \Phi(K)$.

Note that G/N is a Frobenius group with the nonabelian kernel K/N and a complement isomorphic to H. So G/N satisfies the assumptions of this proposition, and in order to complete our proof, we may assume that $N = 1$.

Clearly $N = 1$ implies that K is nilpotent of class 2 and, in particular, $K' \leqslant Z(K)$. By applying Proposition 5 to G/K', it follows that either $|H| = \frac{1}{2}(\frac{|K|}{|K'|} - 1)$ and p is odd or $|H| = \frac{|K|}{|K'|} - 1$ and $p = 2$. In both cases $[K : K'] \leqslant 2|H| + 1$.

By our assumptions, there are at most two H-orbits of irreducible characters of K of any given degree, and if there are two orbits, then the characters in each orbit are the complex conjugates of the characters in the other orbit. This and other facts used later about the irreducible characters of Frobenius groups can be found in Theorem 18.7 in [10].

Now let λ be any nonprincipal irreducible character of K'. Since K/K' is abelian, it follows by Theorem 15 in Chapter 7 of [3] that all the irreducible constituents of λ^K have equal degrees. No two of these can be in the same H-orbit since if, say, $\phi^h = \theta$, where ϕ and θ are distinct irreducible characters of K lying over λ and $1 \neq h \in H$, then since both $\phi_{K'}$ and $\theta_{K'}$ are multiples of λ (as $K' \leqslant Z(K)$), we see that h fixes λ, which is false. It follows that λ^K has either one or two distinct irreducible constituents, and these have degree f, say, and by the Frobenius reciprocity theorem, also multiplicity f in λ^K. Thus either $[K : K'] = f^2$, r is even and there is just one irreducible constituent in λ^K, or else $[K : K'] = 2f^2$ and there are exactly two distinct constituents. In the first case, every nonlinear irreducible character of K vanishes on $K - K'$ and in the latter case, $p = 2$ and r is odd.

Suppose r is even. Since the nonlinear irreducible characters of K vanish on $K - K'$, the second orthogonality relation yields $|C_K(x)| = [K : K']$ for each $x \in K - K'$. As $K' \leqslant Z(K)$, it follows that $p|K'| \leqslant |C_K(x)| = [K : K'] \leqslant 1 + 2|H|$. But the action of H on K' is Frobenius, so $|K'| > |H|$, contradicting the previous inequality.

Thus r is odd, $p = 2$ and $[K : K'] - 1 = |H|$. In this case, each nonprincipal linear character of K' determines two irreducible characters of K of degree f, and so the number of these is even and they form two H-orbits. The number of nonprincipal linear characters of K' is thus $|H|$ and $|K'| = 1 + |H| = [K : K'] = 2^r \geqslant 2^3$. By applying Proposition 5 to $G/Z(K)$, it follows that $[K : K'] - 1 = |H| \leqslant [K : Z(K)] - 1$. Thus $|K'| \leqslant |Z(K)| \leqslant |K'|$, implying $Z(K) = K'$. Finally, if $K = G'$, then H is abelian and hence cyclic. The proof is complete. □

It follows from our proof that if $K = G'$, then G/N has $q - 1$ linear characters, one character of degree $q - 1$ (induced by the nonprincipal linear characters of G'/N) and two conjugate characters of degree $(q - 1)\sqrt{q/2}$ (induced by the characters in $\mathrm{Irr}(G'/N) - \mathrm{Lin}(G'/N)$).

5. Extended Camina pairs

Let G be a nonabelian and nonperfect finite group. In this section we investigate extended Camina pairs (G, G'). Recall that (G, G') is an extended Camina pair if $1 < G' < G$ and $xG' \subseteq \mathrm{class}_G(x) \cup \mathrm{class}_G(x^{-1})$ for each $x \in G - G'$. Camina pairs (G, G'), satisfying $xG' \subseteq \mathrm{class}_G(x)$ for each $x \in G - G'$, are clearly also extended Camina pairs.

