Asymptotic behavior of Timoshenko beam with dissipative boundary feedback

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Abstract

This paper is concerned with the stabilization problem of Timoshenko beam in the presence of linear dissipative boundary feedback controls. Using $C_0$-semigroups theory we establish the existence and the uniqueness of solution of the proposed closed loop system. In order to consider the asymptotic behavior of the closed loop system, we first discuss the existence of nonzero solution of a closely related boundary value problem. Then we derive various necessary and sufficient conditions for the system to be asymptotically stable. Finally, we prove the equivalence between the exponential stability and the asymptotic stability for the closed loop system. © 2002 Elsevier Science (USA). All rights reserved.

Keywords: Timoshenko beam; Boundary feedback; $C_0$-semigroups; Asymptotic stability; Exponential stability

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1. Introduction

The purpose of this paper is to study the stabilization problem of Timoshenko beam with linear dissipative boundary feedback. The vibration motion of a Timoshenko beam is described as follows (see [1]):

\[
\begin{align*}
\rho \ddot{w} + (K(\varphi - w'))' &= 0, \quad 0 < x < \ell, \ t > 0, \\
I_\rho \ddot{\varphi} - (EI\varphi')' + K(\varphi - w') &= 0, \quad 0 < x < \ell, \ t > 0, \\
w(0, t) &= \varphi(0, t) = 0, \quad t > 0, \\
K(\ell)(\varphi(\ell, t) - w'(\ell, t)) &= u_1(t), \quad t > 0, \\
- EI(\ell)\varphi'(\ell, t) &= u_2(t), \quad t > 0.
\end{align*}
\]

(1.1)

Here and afterwards, the prime and the dot always denote derivatives with respect to space and time variables, respectively.

We apply the following linear boundary feedbacks:

\[
\begin{align*}
\dot{u}_1(t) &= \alpha \dot{w}(\ell, t) + d_1 \dot{\varphi}(\ell, t), \\
\dot{u}_2(t) &= d_2 \dot{w}(\ell, t) + \gamma \dot{\varphi}(\ell, t)
\end{align*}
\]

(1.2)

as the controls to the right end of the beam. The physical meanings of all the other variables, functions and coefficients in (1.1) can be found in [1].

Throughout this paper, we always assume that \(\rho, I_\rho, K, EI \in C^1[0, \ell]\) and \(\rho, I_\rho, K, EI \geq c_0\), where \(c_0\) is a fixed positive constant. We also set

\[
F \triangleq \begin{bmatrix} \alpha & d_1 \\ d_2 & \gamma \end{bmatrix}, \quad B \triangleq \begin{bmatrix} \alpha & \frac{d_1 + d_2}{2} \\ \frac{d_1 + d_2}{2} & \gamma \end{bmatrix}.
\]

Up to now, a large number of interesting results on the boundary feedback stabilization of Timoshenko model have been obtained by many investigators (e.g., see [1–10] and references therein). In this paper, we study the asymptotic behavior of Timoshenko beam with linear dissipative boundary feedback controls.

It is well known that this type of controls can stabilize the Timoshenko beam exponentially if \(\text{rank}(B) = 2\). However, as will be shown below, when \(\text{rank}(B) = 1\), the closed loop system may no longer be asymptotically stable.

In [1], it was proven that under the condition of \(\alpha, \gamma > 0\) and \(d_1 = d_2 = 0\), the energy of the closed loop system (1.1)–(1.2) decays exponentially. Recently in [4], under the condition of \(\text{rank}(B) = 2\) and \(d_1 = d_2\), the exponential stability of the energy of the closed loop system (1.1)–(1.2) with variable coefficients is established. Hence, it is natural to ask, under the condition of \(\text{rank}(B) < 2\), what the asymptotic behavior of the closed loop system (1.1)–(1.2) is going to be. This will be the focus of our investigation in the present paper.

The rest of the paper is organized as follows. In Section 2, the well-posedness of the closed loop system under study is given. In Section 3, first some of the results about the exponential decay of the closed loop system are outlined. Then we consider a necessary and sufficient condition on the uniqueness of the solution to a boundary value problem related to the closed loop system. Some necessary
and sufficient conditions for the closed loop system to be asymptotically stable are also derived. Finally in Section 4, it is shown that for the closed loop system (1.1)–(1.2), the exponential stability and the asymptotic stability are equivalent.

2. Well-posedness of the closed loop system

We now incorporate the closed loop system (1.1)–(1.2) into an appropriate function space. To this end, we define a product Hilbert space \( \mathcal{H} \) by

\[
\mathcal{H} = V_0^1 \times L_\rho^2(0, \ell) \times V_0^1 \times L_\rho^2(I, \ell),
\]

where

\[
V_0^k = \{ \varphi \in H^k(0, \ell) \mid \varphi(0) = 0 \}, \quad k = 1, 2,
\]

and \( H^k(0, \ell) \) is the usual Sobolev space of order \( k \). The inner product in \( \mathcal{H} \) is defined by

\[
(Y_1, Y_2)_\mathcal{H} = \int_0^\ell K(\varphi_1 - w_1')(\varphi_2 - w_2') \, dx + \int_0^\ell EI\varphi_1'\varphi_2' \, dx
\]

\[
+ \int_0^\ell \rho z_1 z_2 \, dx + \int_0^\ell I_\rho \psi_1 \psi_2 \, dx
\]

for \( Y_k = [w_k, z_k, \varphi_k, \psi_k]^T \in \mathcal{H}, k = 1, 2. \) Also we define a linear operator \( A \) in \( \mathcal{H} \) by

\[
A \begin{bmatrix} w \\ z \\ \varphi \\ \psi \end{bmatrix} = \begin{bmatrix} -\rho^{-1}(K(\varphi - w'))' \\ \psi \\ I_\rho^{-1}(EI\varphi)' - I_\rho^{-1}K(\varphi - w') \\ \psi \end{bmatrix}, \quad \begin{bmatrix} w \\ z \\ \varphi \\ \psi \end{bmatrix} \in \mathcal{D}(A),
\]

\[
\mathcal{D}(A) = \{ [w, z, \varphi, \psi]^T \in \mathcal{H} \mid w, \varphi \in V_0^1, z, \psi \in V_0^1, K(\ell)(\varphi(\ell) - w'(\ell)) = \alpha z(\ell) + d_1 \psi(\ell), -EI(\ell)\varphi'(\ell) = d_2 z(\ell) + \gamma \psi(\ell) \},
\]

where the superscript \( \tau \) denotes the transpose of a matrix. Then the closed loop system (1.1)–(1.2) can be written as a linear evolution equation in \( \mathcal{H} \) in the form

\[
\frac{dY(t)}{dt} = AY(t),
\]

where \( Y(t) = [w(x, t), \dot{w}(x, t), \varphi(x, t), \dot{\varphi}(x, t)]^T. \)

Lemma 2.1. Assume \( B \geq 0. \) Then \( A \) generates a \( C_0 \)-semigroup \( T(t) \) of contraction in \( \mathcal{H}. \) Moreover, \( A \) has compact resolvent and \( 0 \in \rho(A). \)
For the proof of the case of $d_1 = d_2$, see [4], and in the case of $d_1 \neq d_2$, the proof is similar and hence it is omitted here.

