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Matrix valued orthogonal polynomials of the Jacobi type

Dedicated to Tom Koornwinder on the occasion of his 60th birthday

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ABSTRACT

The main purpose of this paper is to present new families of Jacobi type matrix valued orthogonal polynomials.

1. INTRODUCTION

The work of Tom Koornwinder over the last 35 years or so has had a profound effect on many areas of the theory of orthogonal polynomials. He has opened up new fields and has made important progress in existing ones. Here we consider one of the areas where Tom Koornwinder has made a pioneering contribution: the theory of matrix valued and vector valued orthogonal polynomials.

Starting with the work of M. G. Krein, [K1] and [K2], as well as more recent contributions, including for instance [D1,D2,D3,D4], [DP], [DvA], [Ge], [MS] and [SvA], there is a nice and general theory of matrix valued orthogonal polynomials. These are bound to play an important part in many areas of mathematics, just as their scalar counterparts.

The optimistic statement above would be easier to justify if one had lots of interesting examples of (truly) vector valued or matrix valued orthogonal

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polynomials with rich properties such as the existence of a differential operator for which the orthogonal polynomials in question are eigenfunctions. This problem was first considered in [D1]. The collection of known examples enjoying this extra property is very small. For a few of them, most of them reducible to the scalar case, see [D1], [CMV], [J1] and [J2]. For an example (in arbitrary dimension), not reducible to the scalar case, see [GPT1,GPT2] and the closing paragraph in [GPT3]. For recent progress in this area, including a general method to attack this problem and a relevant hierarchy of examples, see [DG1].

One of the first examples in the vector valued case is found in [Ko] in connection with representations of the group $SU(2) \times SU(2)$. The theory of group representations was, once again, the source of a family of matrix valued examples intimately connected to the matrix valued spherical functions, see [T1] and [GV], for the complex projective plane. For details see [GPT1] and [GPT2] as well as the closing paragraphs in [GPT3].

The aim of this paper is to define the notion of a classical pair $\{W, D\}$ consisting of a matrix valued weight function $W(t)$ and a second order symmetric differential operator D. This is then expressed as a set of differential equations involving $W(t)$ and the coefficients of D. This fits in the general approach in [D1] and can be seen as a matrix version of the original Bochner's problem, see [B]. This is the earliest version of the *bispectral problem*, see [DG] and [GH].

Then we make the strong assumption that D is of hypergeometric type with a leading coefficient of the form $t(1-t)I$ but make no other commutativity assumptions. Also we assume that the weight function is of the form $W(t) = t^{\alpha}(1-t)^{\beta}F(t)$ supported in $[0,1]$, $\alpha, \beta > -1$, where $F(t)$ is a matrix valued polynomial function. In the first non scalar case corresponding to matrices of size two we manage to solve these equations in a few instances, yielding families of examples that do not reduce to the scalar case. We make connections with the results in [GPT1], [GPT2], [GPT3], [G] and [T2].

2. MATRIX AND VECTOR VALUED ORTHOGONAL POLYNOMIALS

Here we gather some standard facts. All matrices below are $r \times r$ matrices and all vectors are row or column vectors of r components.

Given a self adjoint positive definite matrix valued weight function $W(t)$ we can consider the skew symmetric bilinear form defined for any pair of matrix valued functions $P(t)$ and $Q(t)$ by the numerical matrix

$$(P, Q) = (P, Q)_W = \int_{\mathbb{R}} P(t) W(t) Q^*(t) dt,$$

where $Q^*(t)$ denotes the conjugate transpose of $Q(t)$. Also we consider the inner product defined for any pair of row vector valued functions $H(t)$ and $K(t)$ by the complex number

$$(H, K) = (H, K)_W = \int_{\mathbb{R}} H(t) W(t) K^*(t) dt.$$

For a general reference to matrix orthogonal polynomials, see [GLR]. For the case of orthogonal matrix polynomials on the unit circle, see [DGK].

If ${}^iH$ and ${}^jK$ denote the i-th and j-th rows of P and Q, respectively, then we have

$$(P, Q)_{i,j} = ({}^iH, {}^jK).$$

From now on we will assume that (H, H) is finite for all row vector polynomial functions $H = H(t)$. By considering the subspaces V_n of all row vector polynomial functions of degree less or equal to n, it is clear how we can construct inductively an orthonormal basis $\{{}^iH_n\}_{1\le i\le r, 0\le n}$ of $V = \oplus V_n$ such that $\{{}^iH_m\}_{1\le i\le r, 0\le m\le n}$ is a basis of V_n for all $n \ge 0$.