Let (G, G') be an extended Camina pair and let $x \in G - G'$. We say that x is of *type* 1 if $xG' \subseteq \mathrm{class}_G(x)$ and it is of *type 2* if $xG' \subseteq \mathrm{class}_G(x) \cup \mathrm{class}_G(x^{-1})$, but $xG' \nsubseteq \mathrm{class}_G(x)$. If all $x \in G - G'$ are of type 1, then clearly (G, G') is a Camina pair.

Lemma 7. *Let G be a nonabelian and nonperfect finite group and let (G, G') be an extended Camina pair. Then*:

(1) *If $x \in G - G'$ is of type* 1, *then $xG' = \mathrm{class}_G(x)$ and $|C_G(x)| = [G : G']$.*
(2) *If $x \in G - G'$ is of type* 2, *then $xG' = \mathrm{class}_G(x) \cup \mathrm{class}_G(x^{-1})$, $|C_G(x)| = 2[G : G']$, x is not real and $x^2 \in G'$.*
(3) *If $y \in G - G'$ and yG' is a p-element in G/G' for some prime p, then y is a p-element and $C_{G'}(y)$ is a p-group. In particular, elements of type* 2 *are* 2*-elements and their centralizers in G' are* 2*-groups.*
(4) *If (G, G') is not a Camina pair, then both $|G'|$ and $[G : G']$ are even integers.*

Proof. Note that if $a \in G - G'$, then $\mathrm{class}_G(a) \subseteq aG'$ (as $g^{-1}ag = a(a^{-1}g^{-1}ag) \in aG'$ for all $g \in G$).

(1) Here $xG' \subseteq \mathrm{class}_G(x) \subseteq xG'$, yielding $xG' = \mathrm{class}_G(x)$ and $|C_G(x)| = |G|/|\mathrm{class}_G(x)| = [G : G']$.

(2) Here $xG' \subseteq \mathrm{class}_G(x) \cup \mathrm{class}_G(x^{-1})$, but $xG' \nsubseteq \mathrm{class}_G(x)$. Hence x is nonreal and there exists $z \in xG' \cap \mathrm{class}_G(x^{-1})$. Thus

$$\mathrm{class}_G\left(x^{-1}\right) = \mathrm{class}_G(z) \subseteq zG' = xG'$$

and consequently $xG' \subseteq \mathrm{class}_G(x) \cup \mathrm{class}_G(x^{-1}) \subseteq xG'$, forcing $xG' = \mathrm{class}_G(x) \cup \mathrm{class}_G(x^{-1})$. Since $|\mathrm{class}_G(x)| = |\mathrm{class}_G(x^{-1})|$, it follows that $|G'| = 2[G : C_G(x)]$ and hence $|C_G(x)| = 2[G : G']$. Finally, $x^{-1} \in xG'$ implies that $x^2 \in G'$.

(3) Let $y = y_p \times y_{p'}$, where y_p and $y_{p'}$ are the p and p' parts of y, respectively. Since yG' is a p-element in G/G', some p-power of y lies in G' and hence $y_{p'} \in G'$. So $yG' = y_pG'$ and since y is either of type 1 or of type 2, we get that y is conjugate either to y_p or to y_p^{-1}. Either way $o(y) = o(y_p)$, so y is a p-element.

Now, let $g \in C_{G'}(y)$. Then $ygG' = yG'$ and so yg is conjugate either to y or to y^{-1}. Therefore $o(yg) = o(y)$ and $1 = (yg)^{o(y)} = y^{o(y)}g^{o(y)} = g^{o(y)}$, forcing g to be a p-element as well.

Finally, if $x \in G - G'$ is of type 2, then, by (2), $x^2 \in G'$ and as shown above, x is a 2-element and $C_{G'}(x)$ is a 2-group.