According to the semigroup theory of linear operators, we have

**Theorem 2.2.** For any $Y_0 \in \mathcal{H}$, (2.1) has a unique weak solution $Y(t) = T(t)Y_0$, where $T(t)$ is the $C_0$-semigroup generated by $A$. Moreover, if $Y_0 \in \mathcal{D}(A)$, then $Y(t) = T(t)Y_0$ is a strong solution to (2.1).

The system (2.1) is said to be exponentially stable if there exist two positive constants $M, \theta$ such that

$$\|T(t)\| \leq M\|Y_0\|\mathcal{H}e^{-\theta t}, \quad \forall Y_0 \in \mathcal{H},$$

where $T(t)$ is the $C_0$-semigroup generated by $A$ in $\mathcal{H}$. The system (2.1) is said to be asymptotically stable if

$$\lim_{t \to \infty} \|T(t)Y_0\| = 0, \quad \forall Y_0 \in \mathcal{H}.$$

### 3. Asymptotic stability

In this section, we discuss the asymptotic behavior of the closed loop system (2.1) under the condition of $B \geq 0$.

The energy of the closed loop system (2.1) is defined as

$$E(t) = \frac{1}{2} \left[ \int_0^\ell EI|\varphi'|^2 \, dx + \int_0^\ell K|\varphi - w'|^2 \, dx + \int_0^\ell \rho|\dot{w}|^2 \, dx + \int_0^\ell I_\rho|\dot{\varphi}|^2 \, dx \right],$$

where $Y(t) = [w(\cdot, t), \dot{w}(\cdot, t), \varphi(\cdot, t), \dot{\varphi}(\cdot, t)]^T$ is the solution to (2.1). For any given initial data $y_0 \in \mathcal{D}(A)$, the time derivative of $E(t)$ along the corresponding solution $Y(t)$ satisfies

$$\dot{E}(t) = \text{Re}(A^*Y(t), Y(t))_{\mathcal{H}} = -[z(\ell, t), \psi(\ell, t)]B[z(\ell, t), \psi(\ell, t)]^T. \quad (3.1)$$

The following proposition can be found in [4].

**Proposition 3.1.** Assume that $B > 0$. Then there exist positive constants $M, \theta$ such that

$$E(t) \leq M\|Y_0\|\mathcal{H}e^{-\theta t}, \quad \forall Y_0 \in \mathcal{H}.$$
In the sequel, we assume that rank \( \text{rank}(B) = 1 \) and for a real constant \( \omega \), let
\[
\Phi(x) \triangleq \begin{bmatrix} w_1(x) & w_2(x) & w_3(x) & w_4(x) \\
 w'_1(x) & w'_2(x) & w'_3(x) & w'_4(x) \\
 \varphi_1(x) & \varphi_2(x) & \varphi_3(x) & \varphi_4(x) \\
 \varphi'_1(x) & \varphi'_2(x) & \varphi'_3(x) & \varphi'_4(x) \end{bmatrix}
\]
be the real fundamental solution matrix of the system governed by
\[
\begin{cases}
(K(w' - \varphi))' + \omega^2 \rho w = 0, \\
(EI\varphi') - K(\varphi - w') + \omega^2 I\rho \varphi = 0
\end{cases}
\]
with \( \Phi(0) \) being the \( 4 \times 4 \) unit matrix. Then the general solution to the above system can be written as
\[
Y(x) = \Phi(x)C, \quad \text{where} \quad Y(x) = [w(x), w'(x), \varphi(x), \varphi'(x)]^T
\]
and \( C = [c_1, c_2, c_3, c_4]^T \), a constant vector. Here all elements of \( \Phi(x) \) depend on \( \omega \), which is omitted for the simplicity of the notations.

For a given \( \omega \in \mathbb{R} \), we denote
\[
\sigma \triangleq \omega(d_1 - d_2)/2, \quad d \triangleq (d_1 + d_2)/2.
\]
Notice that \( B \) can be decomposed into \( B = [t_1, t_2]^T[t_1, t_2] \) with two real constants \( t_1 \) and \( t_2 \), not all zero when \( \text{rank}(B) = 1 \).

According to the criterion for the asymptotic stability in [5] and the compactness of the resolvent of \( \mathcal{A} \), for the energy of the closed loop system (2.1) to be decay asymptotically, it is necessary and sufficient that \( i\mathbb{R} \cap \sigma_p(\mathcal{A}) = \emptyset \). Now let \( i\omega \in i\mathbb{R} \cap \sigma_p(\mathcal{A}) \) and \( Y \in \mathcal{D}(\mathcal{A}) \) be an eigenfunction of \( \mathcal{A} \) corresponding to \( i\omega \). We have
\[
\operatorname{Re}((i\omega - \mathcal{A})Y, Y)_{\mathcal{H}} = -\operatorname{Re}(\mathcal{A}Y, T)_{\mathcal{H}} = [z(\ell), \psi(\ell)]B[z(\ell), \psi(\ell)]^T = 0,
\]
so it follows that
\[
t_1 z(\ell) + t_2 \psi(\ell) = 0.
\]
Without loss of generality, in the sequel, we always assume \( \alpha \neq 0 \), so that \( t_1 \neq 0 \). Multiplying \( t_1 z(\ell) + t_2 \psi(\ell) = 0 \) by \( t_1 \), taking into account \( \alpha = t_1^2 \) and \( t_1 t_2 = d \), we see that
\[
t_1 z(\ell) + t_2 \psi(\ell) = 0 \quad \iff \quad \alpha z(\ell) + d \psi(\ell) = 0.
\]
What is more, since \( \text{rank}(B) = 1 \) and \( (\alpha, d) \neq 0 \), there exists a real constant \( k \) such that \( (d, \gamma) = k(\alpha, d) \) and hence
\[
dz(\ell) + \gamma \psi(\ell) = 0.
\]
Therefore
\[
dz(\ell) + \gamma \psi(\ell) = (d_2 - d)z(\ell) + (dz(\ell) + \gamma \psi(\ell)) = -2^{-1}(d_1 - d_2)z(\ell)
\]
\[
= -i\sigma w(\ell),
\]
\[
\alpha z(\ell) + d_1 \psi(\ell) = (\alpha z(\ell) + d \psi(\ell)) + (d_1 - d)z(\ell) = 2^{-1}(d_1 - d_2)\psi(\ell)
\]
\[
= i\sigma \varphi(\ell),
\]
where we have used the fact that $\psi(\ell) = i\omega \phi$ and $z(\ell) = i\omega w(\ell)$.

Thus the eigenfunction $Y$ satisfies

\[
\begin{align*}
(K(w' - \varphi))' + \omega^2 \rho w &= 0, \\
(EI\varphi') - K(\varphi - w') + \omega^2 I_\rho \phi &= 0, \\
w(0) &= \varphi(0) = 0, \\
\alpha w(\ell) + d\varphi(\ell) &= EI(\ell)\varphi'(\ell) - i\sigma w(\ell) = 0, \\
K(\ell)(\varphi(\ell) - w'(\ell)) - i\sigma \varphi(\ell) &= 0.
\end{align*}
\]

Conversely, if for some $\omega \in \mathbb{R}$, (3.2) has nonzero solution, then $i\omega \in \sigma_p(A)$.

Therefore, according to [5], if (3.2) admits only zero solution for any $\omega \in \mathbb{R}$, then the system (2.1) is asymptotically stable. If for some $\omega \in \mathbb{R}$, (3.2) has nonzero solution, the (3.2) is not stable.

