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New Lower Bounds for the Hadwiger Numbers of ℓ_p Balls for $p < 2$

K. J. SWANEPOEL
Department of Mathmatics and Applied Mathmatics
University of Pretoria, 0002 Pretoria, South Africa

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Abstract—In this note, we derive an asymptotic lower bound for the size of constant weight binary codes that is exponential in the code length, if both the minimum distance and the weight grow in proportion to the code length. We use this bound to find new lower bounds for the Hadwiger and weak Hadwiger numbers of d-dimensional ℓ_p balls in the case $1 \leq p < 2$.

Keywords—Hadwiger number, p-norm, Minimal spanning tree, MST.

1. INTRODUCTION

We denote d-dimensional real vector space by $\mathbb{R}^d$. A *convex body* $C \subseteq \mathbb{R}^d$ is a closed convex subset of $\mathbb{R}^d$ with nonempty interior. The *Hadwiger number* $h(C)$ (*weak Hadwiger number* $h_w(C)$) of C is the maximum cardinality of a set $S \subseteq \mathbb{R}^d$ such that the sets $\{s + C : s \in S \text{ or } s = 0\}$ have disjoint interiors (are disjoint, respectively). Obviously, $h_w(C) \leq h(C)$.

The Hadwiger number $h(X) = h(B)$ of the unit ball B of a finite dimensional normed space (or *Minkowski space*) X gives a tight upper bound on the maximum degree of a Minimum Spanning Tree (MST) of a set of points in X. More precisely, we have the following. Let $\Delta(T)$ denote the maximum degree of a tree T. Then $h(X) = \max_S \max_T \Delta(T)$, where the first maximum is over all finite point sets S in X, and the second maximum over all MSTs of S [1,2]. Also, as proved in [3], for the weak Hadwiger number $h_w(X) = h_w(B)$ of the unit ball of X it holds that $h_w(X) = \max_S \min_T \Delta(T)$, where again S ranges over all finite point sets in X, and T over all MSTs of S. Thus, for any finite set of points in X, there exists an MST with maximum degree at most $h_w(X)$. The weak Hadwiger number may be substantially smaller than the Hadwiger number, see [3] for examples.

The d-dimensional ℓ_p space ($1 < p < \infty$), denoted by ℓ_p^d, is $\mathbb{R}^d$ with the p-norm

$$\|(x_1, \dots, x_d)\|_p := \left(\sum_{i=1}^{d} |x_i|^p\right)^{1/p}.$$

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In [3], it is shown that the weak Hadwiger number of ℓ_p^d has the lower bound

$$h_w(\ell_p^d) > \begin{cases} 2^{0.0312d+o(d)}, & \text{if } p = 1, \\ 2^{d(1-H(2^{-p}))+o(d)}, & \text{if } 1 < p < \infty, \end{cases}$$

where $H(x) = -x\log_2 x - (1-x)\log_2(1-x)$, $0 < x < 1$ is the *binary entropy function.* We improve these bounds in the range $1 \le p < 2$.

THEOREM 1.

$$h_w(\ell_p^d) > \begin{cases} 2^{0.0941\ldots d}, & \text{for } 1 \le p \le 1.62107\ldots, \\ \left(2^{1-p/2}\sqrt{1-2^{-p}}\right)^{-d}, & \text{for } 1.62107\cdots < p \le 2. \end{cases}$$

Note that both bounds hold in the whole interval $1 \le p \le 2$. We indicate the cut-off point $p = 1.62107\ldots$, below which the first bound is better, and above which the second bound is better.

The first bound follows from a lower bound on constant weight binary codes, derived in Section 2. This lower bound follows essentially from the analogue of the Gilbert-Varshamov lower bound for constant weight codes, noted in [4].

The second bound is found as follows. Use the Wyner lower bound for euclidean spherical codes [5] to find at least $(2^{1-p/2}\sqrt{1-2^{-p}})^{-d}$ unit vectors in Euclidean space ℓ_2^d with minimum Euclidean distance larger than $2^{1-p/2}$. Then use the following well-known (nonlinear) norm-preserving map

$$f : \ell_2^d \to \ell_p^d, \quad (x_i)_{i=1}^d \mapsto \left(|x_i|^{2/p}\operatorname{sgn} x_i\right)_{i=1}^d,$$

which satisfies $\|1/2(f(x)-f(y))\|_p^p \ge \|1/2(x-y)\|_2^2$ if $1 \le p \le 2$ (see [6–8]).

In Section 3, we finish the proof of Theorem 1.