(4) There exists $x \in G - G'$ of type 2, which by (3) is a 2-element. Hence $[G : G']$ is even. Since, by (2), $|G'| = 2[G : C_G(x)]$, it follows that $|G'|$ is also even. □

We are ready now for a general description of extended Camina pairs of type (G, G').

Theorem 8. *Let G be a nonabelian and nonperfect finite group and let (G, G') be an extended Camina pair. Then one of the following holds*:

(1) *G is a p-group for some prime p.*

(2) *G is a Frobenius group with the kernel G'.*
(3) *G/G' is a 2-group. In particular, $C_G(u)$ is a 2-group for every $u \in G - G'$.*

In particular, if (G, G') is not a Camina pair, then (3) *holds.*

Proof. Assume, first, that (G, G') is a Camina pair. By [4], either (1) or (2) holds, or one of G' and G/G' is a p-group for some prime p. So suppose that G is not as described in (1) or (2). If G' is a p-group, then, by Lemma 4.4 in [5], $O_{p'}(G/G') = 1$ and since G/G' is abelian, this forces G to be a p-group, a contradiction. Finally, if G/G' is a (abelian) p-group, then by the corollary in [6], G is a Frobenius group with a complement Q_8 and $[G : G'] = 4$. So G/G' is a 2-group, satisfying (3).

It remains only to prove the "in particular" part. So assume that (G, G') is not a Camina pair and hence there are elements of type 2 in $G - G'$. We shall use Lemma 7 freely, in order to show that G/G' is a 2-group, satisfying (3). Suppose, to the contrary, that p is an odd prime divisor of $|G/G'|$. Let yG' be a nontrivial p-element in G/G'. Then y is a p-element and $C_{G'}(y)$ is a p-group. As $p \neq 2$, y is of type 1, whence $|C_G(y)| = [G : G']$. By Lemma 7(4), $|C_G(y)|$ is even.

Let R be a Sylow 2-subgroup of $C_G(y)$ and let P be a Sylow 2-subgroup of G containing R. Note that $P \cap G'$ is a Sylow 2-subgroup of G' and $P \cap G' \trianglelefteq P$. Furthermore, $R \cap G' \leqslant C_G(y) \cap G' = C_{G'}(y)$. However, $R \cap G'$ is a 2-group and $C_{G'}(y)$ is a p-group, so $R \cap G' = R \cap (P \cap G') = 1$ and R is isomorphic to RG'/G'. Hence R is an abelian group. Furthermore

$$\left|R(P \cap G')\right| = |R||P \cap G'| = \left|C_G(y)\right|_2 |G'|_2 = [G : G']_2 |G'|_2 = |G|_2.$$

Thus $R(P \cap G')$ is a Sylow 2-subgroup of G contained in P, which implies that $P = R(P \cap G')$.

Let $r \in R$ be an involution. Then $R \cap G' = 1$ implies that $r \notin G'$. Also, as r is a real element, r is of type 1, so $|C_G(r)| = [G : G']$. This implies that $|R| = [G : G']_2 = |C_G(r)|_2$ and since R is abelian, we get $R \leqslant C_G(r) \cap P = C_P(r)$. Thus

$$|R| \leqslant \left|C_P(r)\right| \leqslant \left|C_G(r)\right|_2 = |R|$$

which implies that $R = C_P(r)$. In particular, $Z(P) \leqslant R$ and as $R \cap G' = 1$, we get $Z(P) \cap G' = 1$. Hence $Z(P) \cap P' = 1$, which implies that P is abelian. Thus $P \leqslant C_G(r)$ and since $[G : G'] = |C_G(r)|$, $|G'|$ is odd, in contradiction to Lemma 7(4). It follows that if (G, G') is not a Camina pair, then G/G' is a 2-group.

Finally, if (G, G') is an extended Camina pair and G/G' is a 2-group, let $u \in G - G'$. Then, by Lemma 7(3), u is a 2-element and $C_{G'}(u)$ is a 2-group. Thus $C_G(u) - C_{G'}(u)$ consists of 2-elements and hence $C_G(u)$ is also a 2-group. □

6. Proof of Theorem 1

We start the proof of Theorem 1 with two preliminary lemmas.