If P_n is the matrix valued polynomial function whose i-th row is ${}^iH_n$ we have a sequence $\{P_n\}_{n\ge 0}$ of matrix valued polynomials such that P_n is of degree n, the leading coefficient is a nonsingular matrix and

$$(1) \qquad (P_m, P_n) = \delta_{m,n} I.$$

Such a sequence is by definition *an orthonormal sequence of matrix valued polynomials*. If instead of the identity matrix I appearing in (1) we have

$$(P_m, P_n) = \delta_{m,n} M_n,$$

where $M_n = \mathrm{diag}(\mu_n^1, \ldots, \mu_n^r)$ is a positive diagonal matrix, we say that $\{P_n\}_{n\ge 0}$ is a *sequence of orthogonal polynomials*.

One important remark is that given an orthonormal sequence $\{P_n(t)\}_{n\ge 0}$ one gets another one by choosing an arbitrary sequence of unitary matrices U_n and replacing $P_n(t)$ by $U_n P_n(t)$. Also if $\{P_n(t)\}_{n\ge 0}$ is a sequence of orthogonal polynomials then $M_n^{-1/2} P_n$ is an orthonormal one.

Given an orthonormal sequence $\{P_n(t)\}_{n\ge 0}$ one gets by the usual argument a three term recursion relation

$$(2) \qquad tP_n(t) = A_{n+1}P_{n+1}(t) + B_n P_n(t) + C_{n-1}P_{n-1}(t),$$

where A_{n+1} is nonsingular, $B_n^* = B_n$ and $C_{n-1} = A_n^*$.

We now turn our attention to an important class of orthogonal polynomials which we will call *classical matrix valued orthogonal polynomials*.

3. CLASSICAL MATRIX AND VECTOR VALUED ORTHOGONAL POLYNOMIALS

The skew symmetric bilinear form introduced in Section 2 is not the only possible such choice, as noticed for instance in [SvA]. In this section we will also consider the form

$$\langle P, Q \rangle = (P^*, Q^*)^*.$$

The reason for considering this form can be traced back to [GPT1] as will be noticed below. Observe that a sequence $\{P_n\}_{n\ge 0}$ of matrix valued polynomials is orthogonal with respect to $(\cdot,\cdot)$ if and only if the sequence $\{P_n^*\}_{n\ge 0}$ is orthogonal with respect to $\langle \cdot,\cdot \rangle$.

We say that the weight function is *classical* if there exists a second order ordinary differential operator D with matrix valued polynomial coefficients $A_j(t)$ of degree less or equal to j of the form

$$(3) \qquad D = A_2(t)\frac{d^2}{dt^2} + A_1(t)\frac{d}{dt} + A_0(t),$$

such that

$$(4) \qquad \langle DP, Q\rangle = \langle P, DQ\rangle$$

for all matrix valued polynomial functions P and Q.

We refer to such a pair $\{W, D\}$ as a *classical pair.* If $\{W, D\}$ is a classical pair then there exists an orthonormal sequence $\{P_n\}$, with respect to $(\cdot,\cdot)$, of matrix valued polynomials such that

$$(5) \qquad DP_n^* = P_n^* \Lambda_n,$$

where Λ_n is a real valued diagonal matrix.

In fact, let us consider the subspaces E_n of all column vector polynomial functions of degree less or equal to n, equipped with the inner product

$$(6) \qquad \langle H^*, K^*\rangle = \overline{(H, K)} \qquad H^*, K^* \in E_n.$$

Then, by observing that ${}^iH^*$ is the i-th column of P^* if ${}^iH$ is the i-th row of P, and that $\langle DP, Q\rangle = \langle P, DQ\rangle$ for all $P, Q \in V_n$ is equivalent to $\langle DH^*, K^*\rangle = \langle H^*, DK^*\rangle$ for all $H^*, K^* \in E_n$, it is clear how we can find inductively a basis $\{{}^iH_n^*\}_{1\le i\le r, 0\le n}$ of $E = \oplus E_n$ such that $\{{}^iH_m^*\}_{1\le i\le r, 0\le m\le n}$ is a basis of E_n for all $n \geq 0$ and such that

$$\langle {}^iH_m^*, {}^jH_n^*\rangle = \delta_{i,j}\delta_{m,n} \quad \text{and} \quad D\,{}^iH_n^* = \lambda_n^i\,{}^iH_n^*,$$

with $\lambda_n^i \in \mathbb{R}$.