**Lemma 3.2.** Assume that $\text{rank}(B) = 1$, $B \geq 0$ and $0 \neq \omega \in \mathbb{R}$. Then the boundary value problem (3.2) has nontrivial solution if and only if

\[
\begin{align*}
\alpha w_2(\ell) + d\varphi_2(\ell) &= 0, \\
\alpha w_4(\ell) + d\varphi_4(\ell) &= 0, \\
\alpha EI(\ell)\varphi_2'(\ell) + d K(\ell)(\varphi_2(\ell) - w_2'(\ell)) &= 0, \\
\alpha EI(\ell)\varphi_4'(\ell) + d K(\ell)(\varphi_4(\ell) - w_4'(\ell)) &= 0, \\
w_2'(\ell)\varphi_4(\ell) - \varphi_2(\ell)w_4'(\ell) &= 0.
\end{align*}
\]

**Proof.** Define

\[
\Phi_1(x) \triangleq \begin{bmatrix} w_1(x) & w_3(x) \\
w_1'(x) & w_3'(x) \\
\varphi_1(x) & \varphi_3(x) \\
\varphi_1'(x) & \varphi_3'(x) \end{bmatrix}, \quad \Phi_2(x) \triangleq \begin{bmatrix} w_2(x) & w_4(x) \\
w_2'(x) & w_4'(x) \\
\varphi_2(x) & \varphi_4(x) \\
\varphi_2'(x) & \varphi_4'(x) \end{bmatrix}. \]

We know that the general solution $Y(x) = [w(x), w'(x), \varphi(x), \varphi'(x)]^T$ to the first two equations in (3.2) is

\[
Y(x) = \Phi(x)[a_1, a_2, a_3, a_4]^T,
\]

where $a_1$, $a_2$, $a_3$ and $a_4$ are four complex constants.

Since $a_1 = w(0) = 0$, $a_3 = \varphi(0) = 0$, we have

\[
Y(x) = \Phi_2(x)[a_2, a_4]^T. \tag{3.4}
\]

Referring to the boundary conditions of (3.2) at $x = \ell$, we can derive that

\[
\Phi_3(\ell) \begin{bmatrix} a_2 \\
a_4 \end{bmatrix} \triangleq \begin{bmatrix} \alpha w_2(\ell) + d\varphi_2(\ell) \\
\alpha w_4(\ell) + d\varphi_4(\ell) \\
-\alpha w_2(\ell) + d\varphi_2(\ell) \\
-\alpha w_4(\ell) + d\varphi_4(\ell) \end{bmatrix} \begin{bmatrix} a_2 \\
a_4 \end{bmatrix} = 0. \tag{3.5}
\]
Thus it is enough to prove that, under the conditions of the theorem, (3.5) admits nontrivial solution \((a_2, a_4)\) if and only if (3.3) holds true.

For the linear equation (3.5) on \(a_2\) and \(a_4\) to admit nontrivial solution, it is necessary and sufficient that

\[
\text{rank } \Phi_3(\ell) < 2, \tag{3.6}
\]

which implies that

\[
\begin{align*}
\det \begin{bmatrix}
\alpha w_2(\ell) + d\varphi_2(\ell) & \alpha w_4(\ell) + d\varphi_4(\ell) \\
\varphi_2'(\ell) & \varphi_4'(\ell)
\end{bmatrix} &= 0, \\
\det \begin{bmatrix}
\alpha w_2(\ell) + d\varphi_2(\ell) & \alpha w_4(\ell) + d\varphi_4(\ell) \\
\varphi_2(\ell) - w_2'(\ell) & \varphi_4(\ell) - w_4'(\ell)
\end{bmatrix} &= 0, \\
\det \begin{bmatrix}
\alpha w_2(\ell) + d\varphi_2(\ell) & \alpha w_4(\ell) + d\varphi_4(\ell) \\
\varphi_2(\ell) & \varphi_4(\ell)
\end{bmatrix} &= 0, \\
\det \begin{bmatrix}
\alpha w_2(\ell) + d\varphi_2(\ell) & \alpha w_4(\ell) + d\varphi_4(\ell) \\
\varphi_2'(\ell) & \varphi_4'(\ell)
\end{bmatrix} &= 0,
\end{align*}
\tag{3.7}
\]

where we have used the fact that \(i\sigma\) is a pure imaginary number, and \(w_k, \varphi_k\) are all real functions. Now if \((\alpha w_2(\ell) + d\varphi_2(\ell), \alpha w_4(\ell) + d\varphi_4(\ell)) \neq (0, 0)\), then we have

\[
\text{rank } \Phi_2(\ell) = \text{rank } \begin{bmatrix}
\alpha w_2(\ell) + d\varphi_2(\ell) & \alpha w_4(\ell) + d\varphi_4(\ell) \\
w_2'(\ell) & w_4'(\ell) \\
\varphi_2(\ell) & \varphi_4(\ell) \\
\varphi_2'(\ell) & \varphi_4'(\ell)
\end{bmatrix} < 2, \tag{3.8}
\]

from which it follows that

\[
\text{rank } \Phi(\ell) < 4, \tag{3.9}
\]

a contradiction to the definition of the fundamental solution matrix. So if (3.6) holds, we must have

\[
\alpha w_2(\ell) + d\varphi_2(\ell) = 0, \quad \alpha w_4(\ell) + d\varphi_4(\ell) = 0. \tag{3.10}
\]

Thus (3.6) is equivalent to

\[
\begin{align*}
\det \begin{bmatrix}
\text{EI}(\ell)\varphi_2'(\ell) - i\sigma w_2(\ell) & \text{EI}(\ell)\varphi_4'(\ell) - i\sigma w_4(\ell) \\
K(\ell)(\varphi_2(\ell) - w_2'(\ell)) - i\sigma \varphi_2(\ell) & K(\ell)(\varphi_4(\ell) - w_4'(\ell)) - i\sigma \varphi_4(\ell)
\end{bmatrix} &= 0, \\
\det \begin{bmatrix}
\varphi_2'(\ell) & \varphi_4'(\ell) \\
\varphi_2(\ell) - w_2'(\ell) & \varphi_4(\ell) - w_4'(\ell)
\end{bmatrix} &= 0, \\
\det \begin{bmatrix}
w_2'(\ell) & w_4'(\ell) \\
X_2(\ell) & X_4(\ell)
\end{bmatrix} &= 0, \\
\det \begin{bmatrix}
\varphi_2'(\ell) & \varphi_4'(\ell) \\
\varphi_2(\ell) - w_2'(\ell) & \varphi_4(\ell) - w_4'(\ell)
\end{bmatrix} &= 0,
\end{align*}
\tag{3.11}
\]

which, together with (3.10), leads to

\[
\det \begin{bmatrix}
w_2'(\ell) & w_4'(\ell) \\
X_2(\ell) & X_4(\ell)
\end{bmatrix} = 0, \tag{3.12}
\]
where
\[ X_2(\ell) \triangleq \alpha EI(\ell)\varphi'_2(\ell) + dK(\ell)(\varphi_2(\ell) - w'_2(\ell)), \]
\[ X_4(\ell) \triangleq \alpha EI(\ell)\varphi'_4(\ell) + dK(\ell)(\varphi_4(\ell) - w'_4(\ell)). \]