Lemma 9. *Let G be a nonabelian and nonperfect finite group in which $\chi(1) = \theta(1)$ for distinct $\chi, \theta \in \mathrm{Irr}(G) - \mathrm{Lin}(G)$ if and only if $\theta = \overline{\chi}$. Suppose that $x \in G - G'$ and there exists $\chi \in \mathrm{Irr}(G) - \mathrm{Lin}(G)$ such that $\chi(x) \neq 0$. Then the following statements hold*:

(1) *x is nonreal.*
(2) $xG' = \mathrm{class}_G(x) \cup \mathrm{class}_G(x^{-1})$, $|C_G(x)| = 2[G : G']$ *and* $x^2 \in G'$.

(3) $\chi(x) = \pm i\sqrt{[G:G']/2}$ *(here* $i = \sqrt{-1}$*).*
(4) *All characters in* $\mathrm{Irr}(G) - \{\mathrm{Lin}(G) \cup \{\chi, \overline{\chi}\}\}$ *vanish on* x.
(5) χ *vanishes on* $G - \{G' \cup xG'\}$.
(6) *Every linear character takes the values* ± 1 *on* x.

Proof. By Lemma 3, statements (5) and (6) hold and $x^2 \in G'$.

As χ does not vanish on $G - G'$, χ does not vanish on $G - \ker(\mu)$ for some $\mu \in \mathrm{Lin}(G) - \{1_G\}$. Therefore $\mu\chi \neq \chi$, $\mu\overline{\chi} \neq \overline{\chi}$ and by our assumptions $\mu\chi = \overline{\chi}$. Note that $\mu(x)\chi(x) = \overline{\chi}(x)$ implies that $-\chi(x) = \overline{\chi}(x)$ and so $\chi(x) = bi$ for some real nonzero number b. Hence x is nonreal, proving (1).

Let $\theta_1, \theta_2, \dots, \theta_s$ be all the nonlinear irreducible characters of G nonvanishing on x. Set $\theta_j(x) = b_j i$. Then

$$\left|C_G(x)\right| = [G:G'] + \sum_{j=1}^{s}\left|\theta_j(x)\right|^2 = [G:G'] + \sum_{j=1}^{s} b_j^2. \qquad (*)$$

Consider, now, the element x^{-1}. Clearly $\theta_1, \theta_2, \dots, \theta_s$ are all the nonlinear irreducible characters of G nonvanishing on x^{-1}. Moreover, $\theta_j(x^{-1}) = -b_j i$ and by (6), $\lambda(x) = \lambda(x^{-1})$ for all $\lambda \in \mathrm{Lin}(G)$. Applying the second orthogonality relation to x and x^{-1} we obtain:

$$\begin{aligned} 0 &= \sum_{\chi \in \mathrm{Irr}(G)} \chi(x)\overline{\chi\left(x^{-1}\right)} = \sum_{\chi \in \mathrm{Irr}(G)} \left(\chi(x)\right)^2 = [G:G'] + \sum_{j=1}^{s}\left(\theta_j(x)\right)^2 \\ &= [G:G'] + \sum_{j=1}^{s}(b_j i)^2 = [G:G'] - \sum_{j=1}^{s} b_j^2. \end{aligned}$$

So $[G:G'] = \sum_{j=1}^{s} b_j^2$ and therefore, by $(*)$, $|C_G(x)| = 2[G:G']$. It follows that $|\mathrm{class}_G(x)| = |\mathrm{class}_G(x^{-1})| = \frac{|G'|}{2}$. Since $\mathrm{class}_G(x) \cup \mathrm{class}_G(x^{-1}) \subseteq xG' = x^{-1}G'$, we may conclude that $xG' = \mathrm{class}_G(x) \cup \mathrm{class}_G(x^{-1})$. The proof of (2) is now also complete.