Now if P_n is the matrix whose i-th row is ${}^iH_n$ then $\{P_n\}_{n\geq 0}$ is a sequence of orthonormal polynomials with respect to $(\cdot,\cdot)$ such that

$$DP_n^* = P_n^* \Lambda_n,$$

where $\Lambda_n = \mathrm{diag}(\lambda_n^1, \ldots, \lambda_n^r)$.

Conversely, if D is a differential operator as in (3) such that there exists a sequence $\{P_n\}_{n\geq 0}$ of orthogonal polynomials such that $DP_n^* = P_n^* \Lambda_n$ where Λ_n is a diagonal real numerical matrix, then $\{W, D\}$ is a classical pair.

This form of the eigenvalue equation (5) appears naturally in [GPT1] and corresponds to the fact that the *rows* of P_n are eigenfunctions of D.

Expressions (2) and (5) involve a difference operator acting on n and a differential operator acting on t. While these operators commute in the scalar case, they fail to do so in the matrix valued situation. Having D act on P_n^* instead of on P_n in (5) restores this commutativity. Alternatively, one can consider right or left handed differential operators as in [D1].

A look at [GPT1], [GPT2] and the last section of [GPT3] shows that the main diagonals of the matrix valued functions associated to the spherical functions discussed in those references are closely tied to a sequence of vector valued orthogonal polynomials as above. In these papers the objects that appear naturally are these vector valued functions, which are then properly packaged into matrix valued functions.

If S is a nonsingular matrix and $\{W, D\}$ is a classical pair then SWS^* is another self adjoint positive definite weight function. Moreover, if P_n is a sequence of orthogonal polynomials with respect to W such that $DP_n^* = P_n^* \Lambda_n$ with Λ_n a diagonal matrix, then

$$(P_m S^{-1}, P_n S^{-1})_{SWS^*} = (P_m, P_n)_W$$

and

$$\Big((S^*)^{-1} D S^*\Big)(P_n S^{-1})^* = (S^*)^{-1} P_n^* \Lambda_n = (P_n S^{-1})^* \Lambda_n.$$

Therefore $\{\widetilde{W}, \widetilde{D}\} = \{SWS^*, (S^*)^{-1} D S^*\}$ is also a classical pair. In such a case we say that $\{\widetilde{W}, \widetilde{D}\}$ and $\{W, D\}$ are *equivalent*.

This notion of equivalence leads us to distinguish among some situations where the weight matrices are related by a matrix S as above *but the corresponding differential operators are not*. An instance of this phenomenon is given in example 5.1 below.

4. THE CONDITIONS FOR SYMMMETRY

Now we shall assume that the weight function $W = W(t)$ is supported in the interval (a, b). Then it is a simple matter of careful integration by parts to see that the condition of symmetry (4) is equivalent to the following three differential equations

$$\tag{7} \begin{aligned} A_2^* W &= W A_2 \\ A_1^* W &= -W A_1 + 2\frac{d}{dt}(W A_2) \\ A_0^* W &= W A_0 - \frac{d}{dt}(W A_1) + \frac{d^2}{dt^2}(W A_2), \end{aligned}$$

with the boundary conditions

$$\lim_{t \to x} W(t) A_2(t) = 0 = \lim_{t \to x} \big(W(t) A_1(t) - A_1^*(t) W(t)\big),$$

for $x = a, b$.

The second condition can be replaced by

$$\lim_{t \to x} \Big(W(t) A_1(t) - \frac{d}{dt}\big(W(t) A_2(t)\big)\Big) = 0.$$

These equations are quite general and do not depend on the assumptions that the matrix valued coefficients of D are polynomials nor that the interval (a, b) be finite.

In [DG1] a general method to handle these equations is presented. This method should allow one to produce many examples of classical pairs $\{W, D\}$

of differents types in line with the Jacobi, Hermite, Laguerre examples in the scalar case. In the next section we give some interesting families of classical pairs $\{W, D\}$ of the Jacobi type obtained without using this method. These pairs will have a weight function of the form $W(t) = t^{\alpha}(1-t)^{\beta}F(t)$ supported in $[0, 1]$, $\alpha, \beta > -1$, where $F(t)$ is a matrix valued polynomial function of degree d.