We claim that \( X_2(\ell) = X_4(\ell) = 0. \) In fact, suppose \( (X_2(\ell), X_4(\ell)) \neq (0, 0). \) Then in view of (3.10), we have
\[
\text{rank } \Phi_2(\ell) = \text{rank } \begin{bmatrix} w'_2(\ell) & w'_4(\ell) \\ \varphi'_2(\ell) & \varphi'_4(\ell) \\ X_2(\ell) & X_4(\ell) \end{bmatrix} = \text{rank } \begin{bmatrix} w'_2(\ell) & w'_4(\ell) \\ \varphi'_2(\ell) - w'_2(\ell) & \varphi'_4(\ell) - w'_4(\ell) \end{bmatrix} < 2, \tag{3.13}
\]
contrary to \( \text{rank } \Phi(\ell) = 4. \)

It follows that
\[ \alpha w_2(\ell) + d\varphi_2(\ell) = 0, \quad \alpha w_4(\ell) + d\varphi_4(\ell) = 0 \]
and
\[ \alpha EI(\ell)\varphi'_2(\ell) + dK(\ell)(\varphi_2(\ell) - w'_2(\ell)) = 0, \]
\[ \alpha EI(\ell)\varphi'_4(\ell) + dK(\ell)(\varphi_4(\ell) - w'_4(\ell)) = 0. \]

Therefore,
\[
\text{rank } \Phi_2(\ell) = \text{rank } \begin{bmatrix} w'_2(\ell) & w'_4(\ell) \\ \varphi'_2(\ell) & \varphi'_4(\ell) \end{bmatrix} = 2, \tag{3.14}
\]
which implies the last inequality in (3.3).

Now we show that (3.2) has nonzero solution if (3.3) holds. For this purpose, we need only to prove that (3.5) has nonzero solution \((a_2, a_4),\) or equivalently, \( \text{rank } \Phi_3(\ell) < 2. \) But with the first two conditions of (3.3), \( \text{rank } \Phi_3(\ell) = \text{rank } \Phi_4(\ell), \) where
\[ \Phi_4(\ell) = \begin{bmatrix} EI\varphi'_2(\ell) - i\sigma w_2(\ell) & EI\varphi'_4(\ell) - i\sigma w_4(\ell) \\ K(\varphi_2(\ell) - w'_2(\ell)) - i\sigma\varphi_2(\ell) & K(\varphi_4(\ell) - w'_4(\ell)) - i\sigma\varphi_4(\ell) \end{bmatrix}. \]

By using (3.3), we have
\[
\text{Re det } \Phi_4(\ell) = \begin{vmatrix} EI\varphi'_2(\ell) & EI\varphi'_4(\ell) \\ K(\varphi_2(\ell) - w'_2(\ell)) & K(\varphi_4(\ell) - w'_4(\ell)) \end{vmatrix} - \sigma^2 \begin{vmatrix} w_2(\ell) & w_4(\ell) \\ \varphi_2(\ell) & \varphi_4(\ell) \end{vmatrix} = \begin{vmatrix} EI\varphi'_2(\ell) & EI\varphi'_4(\ell) \\ K(\varphi_2(\ell) - w'_2(\ell)) & K(\varphi_4(\ell) - w'_4(\ell)) \end{vmatrix} = 0,
\]
and
\[
\text{Im } \det \Phi_4(\ell) = -\sigma \begin{vmatrix}
EI\varphi_2'(\ell) & EI\varphi_4'(\ell) \\
\varphi_2(\ell) & \varphi_4(\ell)
\end{vmatrix}
- \sigma \begin{vmatrix}
w_2(\ell) & w_4(\ell) \\
K(\varphi_2(\ell) - \varphi_2'(\ell)) & K(\varphi_4(\ell) - \varphi_4'(\ell))
\end{vmatrix}
= \frac{\sigma}{\alpha} \begin{vmatrix}
w_2'(\ell) & w_4'(\ell) \\
X_2(\ell) & X_4(\ell)
\end{vmatrix} = 0,
\]
from which we have proven that \(\text{rank } \Phi_3(\ell) < 2\), and then the proof is complete. \(\Box\)

From now on, we assume that the system discussed in this paper is uniform, i.e., \(\rho, I\rho, K\) and \(EI\) are all positive constants. Let \(\omega \in \mathbb{R}\) nonzero number and define
\[
a = \rho_1^2 \omega^2, \quad b = \rho_2^2 \omega^2, \quad c = K/EI,
\]
\[
\rho_1 = \sqrt{\rho/K}, \quad \rho_2 = \sqrt{I\rho/EI},
\]
\[
\alpha_1 \triangleq i \sqrt{\frac{a + b + \sqrt{(a - b)^2 + 4ac}}{2}},
\]
\[
\alpha_2 \triangleq \begin{cases} 
 i \sqrt{\frac{a + b - \sqrt{(a - b)^2 + 4ac}}{2}}, & \text{if } b > c,
\end{cases}
\]
\[
\beta_1 \triangleq 1 + \alpha_1^2/a = \frac{(a - b) - \sqrt{(a - b)^2 + 4ac}}{2a} < 0,
\]
\[
\beta_2 \triangleq 1 + \alpha_2^2/a = \frac{(a - b) + \sqrt{(a - b)^2 + 4ac}}{2a} > 0.
\]
Here and henceforth, we assume \(b \neq c\); i.e., \(\omega \neq \pm \sqrt{K/I\rho}\), without loss of generality, for it is not difficult to check that \(\pm i \sqrt{K/I\rho} \in \rho(A)\).

**Lemma 3.3** [6]. Assume that \(B \succeq 0\) and \(d_1 = d_2\). Then the energy of the closed loop system (2.1) decays asymptotically to zero for all \((\rho, I\rho, K, EI) > 0\) if and only if \(\text{rank } (B) = 2\).

**Theorem 3.4.** Assume that \(B \succeq 0\), \(d_1 = -d_2\) and \(d_1 \neq 0\). Then the energy of the closed loop system (2.1) decays asymptotically to zero for all \((\rho, I\rho, K, EI) > 0\) if and only if \(\text{rank } (B) > 0\).

**Proof.** By Lemmas 2 and 3 in [6], under the assumption of the theorem, \(A\) has no spectral points in \(i\mathbb{R}\), and the desired conclusion follows immediately. \(\Box\)
Set 
\[ \eta \triangleq \frac{Kd}{EI\alpha}. \]

**Theorem 3.5.** Let \( \omega \in \mathbb{R} \), and assume that \( B \geq 0 \) and \( d_2 \neq \pm d_1 \). Then the energy of the closed loop system (2.1) decays asymptotically if and only if the system on \( \omega \)

\[
\begin{bmatrix}
\eta & -\eta & \alpha_1 \beta_2 & -\alpha_2 \beta_1 \\
\alpha_1 \beta_1 \eta & -\alpha_2 \beta_2 \eta & K\alpha_1 & -K\alpha_2 \\
K\alpha_1 \alpha_2 \beta_2 & -K\alpha_1 \alpha_2 \beta_1 & a\alpha_2 \beta_1 \beta_2 EI\eta & -a\alpha_1 \beta_1 \beta_2 EI\eta \\
\alpha_1 \alpha_2 \beta_1 \beta_2 & -\alpha_1 \alpha_2 \beta_1 \beta_2 & \beta_1 \alpha_2 \eta & -\beta_2 \alpha_2 \eta \\
\end{bmatrix}
\times
\begin{bmatrix}
\cosh \alpha_2 \ell \\
\cosh \alpha_1 \ell \\
\sinh \alpha_1 \ell \\
\sinh \alpha_2 \ell \\
\end{bmatrix}
= 0,
\]

(3.15)

does not have any nonzero solution.