2. CONSTANT WEIGHT BINARY CODES

We let $A(n,d,w)$ denote the largest cardinality of a constant weight binary code of length n, weight w, and minimum (Hamming) distance strictly larger than the real number d. Define $f(\alpha,\beta) := H(\beta) - \beta H(\alpha) - (1-\beta)H(\alpha\beta/(1-\beta))$ for $\alpha,\beta > 0, \alpha+\beta < 1$. Note that since H' is a decreasing function, $-H$ is strictly convex on $(0,1)$, hence we may take convex combinations to obtain

$$\begin{aligned} &\beta H(\alpha) + (1-\beta)H\left(\frac{\alpha\beta}{1-\beta}\right) \\ &= \beta H(1-\alpha) + (1-\beta)H\left(\frac{\alpha\beta}{1-\beta}\right) \\ &< H\left(\beta(1-\alpha) + \frac{(1-\beta)\alpha\beta}{1-\beta}\right) \\ &= H(\beta) \end{aligned}$$

and it follows that f is strictly positive.

THEOREM 2. *For each $\alpha,\beta > 0$ such that $\alpha+\beta < 1$ there exist $c > 0$ and $n_0 \ge 1$ such that $A(n, 2\alpha\beta n, \lfloor\beta n\rfloor) > c2^{f(\alpha,\beta)n}/\sqrt{n}$ for all $n \ge n_0$.*

PROOF. Given $n \ge 1$, define a graph on $\{x \in \{0,1\}^n : x \text{ has weight } w\}$, where $w := \lfloor\beta n\rfloor$, by joining two words if they are at Hamming distance $\le 2\alpha\beta n$. This graph is regular, of degree

$$m := \sum_{1\le k\le\alpha\beta n} \binom{w}{k}\binom{n-w}{k}.$$

We now bound m from above. Note that $\binom{w}{k-1}\binom{n-w}{k-1} \leq \binom{w}{k}\binom{n-w}{k}$ iff $k \leq (w+1)(n-w+1)/(n+2)$. But we have $k \leq \alpha\beta n$, and it is easily checked that

$$\alpha\beta n < \frac{(\beta n - \theta + 1)(n - \beta n + \theta + 1)}{n+2} \text{ for any } 0 \leq \theta < 1.$$

Thus, $k < (w+1)(n-w+1)/(n+2)$, the last term in the sum is the largest, and

$$m < \lfloor \alpha\beta n \rfloor \binom{w}{\lfloor \alpha\beta n \rfloor} \binom{n-w}{\lfloor \alpha\beta n \rfloor}.$$

Therefore,

$$A(n, 2\alpha\beta n, w) \geq \frac{\binom{n}{w}}{\lfloor \alpha\beta n \rfloor \binom{w}{\lfloor \alpha\beta n \rfloor} \binom{n-w}{\lfloor \alpha\beta n \rfloor}} =: G(\alpha, \beta, n). \tag{1}$$

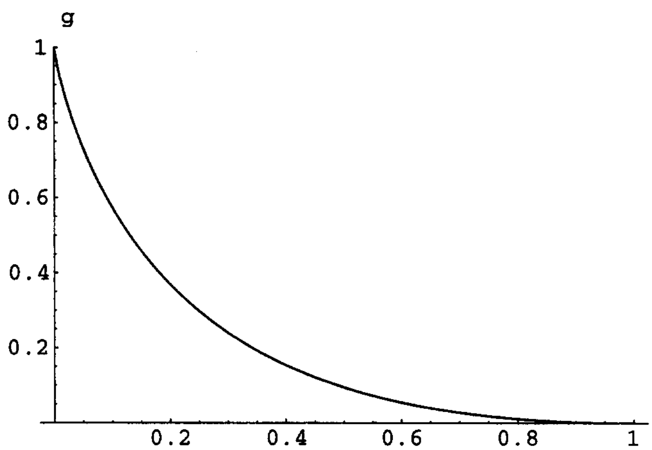
By Stirling's formula we now obtain

$$\begin{aligned} \log_2 G(\alpha, \beta, n) &= n\left(H(\beta) - \beta H(\alpha) - (1-\beta) H\left(\frac{\alpha\beta}{1-\beta}\right)\right) \\ &\quad - \frac{1}{2}\log_2 n + O(1) \\ &= f(\alpha, \beta) n - \frac{1}{2}\log_2 n + O(1). \end{aligned}$$

For fixed α, we now want to choose a β so as to maximize $f(\alpha, \beta)$. Define $g(\alpha) = \max_{0<\beta<1-\alpha} f(\alpha, \beta)$. We now give an analysis of g, showing that it is smooth and strictly decreasing on $(0, 1)$. The approximate values of g were calculated and graphed using *Mathematica.* See Figure 1.