In [GPT1], [GPT2] and [GPT3] we found pairs $\{W, D\}$ for any value of r, $\alpha = n \in \mathbb{Z}$ and $\beta = 1$. In [G] a pair $\{W, D\}$ with $r = 2$ and α, β arbitrary is exhibited. In the next section and in the case $r = 2$ we discuss three *one parameter families* of examples. In the first one $F(t)$ has degree 1. The second one includes the example in [G], and as well as the third example, features an $F(t)$ of degree 2. In each case a sequence of matrix valued orthogonal polynomials with respect to W satisfying $DP_n^* = P_n^* \Lambda_n$ is given in terms of the recently developed concept of *the matrix valued hypergeometric function*, see [T2]. Notice that in [G] ones gets away from *group values.*

5. JACOBI TYPE EXAMPLES

In this section we will give explicit classical pairs $\{W, D\}$ with

$$W(t) = t^{\alpha}(1-t)^{\beta}F(t) \qquad \alpha, \beta > -1, \quad 0 < t < 1,$$

and

$$D = t(1-t)\frac{d^2}{dt^2} + (X - tU)\frac{d}{dt} + V,$$

where X, U, V are constant matrices. In this case it is easy to observe that the first boundary conditions are automatically satisfied, while the second ones follow from the differential equations when $\alpha \neq 0$ and $\beta \neq 0$, respectively.

All matrices will be of size two. In 5.1 $F(t)$ will be of degree one and in 5.2, 5.3 it will be of degree two,

$$F(t) = t^2 F_2 + tF_1 + F_0$$

with F_0, F_1, F_2 constant self adjoint matrices.

In cases 5.2 an 5.3 the equations (7) lead to a system of quadratic equations in a total of $21 = 3 \times 3 + 3 \times 4$ unknowns, an imposible task. Nevertheless we have managed to find in each case a one parameter family of solutions by imposing a number of mild assumptions on the pair $\{W, D\}$ and by replacing it by an equivalent one. In case 5.1 we have 18 unknowns and we were able to find a complete solution. We list these conditions below.

We assume that $\{W, D\}$ has a sequence of matrix valued orthogonal polynomials $\{P_n\}_{n \geq 0}$ such that $P_0(t) = I$ and $DP_n^* = P_n^* \Lambda_n$, Λ_n diagonal. By a proper conjugation by a unitary matrix we can take U to be upper triangular. This implies that V is diagonal and that Λ_n is a diagonal matrix polynomial function of n of degree 2, with leading coefficient equal to -1.

With these requirements at hand we get a more manageable system of quadratic equations. In 5.2 and 5.3 we further choose the $(1,2)$ entry of U to be -1. In these cases, as stated earlier, we find two families depending on a free parameter denoted by u. We determine in each case the range of u giving a positive definite $F(t)$.

Finally we show how in each case certain choice of orthogonal polynomials associated to the classical pair $\{W, D\}$ is given in terms of an appropriate matrix valued hypergeometric function introduced in [T2]. Such an expression was already obtained in that paper for the example in [G] which happens to be included in the one parameter family given in 5.2.

Let V be a finite dimensional complex vector space. Given $A, B, C \in \mathrm{End}(V)$ let us consider the following hypergeometric equation

$$(8) \qquad z(1-z)\Phi'' + (C - z(1+A+B))\Phi' - AB\Phi = 0,$$

where Φ denotes a function on $\mathbb{C}$ with values in V.

If the eigenvalues of C are not among $0, -1, -2, \ldots$ then we introduce the following notation

$$(C, A, B)_{m+1} = (C+m)^{-1}(A+m)(B+m)(C+m-1)^{-1}(A+m-1)(B+m-1)\cdots C^{-1}AB.$$

for all $m \geq 0$ and $(C, A, B)_0 = 1$. Then we define

$${}_2F_1\left(\begin{matrix} A,\ B \\ C \end{matrix} ; z\right) = \sum_{n\geq 0} \frac{z^n}{n!}(C, A, B)_n.$$

In [T2] it is proved that ${}_2F_1\left(\begin{smallmatrix} A,B \\ C \end{smallmatrix} ; z\right)$ is analytic on $|z| < 1$ with values in $\mathrm{End}(V)$, and that the analytic solutions at $z = 0$ of (8) are given by

$$\Phi(z) = {}_2F_1\left(\begin{matrix} A,\ B \\ C \end{matrix} ; z\right)\Phi(0).$$

5.1. The following family of classical pairs contains, up to equivalence, all classical pairs of size two where $F(t)$ is of degree one and which are irreducible, in the sense that they are not equivalent to a direct sum of classical pairs of size one.