Next we compute $\sum_{g \in G} |\chi(g)|^2$, using (2), (5) and the fact that $\chi(x) = bi$:

$$|G| = \sum_{g \in G}\left|\chi(g)\right|^2 = b^2|G'|/2 + b^2|G'|/2 + \sum_{g \in G'}\left|\chi(g)\right|^2 = b^2|G'| + \sum_{g \in G'}\left|\chi(g)\right|^2.$$

We also note that $\mu\chi = \overline{\chi}$ implies that $\chi_{G'} = \overline{\chi}_{G'}$, so χ is real on G'. Thus

$$\begin{aligned} 0 = |G|(\chi, \overline{\chi}) &= \sum_{g \in G}\left(\chi(g)\right)^2 = (bi)^2|G'|/2 + (-bi)^2|G'|/2 + \sum_{g \in G'}\left(\chi(g)\right)^2 \\ &= -b^2|G'| + \sum_{g \in G'}\left|\chi(g)\right|^2. \end{aligned}$$

We conclude from these computation that $b^2 = \frac{[G:G']}{2}$, which implies (3).

Finally,

$$\left|C_G(x)\right| = 2[G:G'] = [G:G'] + 2b^2 = [G:G'] + \left|\chi(x)\right|^2 + \left|\overline{\chi}(x)\right|^2.$$

This means that all elements of $\mathrm{Irr}(G) - \mathrm{Lin}(G)$ other than χ and $\overline{\chi}$ vanish on x, proving (4) and finishing the proof of the lemma. □

Lemma 10. *Let G be a nonabelian and nonperfect finite group in which $\chi(1) = \theta(1)$ for distinct $\chi, \theta \in \mathrm{Irr}(G) - \mathrm{Lin}(G)$ if and only if $\theta = \overline{\chi}$. Then the following statements hold*:

(1) *(G, G') is an extended Camina pair.*
(2) *If $x \in G - G'$ is of type 2, then there exists $\chi \in \mathrm{Irr}(G) - \mathrm{Lin}(G)$ such that $\chi(x) \neq 0$, and therefore x and χ satisfy the six statements of Lemma 9.*

Proof. Let $x \in G - G'$. If every $\chi \in \mathrm{Irr}(G) - \mathrm{Lin}(G)$ vanishes on x, then $\mathrm{class}_G(x) = xG'$. If, on the other hand, there exists $\chi \in \mathrm{Irr}(G) - \mathrm{Lin}(G)$ satisfying $\chi(x) \neq 0$, then by Lemma 9(2) $\mathrm{class}_G(x) \cup \mathrm{class}_G(x^{-1}) = xG'$. Therefore (G, G') is an extended Camina pair. If $x \in G - G'$ is of type 2, then there exists $\chi \in \mathrm{Irr}(G) - \mathrm{Lin}(G)$ satisfying $\chi(x) \neq 0$ and by Lemma 9 the second claim of Lemma 10 holds. □

Next we quote two lemmas which will be needed for the proof of Theorem 1. Here $O(G)$ denotes the largest normal subgroup of G of odd order.

Lemma 11. *Let G be a finite group and suppose that G has a nonreal element g of order 4 satisfying $C_G(g) = \langle g\rangle$. Then $G = O(G)\langle g\rangle$.*

Proof. See Lemma 2.9 in [1]. □

Lemma 12. *Let P be a finite p-group of class at most 2 and suppose that P acts on some nontrivial finite p'-group Q such that $C_P(x) \leqslant P'$ for all $1 \neq x \in Q$. Then P is either cyclic or isomorphic to Q_8.*

Proof. See Lemma 10.1, p. 245 in [13]. □

We are ready now for the proof of Theorem 1.

Proof of Theorem 1. By Lemma 10(1), (G, G') is an extended Camina pair. We shall break the proof into a series of steps.