**Proof.** Based on the criterion for the asymptotic stability of \( C_0 \)-semigroups in [5], for the energy of the closed loop system (2.1) to decay asymptotically, it is necessary and sufficient that (3.2) admits only zero solution. We now consider the solvability of (3.2). For this, we first list the elements of the fundamental solution matrix \( \Phi(x) \):

\[
w_1(x) = \frac{1}{\beta_2 - \beta_1} (\beta_2 \cosh \alpha_1 x - \beta_1 \cosh \alpha_2 x),
\]

\[
w_2(x) = \frac{1}{a(\beta_1 - \beta_2)} (\alpha_1 \beta_2 \sinh \alpha_1 x - \alpha_2 \beta_1 \sinh \alpha_2 x),
\]

\[
w_3(x) = \frac{-\alpha_1 \alpha_2}{a^2(\beta_1 - \beta_2)} (\alpha_2 \sinh \alpha_1 x - \alpha_1 \sinh \alpha_2 x),
\]

\[
w_4(x) = \frac{1}{a(\beta_2 - \beta_1)} (-\cosh \alpha_1 x + \cosh \alpha_2 x),
\]

\[
w_1'(x) = \frac{1}{\beta_2 - \beta_1} (\alpha_1 \beta_2 \sinh \alpha_1 x - \alpha_2 \beta_1 \sinh \alpha_2 x),
\]

\[
w_2'(x) = \frac{1}{a(\beta_1 - \beta_2)} (\alpha_1^2 \beta_2 \cosh \alpha_1 x - \alpha_2^2 \beta_1 \cosh \alpha_2 x),
\]

\[
w_3'(x) = \frac{c - b}{a(\beta_1 - \beta_2)} (\cosh \alpha_1 x - \cosh \alpha_2 x),
\]

\[
w_4'(x) = \frac{1}{a(\beta_2 - \beta_1)} (-\alpha_1 \sinh \alpha_1 x + \alpha_2 \sinh \alpha_2 x),
\]
\[ \varphi_1(x) = \frac{c}{\beta_1 - \beta_2} (\alpha_1^{-1} \sinh \alpha_1 x - \alpha_2^{-1} \sinh \alpha_2 x), \]
\[ \varphi_2(x) = \frac{c}{a(\beta_2 - \beta_1)} (\cosh \alpha_1 x - \cosh \alpha_2 x), \]
\[ \varphi_3(x) = \frac{1}{a(\beta_2 - \beta_1)} (\alpha_2^2 \beta_1 \cosh \alpha_1 x - \alpha_1^2 \beta_2 \cosh \alpha_2 x), \]
\[ \varphi_4(x) = \frac{1}{\beta_2 - \beta_1} (-\alpha_1^{-1} \beta_1 \sinh \alpha_1 x + \alpha_2^{-1} \beta_2 \sinh \alpha_2 x), \]
\[ \varphi'_1(x) = \frac{c}{\beta_1 - \beta_2} (\cosh \alpha_1 x - \cosh \alpha_2 x), \]
\[ \varphi'_2(x) = \frac{c}{a(\beta_2 - \beta_1)} (\alpha_1 \sinh \alpha_1 x - \alpha_2 \sinh \alpha_2 x), \]
\[ \varphi'_3(x) = \frac{1}{a(\beta_2 - \beta_1)} (\alpha_1 \alpha_2^2 \beta_1 \sinh \alpha_1 x - \alpha_1^2 \alpha_2 \beta_2 \sinh \alpha_2 x), \]
\[ \varphi'_4(x) = \frac{1}{\beta_2 - \beta_1} (-\beta_1 \cosh \alpha_1 x + \beta_2 \cosh \alpha_2 x). \]

By a lengthy calculation, it can be shown that (3.15) is equivalent to (3.3) and, as a result, the desired result follows immediately from Lemma 3.2. \[\square\]

By a further analysis on the transcendental equation (3.15), we have

**Theorem 3.6.** Let \( \omega \in \mathbb{R} \), and assume that rank(\( B \)) = 1, \( d_2 \neq \pm d_1 \neq 0 \) and \( B \geq 0 \).
Then the energy of the closed loop system (2.1) decays asymptotically if and only if the system on \( \omega \)
\[
\begin{align*}
\alpha_2 \beta_1 (\eta - \alpha_1^2 \beta_2^2) \sinh \alpha_1 \ell &= \alpha_1 \beta_2 [\eta^2 - \alpha_2^2 \beta_1^2] \sinh \alpha_2 \ell, \\
\alpha_2 \beta_1 (\eta^2 - \alpha_1^2 \beta_2^2) \cosh \alpha_1 \ell &= (\eta^3 + [c - \beta_2 (a - b + c)] \eta) \sinh \alpha_2 \ell, \\
\alpha_2 \beta_1 (\eta^2 - \alpha_1^2 \beta_2^2) \cosh \alpha_2 \ell &= (\eta^3 + [c - \beta_1 (a - b + c)] \eta) \sinh \alpha_2 \ell, \\
(c \alpha_2 + a \alpha_2^{-1} \alpha_1^2 \beta_2^2) \sinh \alpha_2 \ell \cosh \alpha_1 \ell + (c \alpha_1 + a \alpha_1^{-1} \alpha_2^2 \beta_1^2) \sinh \alpha_1 \ell \cosh \alpha_2 \ell &\neq 0
\end{align*}
\]

(3.16)
does not have nonzero solution.

4. Exponential stability

The following lemma is vital to proving the main result in this paper.

**Lemma 4.1.** Assume that \( B \geq 0 \), rank(\( B \)) > 0, \( d_1 \neq -d_2 \) and \( \sigma(A) \cap i\mathbb{R} = \emptyset \).
Then there exists a positive constant \( M \) such that
\[ \| R(i \omega, A) \| \leq M, \quad \forall \omega \in \mathbb{R}, \]
where \( R(\lambda, A) = (A - \lambda)^{-1} \) is the resolvent of \( A \).

**Proof.** Using the continuity of the resolvent, we need only to prove Lemma 4.1 for positive and sufficiently large \( \omega \). For \( \lambda = i \omega \) and \([f, g, f_1, g_1]^r \in \mathcal{H} \), let \([w, z, \varphi, \psi]^r \in D(A)\) satisfy \((A - i \omega)[w, z, \varphi, \psi]^r = [f, g, f_1, g_1]^r\), i.e.,

\[
\begin{aligned}
  z - i \omega w &= f, \\
  K(w' - \varphi)' - i \omega \varphi z &= \rho g, \\
  \psi - i \omega \varphi &= f_1, \\
  &\quad E(\varphi'') + K(w' - \varphi) - i \omega I_\rho \varphi = I_\rho g_1, \\
  w(0) &= 0, \quad \varphi(0) = 0, \quad z(0) = 0, \quad \psi(0) = 0, \\
  K(\varphi(\ell) - w'(\ell)) &= a z(\ell) + d_1 \psi(\ell), \\
  -E(\varphi'(\ell)) &= d_2 z(\ell) + \gamma \psi(\ell).
\end{aligned}
\]

Eliminating \( z \) and \( \psi \) from (4.1), we have

\[
\begin{aligned}
  &\quad K(w'' - \varphi') + \omega^2 \rho w = i \omega f + \rho g, \\
  &\quad E(\varphi'') + K(w' - \varphi) + \omega^2 I_\rho \varphi = i \omega I_\rho f_1 + I_\rho g_1, \\
  w(0) &= 0, \quad \varphi(0) = 0,
\end{aligned}
\]

(4.2)

The general solution to the first two equations of (4.2) is

\[
Y(x) = Y_1(x) + Y_2(x),
\]

(4.3)

where

\[
Y(x) = [w, w', \varphi, \psi]^r, \quad Y_k(x) = [\tilde{w}_k, \tilde{w}_k', \tilde{\varphi}_k, \tilde{\varphi}_k']^r, \quad k = 1, 2,
\]

\[
Y_2(x) = \int_0^x \Phi(x - s) \left[0, \rho_1^2 (i \omega f(s) + g(s)), 0, \rho_2^2 (i \omega f_1(s) + g_1(s)) \right]^r \, ds,
\]

\[
Y_1 = \Phi(x) Y(0), \quad Y(0) = [a_1, a_2, a_3, a_4]^r, \quad a_k \in \mathbb{C}, \quad k = 1, 2, 3, 4,
\]

and \( \Phi(x) \) is the state transition matrix to the first two equations of (3.2) as defined in Section 3.