Let $\{W, D\}$ be the following family of classical pairs parametrized by a real number u and given by

$$W(t) = t^{\alpha}(1-t)^{\beta}F(t), \qquad F(t) = tF_1 + F_0,$$
$$D = t(1-t)\frac{d^2}{dt^2} + (X - tU)\frac{d}{dt} + V,$$

where

$$F_1 = \frac{\alpha+\beta+2}{\alpha+1}\begin{pmatrix} 0 & -1 \\ -1 & \frac{\beta-\alpha}{\alpha+1} \end{pmatrix}, \qquad F_0 = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix},$$

$$V = \begin{pmatrix} 0 & 0 \\ 0 & -(\alpha+\beta+2)(1+u) \end{pmatrix}, \quad X = \begin{pmatrix} \alpha+1-\frac{(\alpha+1)u}{\alpha+\beta+2} & -\frac{(\alpha+1)u}{\alpha+\beta+2}-1 \\ \frac{(\alpha+1)u}{\alpha+\beta+2} & \alpha+2+\frac{(\alpha+1)u}{\alpha+\beta+2} \end{pmatrix},$$

$$U = \begin{pmatrix} \alpha+\beta+2-u & -\frac{(\alpha-\beta)(1+u)}{\alpha+1} \\ 0 & \alpha+\beta+4+u \end{pmatrix}.$$

A sequence of matrix valued orthogonal polynomials P_n is then obtained by solving the following differential equation

$$(9) \qquad t(1-t)\frac{d^2}{dt^2}P_n^* + (X - tU)\frac{d}{dt}P_n^* + VP_n^* - P_n^*\Lambda_n = 0,$$

where

$$\Lambda_n = \begin{pmatrix} -n(n+\alpha+\beta+1-u) & 0 \\ 0 & -(n+\alpha+\beta+2)(n+1+u) \end{pmatrix}.$$

Notice that the weight matrix $W(t)$ is *independent* of the parameter u, and that the family of differential operators (depending on u) is a *commutative family* having a sequence of matrix valued orthogonal polynomials as a set of common eigenfunctions. It is also clear that from this family one can obtain a *first order* differential operator having our P_n^* as eigenfunctions. This connects this example with results in [GI], and the observation above is due to P. Iliev.

The term $P_n^*\Lambda_n$ in (9) forces us to consider this equation as a differential equation on functions which take values on $\mathbb{C}^4$ and to consider the left and right multiplication by matrices in $M(2,\mathbb{C})$ as linear maps in $\mathbb{C}^4$. Thus instead of (9) we shall consider the following equivalent differential equation

$$(10) \qquad t(1-t)\frac{d^2}{dt^2}P_n^* + (C - t\tilde{U})\frac{d}{dt}P_n^* - \tilde{T}P_n^* = 0,$$

where

$$C = \begin{pmatrix} \left(\alpha+1-\frac{(\alpha+1)u}{\alpha+\beta+2}\right)I & -\left(1+\frac{(\alpha+1)u}{\alpha+\beta+2}\right)I \\ \frac{(\alpha+1)u}{\alpha+\beta+2}I & \left(\alpha+2+\frac{(\alpha+1)u}{\alpha+\beta+2}\right)I \end{pmatrix},$$

$$\tilde{U} = \begin{pmatrix} (\alpha+\beta+2-u)I & -\frac{(\alpha-\beta)(1+u)}{\alpha+1}I \\ 0 & (\alpha+\beta+4+u)I \end{pmatrix},$$

$$\tilde{T} = \begin{pmatrix} t_1 & 0 & 0 & 0 \\ 0 & t_2 & 0 & 0 \\ 0 & 0 & t_3 & 0 \\ 0 & 0 & 0 & t_4 \end{pmatrix},$$

where I in the matrices above denotes the 2×2 identity matrix, and

$$t_1 = -n(n+\alpha+\beta+1-u), \quad t_2 = -(n+\alpha+\beta+2)(n+1+u),$$
$$t_3 = -(n+\alpha+\beta+2)(n-1-u), \quad t_4 = -n(n+\alpha+\beta+3+u).$$