α	$g(\alpha)$
0	1
0.1	0.574061
0.2	0.369609
0.3	0.240239
0.4	0.153444
0.5	0.094158
0.6	0.053899
0.7	0.027378
0.8	0.011074
0.9	0.002536
1	0

(a)



(b)

Figure 1.

For fixed $0 < \alpha < 1$,

$$\lim_{\beta \to 0_+} f(\alpha, \beta) = \lim_{\beta \to 1-\alpha_-} f(\alpha, \beta) = 0,$$

$$\lim_{\beta \to 1-\alpha_-} \frac{\partial f}{\partial \beta} = 0,$$

and

$$\frac{\partial^2 f}{\partial \beta^2} > 0, \qquad \text{iff } \beta > \frac{1-\alpha}{1+\alpha},$$

as can be verified by calculation. Thus, as β increases from 0, $\frac{\partial f}{\partial \beta}$ decreases until β reaches $(1-\alpha)/(1+\alpha)$, then increases to 0 as $\beta \to 1-\alpha_-$. Thus, $\frac{\partial f}{\partial \beta} = 0$ at a unique $\beta = h(\alpha) < (1-\alpha)/(1+\alpha)$ in the interval $0 < \beta < 1-\alpha$, and $f(\alpha, \cdot)$ attains its maximum here. The implicit function theorem gives that h is smooth. Since now $g(\alpha) = f(\alpha, h(\alpha))$, g is also smooth. Since $\frac{\partial f}{\partial \beta}|_{\beta=h(\alpha)} = 0$, we obtain

$$g'(\alpha) = \frac{\partial f}{\partial \alpha} = -\beta\left(H'(\alpha) + H'\left(\frac{\alpha\beta}{1-\beta}\right)\right).$$

Also, $g(0_+) = 1$ and $g(1_-) = 0$. Note that $H'(1-\alpha) = -H'(\alpha)$. Since $1-\beta > \alpha$, $-H'(1-\beta) > -H'(\alpha)$ and $\alpha\beta/(1-\beta) < \beta$. Thus, $H'(\alpha\beta/(1-\beta)) > H'(\beta) = -H'(1-\beta) > -H'(\alpha)$, and $\frac{\partial f}{\partial \alpha} < 0$. It follows that g is strictly decreasing.

3. PROOF OF THEOREM 1

We now finish the proof of Theorem 1. Let $\beta = h(1/2)$ and $\gamma = 0.094158 < g(1/2)$. To bound the weak Hadwiger number from below, it is sufficient to find a set of at least $2^{\gamma d}$ unit vectors in ℓ_p^d, such that the distance between any two vectors is strictly greater than 1, see [3]. By Theorem 2 there is a binary code of length d, weight $\lfloor \beta d \rfloor$, minimum distance greater than βd, and size at least $2^{\gamma d}$. Considering the words in the code as $(0,1)$ vectors in $\mathbb{R}^d$, we find $\|x\|_p = \lfloor \beta d \rfloor^{1/p}$ for each word x, and $\|x-y\|_p > (\beta d)^{1/p}$ for distinct words x and y. A rescaling gives the required unit vectors.

REFERENCES

1. D. Cieslik, Knotengrade kürzester Bäume in endlichdimensionalen Banachräumen, *Rostock Math. Kolloq.* **39**, 89–93, (1990).
2. D. Cieslik, The 1-Steiner-minimal-tree problem in Minkowski-spaces, *Optimization* **22**, 291–296, (1991).
3. G. Robins and J.S. Salowe, Low-degree minimum spanning trees, *Discrete Comput. Geom.* **14**, 151–165, (1995).
4. R.L. Graham and N.J.A. Sloane, Lower bounds for constant weight codes, *IEEE Trans. Inform. Theory* **26**, 37–43, (1980).
5. A.D. Wyner, Capabilities of bounded discrepancy decoding, *Bell System Tech. J.* **44**, 1061–1122, (1965).
6. J.G.v.d. Corput and G. Schaake, Anwendung einer Blichfeldtschen Beweismethode in der Geometrie der Zahlen, *Acta Arithmetica* **2**, 152–160, (1937).
7. R.A. Rankin, On sums of powers of linear forms I, II, *Ann. of Math.* **50**, 691–704, (1949).
8. R.A. Rankin, The closest packing of spherical caps in n dimensions, *Proc. Glasgow Math. Assoc.* **2**, 139–144, (1955).