Step 1. Proof of the theorem in the case that either G is a p-group for some prime p or (G, G') is a Camina pair.

Proof. If G is a p-group, then G is one of the groups of Theorem 2. Only groups in conclusions (1), (2) and (4) satisfy our assumptions, and these groups appear in conclusions (1)–(3) of our theorem. So assume that (G, G') is a Camina pair and G is not a p-group. In this case [4] implies that either G is a Frobenius group with the kernel G' or one of G' and G/G' is a p-group for some prime p.

Consider, first, the case when G is a Frobenius group with the kernel G'. Then, either G' is abelian and Proposition 5 implies that G is included in conclusion (4) of our theorem, or G' is nonabelian, and by Proposition 6 G is included in conclusion (6) of our theorem.

If G' is a p-group, then, by Lemma 4.4 of [5], $O_{p'}(G/G') = 1$ and as G/G' is abelian, this forces G to be a p-group. This case had been discussed above. Finally, if G/G' is a (abelian) p-group, then, by the Corollary in [6], G is a Frobenius group with a complement Q_8. Hence the kernel K of G is abelian, and by Proposition 5 G is of order 72 and is included in conclusion (5) of our theorem. □

From now on we shall assume that neither G is a p-group nor (G, G') is a Camina pair. In particular, $G - G'$ contains elements of type 2 and Theorem 8 implies that G/G' is a 2-group and $C_G(u)$ is a 2-group for every $u \in G - G'$. Our aim is to reach a contradiction.

Step 2. $[G : G'] \neq 2$.

Proof. Suppose that $[G : G'] = 2$ and let $x \in G - G'$ be of type 2. By Lemma 10(2) x is not real, so, in particular, x is not an involution and $|C_G(x)| = 2[G : G'] = 4$. Hence x is of order 4 and $C_G(x) = \langle x \rangle$. It follows then by Lemma 11 that $|G/O(G)| = 4$ and $G' \leqslant O(G)$, contradicting our assumption that $[G : G'] = 2$. So $[G : G'] \neq 2$. □

Step 3. We show that $\chi_{G'}$ is not irreducible for each $\chi \in \mathrm{Irr}(G) - \mathrm{Lin}(G)$.

Proof. Suppose, to the contrary, that $\chi_{G'}$ is irreducible for some $\chi \in \mathrm{Irr}(G) - \mathrm{Lin}(G)$. If χ vanishes on $G - G'$, then

$$|G| = \sum_{g \in G} \left|\chi(g)\right|^2 = \sum_{g \in G'} \left|\chi(g)\right|^2 = |G'|$$

which is impossible. So there exists $y \in G - G'$ such that $\chi(y) \neq 0$ and by Lemma 9(2) $|\mathrm{class}_G(y)| = |\mathrm{class}_G(y^{-1})| = \frac{|G'|}{2}$. By Lemma 9, $\chi(y) = \pm i\sqrt{[G : G']/2}$ and χ vanishes on $G - \{G' \cup \mathrm{class}_G(y) \cup \mathrm{class}_G(y^{-1})\}$. So

$$\begin{aligned} |G| = \sum_{g \in G} \left|\chi(g)\right|^2 &= \left| i\sqrt{\frac{[G : G']}{2}}\right|^2 \cdot \frac{|G'|}{2} \cdot 2 + \sum_{g \in G'} \left|\chi(g)\right|^2 \\ &= \frac{[G : G']|G'|}{2} + |G'| = \frac{|G|}{2} + |G'|. \end{aligned}$$

This implies $[G : G'] = 2$, contradicting Step 2. □

Step 4. $O(G)$ is the normal 2-complement of G, $O(G) \subseteq G'$ and every character in $\mathrm{Irr}(G) - \mathrm{Irr}(G/O(G))$ vanishes on $G - G'$.