From the left boundary condition of \( Y(x) \), we get immediately that \( a_1 = a_3 = 0 \). From the right boundary condition of \( Y(x) \), we can deduce that

\[
\begin{aligned}
  K(\tilde{\varphi}_1(\ell) - \tilde{w}_1'(\ell)) - a i \omega \tilde{w}_1(\ell) - d_1 i \omega \tilde{\varphi}_1(\ell) &= h_1(\ell), \\
  E(\tilde{\varphi}_1'(\ell) + d_2 i \omega \tilde{w}_2(\ell) + \gamma i \omega \tilde{\varphi}_2(\ell) &= h_2(\ell),
\end{aligned}
\]

(4.4)

where

\[
\begin{aligned}
  h_1(\ell) &\triangleq -K(\tilde{\varphi}_2(\ell) - \tilde{w}_2'(\ell)) + a i \omega \tilde{w}_2(\ell) + d_1 i \omega \tilde{\varphi}_2(\ell) + \alpha f(\ell) + d_1 f_1(\ell), \\
  h_2(\ell) &\triangleq -E(\tilde{\varphi}_2'(\ell) - d_2 i \omega \tilde{w}_2(\ell) - \gamma i \omega \tilde{\varphi}_2(\ell) - d_2 f(\ell) - \gamma f_1(\ell).
\end{aligned}
\]
For $Y_2(x) = [\tilde{w}_2(x), \tilde{w}_2'(x), \tilde{\varphi}_2(x), \tilde{\varphi}_2'(x)]^T$, we have

$$\tilde{w}_2(x) = \frac{1}{\alpha_1^2 - \alpha_2^2} \left[ \int_0^x \left( \left[ \alpha_1 \beta_1 S_1(x-s) - \alpha_2 \beta_1 S_2(x-s) \right] \rho_1^2 g(s) 
+ \left[ C_1(x-s) - C_2(x-s) \right] \rho_2^2 g_1(s) \right) ds 
+ \int_0^x \left( \left[ \beta_2 C_1(x-s) - \beta_1 C_2(x-s) \right] \rho_1^2 i \omega f'(s) \right) ds \right] + \frac{i \omega}{a} \rho_2^2 f(x),$$

$$\tilde{w}_2'(x) = \frac{1}{\alpha_1^2 - \alpha_2^2} \left[ \int_0^x \left( \left[ \alpha_1^2 \beta_2 C_1(x-s) - \alpha_2^2 \beta_1 C_2(x-s) \right] \rho_1^2 g(s) 
+ \left[ \alpha_1 S_1(x-s) - \alpha_2 S_2(x-s) \right] \rho_2^2 g_1(s) \right) ds 
+ \int_0^x \left( \left[ C_1(x-s) - C_2(x-s) \right] \rho_2^2 i \omega f'_1(s) \right) ds \right],$$

$$\tilde{\varphi}_2(x) = \frac{1}{\alpha_1^2 - \alpha_2^2} \left[ \int_0^x \left( c \left[ C_2(x-s) - C_1(x-s) \right] \rho_1^2 g(s) 
+ \frac{a}{\alpha_1 \alpha_2} \left[ \alpha_2 \beta_1 S_1(x-s) - \alpha_2 \beta_2 S_2(x-s) \right] \rho_2^2 g_1(s) \right) ds 
+ \int_0^x \left( c \left[ \alpha_2 S_2(x-s) - \alpha_1 S_1(x-s) \right] \rho_1^2 i \omega f'(s) \right) ds \right] + \frac{i \omega}{b-c} \rho_2^2 f_1(x),$$

$$\tilde{\varphi}_2'(x) = \frac{1}{\alpha_1^2 - \alpha_2^2} \left[ \int_0^x \left( c \left[ \alpha_2 S_2(x-s) - \alpha_1 S_1(x-s) \right] \rho_1^2 g(s) 
+ \frac{a}{\alpha_1 \alpha_2} \left[ \alpha_2 \beta_2 S_2(x-s) - \alpha_1 \beta_1 S_1(x-s) \right] \rho_2^2 g_1(s) \right) ds 
+ \int_0^x \left( c \left[ \alpha_2^2 S_2(x-s) - \alpha_1^2 S_1(x-s) \right] \rho_2^2 i \omega f'_1(s) \right) ds \right].$$
\[ + a \left[ \beta_1 C_1(x-s) - \beta_2 C_2(x-s) \right] \rho_2^2 g_1(s) \right] ds + \int_0^x \left( c \left[ C_2(x-s) - C_1(x-s) \right] \rho_1^2 i \omega f'(s) \rho_2^2 g_1(s) \right) ds \right),

where and afterwards, for simplicity, we write \( S_k(x) = \sinh \alpha_k x, \ C_k(x) = \cosh \alpha_k x, \ k = 1, 2. \) Therefore,

\[
\begin{align*}
\tilde{w}_2(x) &= O(\omega^{-1}(\|f'\| + \|f'_1\| + \|g\| + \|g_1\|)), \\
\tilde{w}'_2(x) &= O(\|f'\| + \|f'_1\| + \|g\| + \|g_1\|), \\
\tilde{\varphi}_2(x) &= O(\omega^{-1}(\|f'\| + \|f'_1\| + \|g\| + \|g_1\|)), \\
\tilde{\varphi}'_2(x) &= O(\|f'\| + \|f'_1\| + \|g\| + \|g_1\|)).
\end{align*}
\]