It is easy to verify that the matrices A and B given below satisfy: $\tilde{U} = 1 + A + B$, $\tilde{T} = AB$.

$$A = \begin{pmatrix} x_1 & 0 & \frac{(\alpha-\beta)(1+u)x_1}{(\alpha+1)(x_1+x_3-\alpha-\beta-3-u)} & 0 \\ 0 & y_1 & 0 & \frac{(\alpha-\beta)(1+u)y_1}{(\alpha+1)(y_1+y_3-\alpha-\beta-3-u)} \\ 0 & 0 & x_3 & 0 \\ 0 & 0 & 0 & y_3 \end{pmatrix},$$
$$B = \tilde{U} - A - 1.$$

The parameters x_1, x_3, y_1, y_3 are only subject to the following conditions

$$x_1 = -n \text{ or } x_1 = n+\alpha+\beta+1-u,$$
$$y_1 = n+\alpha+\beta+2 \text{ or } y_1 = -n-1-u,$$
$$x_3 = n+\alpha+\beta+2 \text{ or } x_3 = -n+1+u,$$
$$y_3 = -n \text{ or } y_3 = n+\alpha+\beta+3+u.$$

It is important to notice that spec(C) $= \{\alpha+1, \alpha+2\}$ where each eigenvalue has multiplicity 2. We also remark that $(A+k)(B+k)$ is, generically, non-singular for $k \neq n$, and that the kernel of $(A+n)(B+n)$ is two dimensional. Thus ${}_2F_1\left({A,B \atop C}; z\right)$ is not a polynomial function, as in the classical case, but nevertheless we have

$$P_n^*(t) = {}_2F_1\left({A, B \atop C}; t\right) P_n^*(0).$$

5.2. Let $\{W, D\}$ be the following family of classical pairs parametrized by a real number u such that $\frac{1}{u} > 1 + \frac{\alpha+1}{\beta+1}$ and given by

$$F(t) = t^2F_2 + tF_1 + F_0, \qquad D = t(1-t)\frac{d^2}{dt^2} + (X - tU)\frac{d}{dt} + V,$$

where

$$F_2 = \begin{pmatrix} 0 & 0 \\ 0 & \frac{1}{u^2} \end{pmatrix}, \quad F_1 = \begin{pmatrix} \frac{\beta+1}{(\alpha+1)u} - \frac{\alpha+\beta+2}{\alpha+1} & \frac{\alpha+\beta+2}{(\alpha+1)u} - \frac{\alpha+\beta+2}{\alpha+1} \\ \frac{\alpha+\beta+2}{(\alpha+1)u} - \frac{\alpha+\beta+2}{\alpha+1} & \frac{2\alpha+\beta+3}{(\alpha+1)u} - \frac{2}{u^2} - \frac{\alpha+\beta+2}{\alpha+1} \end{pmatrix},$$
$$F_0 = \begin{pmatrix} 1 & 1-\frac{1}{u} \\ 1-\frac{1}{u} & \left(1-\frac{1}{u}\right)^2 \end{pmatrix},$$
$$X = \begin{pmatrix} \alpha+2-u & 1-u \\ u & \alpha+1+u \end{pmatrix}, \quad U = \begin{pmatrix} \alpha+\beta+3 & -1 \\ 0 & \alpha+\beta+4 \end{pmatrix},$$
$$V = \begin{pmatrix} 0 & 0 \\ 0 & -\frac{\alpha+1}{1-u} \end{pmatrix}.$$

The example given in [G] corresponds to $u = (\beta+1)/(2\alpha+\beta+3)$.