Proof. If $\chi \in \mathrm{Irr}(G) - \mathrm{Lin}(G)$, then $\chi_{G'} = e(\theta_1 + \theta_2 + \cdots + \theta_t)$, where $\theta_i \in \mathrm{Irr}(G')$ have the same degree and since $\chi_{G'}$ is reducible by Step 3, we have $et > 1$. Now $\chi(1) = et\theta(1)$ and both e and t divide $[G : G']$ (see [10, p. 82]), which is a power of 2. So $\chi(1)$ is even for each $\chi \in \mathrm{Irr}(G) - \mathrm{Lin}(G)$ and by Thompson's theorem (see [10, Corollary 12.2, p. 199]) G has a normal 2-complement, which is equal to $O(G)$. As G/G' is a 2-group, $O(G) \subseteq G'$. Finally, let $u \in G - G'$. By Theorem 8(3), $C_G(u)$ is a 2-group and hence $C_G(u) \cap O(G) = 1$. It follows that $|C_{G/O(G)}(uO(G))| \geqslant |C_G(u)|$. Since the opposite inequality always holds (see [10, Corollary 2.24, p. 26]), we get equality and consequently every character in $\mathrm{Irr}(G) - \mathrm{Irr}(G/O(G))$ vanishes on u. The proof of Step 4 is complete. □

Let T be a Sylow 2-subgroup of G. By Step 4 $G = O(G)T$, a semi-direct product, and since G is not a p-group, $O(G) > 1$.

Step 5. $T' = G' \cap T$.

Proof. Since by Lemma 7(3) elements of type 2 are 2-elements, T contains an element v of type 2. By Lemma 10(2), there exists $\chi \in \mathrm{Irr}(G) - \mathrm{Lin}(G)$ such that $\chi(v)$ is nonreal. By Step 4 $\chi \in \mathrm{Irr}(G/O(G))$. Since $O(G) \leqslant \ker(\chi)$, we get

$$|G| = \sum_{t\in T} \sum_{h\in O(G)} \left|\chi(ht)\right|^2 = \sum_{t\in T} \sum_{h\in O(G)} \left|\chi(t)\right|^2 = \left|O(G)\right| \sum_{t\in T} \left|\chi(t)\right|^2.$$

It follows that $|T| = \frac{|G|}{|O(G)|} = \sum_{t\in T} |\chi(t)|^2$ and so $\chi_T \in \mathrm{Irr}(T) - \mathrm{Lin}(T)$. Moreover, χ_T is nonreal and T is nonabelian.

Clearly $G/O(G) \cong T$ satisfies the assumptions of Theorem 1 and since T is a 2-group, it must be one of the four 2-groups of the conclusion of Theorem 2. However, only one of them (of type (2)) has a nonreal nonlinear irreducible character. Thus $|T'| = 2$ and $Z(T)$ is cyclic of order 4. Moreover, by Theorem 7.5 in [10, p. 82], the nonreal nonlinear irreducible characters of T get nonreal values only on the two elements of order 4 of $Z(T)$. Thus $v \in Z(T)$.

As v is of type 2, we know by Lemma 10(2) that $|C_G(v)| = 2[G : G']$. Since $v \in Z(T)$ and $C_G(v)$ is a 2-group, it follows that $|C_G(v)| = 2[G : G'] = |T|$. But $G = O(G)T = G'T$, so

$$|T| = 2\left|\frac{G'T}{G'}\right| = 2\left|\frac{T}{G' \cap T}\right|,$$

which implies that $|G' \cap T| = 2 = |T'|$. Hence $T' = G' \cap T$, proving Step 5. □

Step 6. The final contradiction.

Proof. Let $b \in O(G) - \{1\}$ and let $c \in C_T(b)$. Then $b \in C_G(c)$, so $C_G(c)$ is not a 2-group. Therefore, by Theorem 8(3), $c \in G'$. By Step 5, $C_T(b) \leqslant G' \cap T = T'$ and Lemma 12 implies that T is either cyclic or isomorphic to Q_8. This is a contradiction, since T is nonabelian and $|Z(T)| = 4$. □

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