Similarly, we have

\[
h_1(\ell) = \frac{Kai\omega}{\alpha_1^2 - \alpha_2^2} \left[ \int_0^\ell \left( \frac{\beta_1}{\alpha_2} S_2(s) - \frac{\beta_2}{\alpha_1} S_1(s) \right) \rho_1^2 f'((\ell - s) + \left( \frac{C_2(s)}{\alpha_2^2} - \frac{C_1(s)}{\alpha_1^2} \right) \rho_2^2 f'_1((\ell - s) \right] ds
\]

\[
- \frac{\omega^2}{\alpha_1^2 - \alpha_2^2} \int_0^\ell \left[ \alpha \left( \frac{\beta_2 C_1(s) - \beta_1 C_2(s)}{\alpha_2} \right) \right] \rho_1^2 f'(\ell - s) ds
\]

\[
+ d_1 c \left( \frac{S_2(s)}{\alpha_2} - \frac{S_1(s)}{\alpha_1} \right) \right] \rho_1^2 f'(\ell - s) ds
\]

\[
- \frac{\omega^2}{\alpha_1^2 - \alpha_2^2} \int_0^\ell \left[ \alpha \left( \frac{S_1(s)}{\alpha_1} - \frac{S_2(s)}{\alpha_2} \right) \right] \rho_1^2 f'(\ell - s) ds
\]

\[
+ ad_1 \left( \frac{\beta_1 C_1(s)}{\alpha_2} - \frac{\beta_2 C_2(s)}{\alpha_1} \right) \right] \rho_1^2 f'_1((\ell - s)) ds
\]

\[
+ \frac{aK}{\alpha_1^2 - \alpha_2^2} \int_0^\ell \left[ (\beta_1 C_2(s) - \beta_2 C_1(s)) \rho_1^2 g((\ell - s) \right] ds
\]

\[
+ \left( \frac{S_2(s)}{\alpha_2} - \frac{S_1(s)}{\alpha_1} \right) \rho_2^2 g_1((\ell - s) \right] ds
\]

\[
- \frac{i\omega}{\alpha_1^2 - \alpha_2^2} \int_0^\ell \left[ \alpha(\alpha_2 \beta_1 S_2(s) - \alpha_1 \beta_2 S_1(s)) \right]
\]
\[ + d_1 c (C_1(s) - C_2(s)) \rho_1^2 g(\ell - s) \, ds \]
\[- \frac{i \omega}{\alpha_1^2 - \alpha_2^2} \int_0^\ell \left[ \alpha (C_2(s) - C_1(s)) \right] \rho_2^2 g_1(\ell - s) \, ds \]
\[ + a d_1 \left( \frac{\beta_2 S_2(s)}{\alpha_2} - \frac{\beta_1 S_1(s)}{\alpha_1} \right) \rho_2^2 g_1(\ell - s) \, ds \]
\[ + \left( d_1 - \frac{d_1 \omega^2 + K i \omega}{b - c} \rho_2^2 \right) f_1(\ell), \quad (4.6) \]
\[ h_2(\ell) = \frac{E i \omega}{\alpha_1^2 - \alpha_2^2} \int_0^\ell \left[ (C_1(s) - C_2(s)) \rho_1^2 f'(\ell - s) \right] ds \]
\[ + a \left( \frac{\beta_2 S_2(s)}{\alpha_2} - \frac{\beta_1 S_1(s)}{\alpha_1} \right) \rho_2^2 f_1(\ell - s) \, ds \]
\[ + \frac{\omega^2}{\alpha_1^2 - \alpha_2^2} \int_0^\ell \left[ d_2 \left( \beta_2 C_1(s) - \beta_1 C_2(s) \right) \rho_1^2 f'(\ell - s) \right] ds \]
\[ + \gamma c \left( \frac{S_2(s)}{\alpha_2} - \frac{S_1(s)}{\alpha_1} \right) \rho_2^2 f_1(\ell - s) \, ds \]
\[ + \frac{\omega^2}{\alpha_1^2 - \alpha_2^2} \int_0^\ell \left[ d_2 \left( \frac{S_1(s)}{\alpha_1} - \frac{S_2(s)}{\alpha_2} \right) \rho_1^2 f'(\ell - s) \right] ds \]
\[ + \frac{E i}{\alpha_1^2 - \alpha_2^2} \int_0^\ell \left[ c \left( \alpha_1 S_1(s) - \alpha_2 S_2(s) \right) \rho_1^2 g(\ell - s) \right] ds \]
\[ + a \left( \beta_2 C_2(s) - \beta_1 C_1(s) \right) \rho_2^2 g_1(\ell - s) \, ds \]
\[ + \frac{i \omega}{\alpha_1^2 - \alpha_2^2} \int_0^\ell \left[ d_2 \left( \alpha_2 \beta_1 S_2(s) - \alpha_1 \beta_2 S_1(s) \right) \right] \rho_1^2 g(\ell - s) \, ds \]
\[ + \gamma c \left( C_1(s) - C_2(s) \right) \rho_1^2 g(\ell - s) \, ds \]
\[ + \frac{i \omega}{\alpha_1^2 - \alpha_2^2} \int_0^\ell \left[ d_2 \left( C_2(s) - C_1(s) \right) \right] \rho_1^2 g(\ell - s) \, ds \]
+ aγ \left( \frac{β_2 S_2(s)}{α_2} - \frac{β_1 S_1(s)}{α_1} \right) \rho_2^2 g_1(ℓ - s) ds \\
- γ \left( 1 - \frac{ω^2 ρ_2^2}{ρ_2^2 ω^2 - c} \right) f_1(ℓ). 
(4.7)

From (4.4), we have

\begin{align*}
    a_2 &= G^{-1}(g_{11} h_1(1) + g_{12} h_2(1)), \\
    a_4 &= G^{-1}(g_{21} h_1(1) + g_{22} h_2(1)), 
\end{align*}

where

\begin{align*}
    G &\triangleq \frac{1}{α_1^2 - α_2^2} \begin{vmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{vmatrix} \triangleq \frac{1}{α_1^2 - α_2^2} G_1, \\
    g_{11} &= (EIαβ_1 + d_2iω)C_1 - (EIαβ_2 + d_2iω)C_2 + aα^{-1}_1 γ S_1 β_1 iω \\
    &\quad - aα^{-1}_2 γ S_2 β_2 iω, \\
    g_{12} &= -\left( \frac{aKβ_1}{α_1} - Kα_1 - ad_1β_1 iω \right) S_1 + \left( \frac{aKβ_2}{α_2} - Kα_2 - ad_1β_2 iω \right) S_2 \\
    &\quad + αC_1 iω - αC_2 iω, \\
    g_{21} &= (EIcα_1 - d_2α_1 β_2 iω)S_1 - (EIcα_2 - d_2α_2 β_1 iω)S_2 + cγ C_1 iω \\
    &\quad - cγ C_2 iω, \\
    g_{22} &= -(Kc + Kα_1^2 β_2 - d_1 ciω)C_1 + (Kc + Kα_2^2 β_1 - d_1 ciω)C_2 \\
    &\quad - αα_1 β_2 S_1 iω + αα_2 β_1 S_2 iω, \\
    S_1 &= \sinh α_1 ℓ, \quad S_2 &= \sinh α_2 ℓ, \\
    C_1 &= \cosh α_1 ℓ, \quad C_2 &= \cosh α_2 ℓ.
\end{align*}