A sequence of matrix valued orthogonal polynomials P_n is then obtained by solving the following differential equation

$$t(1-t)\frac{d^2}{dt^2}P_n^* + (X - tU)\frac{d}{dt}P_n^* + VP_n^* - P_n^*\Lambda_n = 0, \tag{11}$$

where

$$\Lambda_n = \begin{pmatrix} -n^2-(\alpha+\beta+2)n & 0 \\ 0 & -n^2-(\alpha+\beta+3)n-\frac{\alpha+1}{1-u} \end{pmatrix}.$$

The term $P_n^*\Lambda_n$ in (11) forces us to consider this equation as a differential equation on functions which take values on $\mathbb{C}^4$ and to consider the left and right multiplication by matrices in $M(2,\mathbb{C})$ as linear maps in $\mathbb{C}^4$. Thus instead of (11) we shall consider the following equivalent differential equation

$$t(1-t)\frac{d^2}{dt^2}P_n^* + (C - t\tilde{U})\frac{d}{dt}P_n^* - \tilde{T}P_n^* = 0, \tag{12}$$

where

$$C = \begin{pmatrix} (\alpha+2-u)I & (1-u)I \\ uI & (\alpha+1+u)I \end{pmatrix},$$

$$\tilde{U} = \begin{pmatrix} (\beta+\alpha+3)I & -I \\ 0 & (\alpha+\beta+4)I \end{pmatrix},$$

$$\tilde{T} = \begin{pmatrix} t_1 & 0 & 0 & 0 \\ 0 & t_2 & 0 & 0 \\ 0 & 0 & t_3 & 0 \\ 0 & 0 & 0 & t_4 \end{pmatrix},$$

where I in the matrices above denotes the 2×2 identity matrix, and

$$t_1 = -n(n+\alpha+\beta+2), \quad t_2 = -n(n+\alpha+\beta+3) - \frac{\alpha+1}{1-u},$$
$$t_3 = -n(n+\alpha+\beta+2) + \frac{\alpha+1}{1-u}, \quad t_4 = -n(n+\alpha+\beta+3).$$

It is easy to verify that the matrices A and B given below satisfy: $\tilde{U} = 1 + A + B$, $\tilde{T} = AB$.

$$A = \begin{pmatrix} x_1 & 0 & \frac{-x_1}{x_1+x_3-\alpha-\beta-3} & 0 \\ 0 & y_1 & 0 & \frac{-y_1}{y_1+y_3-\alpha-\beta-3} \\ 0 & 0 & x_3 & 0 \\ 0 & 0 & 0 & y_3 \end{pmatrix},$$

$$B = \tilde{U} - A - 1.$$

The parameters x_1, x_3, y_1, y_3 are only subject to the following conditions

$$
\begin{aligned}
&x_1 = -n \text{ or } x_1 = n+\alpha+\beta+2,\\
&y_3 = -n \text{ or } y_3 = n+\alpha+\beta+3,\\
&(u-1)y_1^2 - (\alpha+\beta+2)(u-1)y_1 - n(n+\alpha+\beta+3)(u-1)+\alpha+1 = 0,\\
&(u-1)x_3^2 - (\alpha+\beta+3)(u-1)x_3 - n(n+\alpha+\beta+3)(u-1)-\alpha-1 = 0.
\end{aligned}
$$

It is important to notice that $\operatorname{spec}(C) = \{\alpha+1, \alpha+2\}$ where each eigenvalue has multiplicity 2. We also remark that $(A+k)(B+k)$ is, generically, non-singular for $k \neq n$, and that the kernel of $(A+n)(B+n)$ is two dimensional. Thus ${}_2F_1\left({A;B \atop C}; z\right)$ is not a polynomial function, as in the classical case, but nevertheless we have

$$P_n^*(t) = {}_2F_1\left({A;B \atop C}; t\right)P_n^*(0).$$

5.3. Let $\{W, D\}$ be the following family of classical pairs parametrized by a real number u such that $-\frac{1}{u} > 1 + \frac{\beta+1}{\alpha+1}$ and given by

$$F(t) = t^2F_2 + tF_1 + F_0, \qquad D = t(1-t)\frac{d^2}{dt^2} + (X - tU)\frac{d}{dt} + V,$$

where

$$
F_2 = \begin{pmatrix} 0 & 0 \\ 0 & \frac{\beta+1}{\alpha+1}\left(\frac{1}{1+u} - \frac{1}{u}\right) \end{pmatrix},
$$

$$
F_1 = \begin{pmatrix} -\frac{\alpha+\beta+2}{\alpha+1} + \frac{\beta+1}{(\alpha+1)((1+u)} & -\frac{\alpha+\beta+2}{\alpha+1} \\ -\frac{\alpha+\beta+2}{\alpha+1} & -\frac{\alpha+\beta+2}{\alpha+1} - \frac{\beta+1}{(\alpha+1)(1+u)} \end{pmatrix}, \qquad F_0 = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix},
$$

$$
X = \begin{pmatrix} \alpha+1-u & -2-u \\ u & \alpha+3+u \end{pmatrix}, \qquad U = \begin{pmatrix} \alpha+\beta+3 & -1 \\ 0 & \alpha+\beta+4 \end{pmatrix},
$$

$$
V = \begin{pmatrix} 0 & 0 \\ 0 & -\frac{\beta+1}{1+u} \end{pmatrix}.
$$

A sequence of matrix valued orthogonal polynomials P_n is then obtained by solving the following differential equation

$$
t(1-t)\frac{d^2}{dt^2}P_n^* + (X - tU)\frac{d}{dt}P_n^* + VP_n^* - P_n^*\Lambda_n = 0, \tag{13}
$$

where

$$
\Lambda_n = \begin{pmatrix} -n^2 - (\alpha+\beta+2)n & 0 \\ 0 & -n^2 - (\alpha+\beta+1)n - \frac{\beta+1}{1+u} \end{pmatrix}.
$$

The term $P_n^*\Lambda_n$ in (13) forces us to consider this equation as a differential equation on functions which take values on $\mathbb{C}^4$ and to consider the left and right multiplication by matrices in $M(2, \mathbb{C})$ as linear maps in $\mathbb{C}^4$. Thus instead of (13) we shall consider the following equivalent differential equation

$$
t(1-t)\frac{d^2}{dt^2}P_n^* + (C - t\widetilde{U})\frac{d}{dt}P_n^* - \widetilde{T}P_n^* = 0, \tag{14}
$$

where

$$C = \begin{pmatrix} (\alpha+1-u)I & (-2-u)I \\ uI & (\alpha+3+u)I \end{pmatrix},$$

$$\tilde{U} = \begin{pmatrix} (\beta+\alpha+3)I & -I \\ 0 & (\alpha+\beta+4)I \end{pmatrix},$$

$$\tilde{T} = \begin{pmatrix} t_1 & 0 & 0 & 0 \\ 0 & t_2 & 0 & 0 \\ 0 & 0 & t_3 & 0 \\ 0 & 0 & 0 & t_4 \end{pmatrix},$$

where I in the matrices above denotes the 2×2 identity matrix, and

$$t_1 = -n(n+\alpha+\beta+2), \quad t_2 = -n(n+\alpha+\beta+3) - \frac{\beta+1}{1+u},$$
$$t_3 = -n(n+\alpha+\beta+2) + \frac{\beta+1}{1+u}, \quad t_4 = -n(n+\alpha+\beta+3).$$

It is easy to verify that the matrices A and B given below satisfy: $\tilde{U} = 1 + A + B$, $\tilde{T} = AB$.

$$A = \begin{pmatrix} x_1 & 0 & \frac{x_1}{x_1+x_3-\alpha-\beta-3} & 0 \\ 0 & y_1 & 0 & \frac{-y_1}{y_1+y_3-\alpha-\beta-3} \\ 0 & 0 & x_3 & 0 \\ 0 & 0 & 0 & y_3 \end{pmatrix},$$
$$B = \tilde{U} - A - 1.$$

The parameters x_1, x_3, y_1, y_3 are only subject to the following conditions

$$x_1 = -n \text{ or } x_1 = n+\alpha+\beta+2,$$
$$y_3 = -n \text{ or } y_3 = n+\alpha+\beta+3,$$
$$(u+1)y_1^2 - (\alpha+\beta+2)(u+1)y_1 - n(n+\alpha+\beta+3)(u+1) - \beta - 1 = 0,$$
$$(u+1)x_3^2 - (\alpha+\beta+3)(u+1)x_3 - n(n+\alpha+\beta+2)(u+1) + \beta + 1 = 0.$$

It is important to notice that spec(C) $= \{\alpha+1, \alpha+3\}$ where each eigenvalue has multiplicity 2. We also remark that $(A+k)(B+k)$ is, generically, non-singular for $k \neq n$, and that the kernel of $(A+n)(B+n)$ is two dimensional. Thus ${}_2F_1\left({A,B \atop C};z\right)$ is not a polynomial function, as in the classical case, but nevertheless we have

$$P_n^*(t) = {}_2F_1\left({A,B \atop C};t\right)P_n^*(0).$$

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