By virtue of the definition of \( a, b, c, α_1, α_2, β_1, β_2 \) and through a lengthy calculation, we obtain

\begin{align*}
    G_1 &= 2(|F|c - K^2 ρ_1^2)ω^2 \\
    &\quad - 2\left[ |F|ω^2 + 2K^2 ρ_1^2 ω^2 + EI K (ρ_1^2 - ρ_2^2)^2 ω^4 \right] C_1 C_2 \\
    &\quad - \frac{ω^4 S_1 S_2}{\sqrt{ρ_1^2(ρ_2^2 ω^2 - c)ω^2}} \left[ |F|((ρ_1^2 - ρ_2^2)^2 ω^2 + (ρ_1^2 - ρ_2^2)c + 2ρ_2^2 c) \\
    &\quad + ρ_2^2 (ρ_1^2 + ρ_2^2) K^2 \right] \\
    &\quad + \frac{\sqrt{2EIωC_1 S_2}}{2\sqrt{(ρ_1^2 + ρ_2^2)ω^2 - (ρ_1^2 - ρ_2^2)^2 ω^4 + 4ρ_1^2 cω^2}} \\
    &\quad \times \left[ (|F|^2(ρ_1^2 - ρ_2^2)ω^2 + 2c)\sqrt{(ρ_1^2 - ρ_2^2)^2 ω^4 + 4ρ_1^2 cω^2} \right]
\end{align*}
\[-(\rho_1^2 - \rho_2^2)\omega^4 - 4\rho_1^2\omega^2 \alpha \]
\[-\left( (\rho_1^2 - \rho_2^2)\omega^4 + 4\rho_1^2\omega^2 \right) \]
\[+ (\rho_1^2 - \rho_2^2)\omega^2 \sqrt{(\rho_1^2 - \rho_2^2)\omega^4 + 4\rho_1^2\omega^2} \gamma c \]
\[-\frac{\sqrt{2EIC_2S_1\omega}}{2\sqrt{(\rho_1^2 + \rho_2^2)\omega^2 - \sqrt{(\rho_1^2 - \rho_2^2)\omega^4 + 4\rho_1^2\omega^2}} \times \left[ \left[ (\rho_1^2 - \rho_2^2)\omega^2 + 2c \right] \sqrt{(\rho_1^2 - \rho_2^2)\omega^4 + 4\rho_1^2\omega^2} \right] \]
\[+ (\rho_1^2 - \rho_2^2)\omega^2 \sqrt{(\rho_1^2 - \rho_2^2)\omega^4 + 4\rho_1^2\omega^2} \gamma c \]
\[+ (\rho_1^2 - \rho_2^2)K(d_1 + d_2)\omega^3(1 - C_1C_2) \]
\[-\frac{\rho_1 \omega^2}{\sqrt{\rho_1^2\omega^2 - c}} \left[ ((\rho_1^2 - \rho_2^2)\omega^2 - 2c)(d_1 + d_2)S_1S_2i \right]. \]

Thus it follows that when \(\omega\) is large

\[ G_1 = \begin{cases} 
-(\rho_1^2 - \rho_2^2)^2(EIKC_1C_2 + |F|\rho_1^{-1}\rho_2^{-1}S_1S_2 \\
+ \gamma K\rho_2^{-1}C_1S_2 + \alpha EI\rho_1^{-1}C_2S_1)\omega^4 + O(\omega^3), & \text{if } \rho_1 > \rho_2, \\
-(\rho_1^2 - \rho_2^2)^2(EIKC_1C_2 + |F|\rho_1^{-1}\rho_2^{-1}S_1S_2 \\
+ \gamma K\rho_1^{-1}C_1S_2 + \alpha EI\rho_2^{-1}C_2S_1)\omega^4 + O(\omega^3), & \text{if } \rho_1 < \rho_2, \\
2[|F|c - \rho_1^2 K^2 - (|F|c + \rho_1^2 K^2)C_1C_2 \\
- (|F|c + \rho_1^2 K^2)S_1S_2 \\
- \rho_1 c(\alpha EI + \gamma K)(C_1S_2 + C_2S_1)i]\omega^2 + O(\omega), & \text{if } \rho_1 = \rho_2. 
\end{cases} \]

From the definition of \(\alpha_1, \alpha_2\), we know that when \(\omega\) is large, \(\alpha_1, \alpha_2 \in i\mathbb{R}\), and thus

\[ G_1 = \begin{cases} 
-(\rho_1^2 - \rho_2^2)^2(EIKc_1c_2 - |F|\rho_1^{-1}\rho_2^{-1}s_1s_2 \\
+ \gamma K\rho_2^{-1}c_2i + \alpha EI\rho_1^{-1}c_2s_1i)\omega^4 + O(\omega^3), & \text{if } \rho_1 > \rho_2, \\
-(\rho_1^2 - \rho_2^2)^2(EIKc_1c_2 - |F|\rho_1^{-1}\rho_2^{-1}s_1s_2 \\
+ \gamma K\rho_1^{-1}c_1s_2i + \alpha EI\rho_2^{-1}c_2s_1i)\omega^4 + O(\omega^3), & \text{if } \rho_1 < \rho_2, \\
2[|F|c - \rho_1^2 K^2 - (|F|c + \rho_1^2 K^2)c_1c_2 \\
+ (|F|c + \rho_1^2 K^2)s_1s_2 \\
- \rho_1 c(\alpha EI + \gamma K)(c_1s_2 + c_2s_1)i]\omega^2 + O(\omega), & \text{if } \rho_1 = \rho_2. 
\end{cases} \] (4.9)
where and afterwards, for simplicity, we denote \( s_k = \sin(-i\alpha_k\ell) \), \( c_k = \cos(-i\alpha_k\ell) \) for \( k = 1, 2 \).

Now we prove that when \( \omega \) is sufficiently large, there exists a positive \( \delta \) such that

\[
|G_1| \geq \begin{cases} 
\delta \omega^4, & \text{if } \rho_1 \neq \rho_2, \\
\delta \omega^2, & \text{if } \rho_1 = \rho_2.
\end{cases}
\tag{4.10}
\]

In fact, in the case of \( \rho_1 > \rho_2 \), from (4.9), when \( \omega \) is large, for \( N > 1 \),

\[
\frac{|G_1|^2}{(\rho_1^2 - \rho_2^2)^4} = \left[ \left( EIKc_1c_2 - \frac{|F|}{\rho_1\rho_2} s_1s_2 \right)^2 + \left( \frac{\gamma K}{\rho_2} c_1s_2 + \frac{\alpha EI}{\rho_1} c_2s_1 \right)^2 \right] \omega^8
\]

\[+ O(\omega^7)
\]

\[= \left[ EIK^2c_1^2c_2^2 + \frac{|F|^2}{\rho_1^2 \rho_2^2} s_1^2 s_2^2 + \frac{\gamma^2 K^2}{\rho_2^2} c_1^2 s_2^2
\]

\[+ \frac{\alpha^2 EI^2}{\rho_2^2} c_2^2 s_2^2 - \frac{2EIK}{\rho_1 \rho_2} d_1 d_2 c_1 c_2 s_1 s_2 \right] \omega^8 + O(\omega^7)
\]

\[\geq \frac{1}{N}\left[ EIK^2c_1^2c_2^2 + \frac{|F|^2}{\rho_1^2 \rho_2^2} s_1^2 s_2^2 + \frac{\gamma^2 K^2}{\rho_2^2} c_1^2 s_2^2 + \frac{\alpha^2 EI^2}{\rho_1^2} c_2^2 s_2^2
\]

\[+ \frac{2EIK}{\rho_1 \rho_2} \left[ \frac{N - 1}{N} (2\alpha \gamma - d_1 d_2)|c_1 c_2 s_1 s_2| - d_1 d_2 c_1 c_2 s_1 s_2 \right] \omega^8 + O(\omega^7).
\]

Noticing that by the assumption of \( F \) and \( B \), we have \( \alpha > 0 \), \( \gamma > 0 \), \( \alpha \gamma - d_1 d_2 > 0 \). Hence for a fixed large \( N \), we get

\[
\frac{|G_1|^2}{(\rho_1^2 - \rho_2^2)^4} \geq \frac{1}{N}\left[ EIK^2c_1^2c_2^2 + \frac{|F|^2}{\rho_1^2 \rho_2^2} s_1^2 s_2^2 + \frac{\gamma^2 K^2}{\rho_2^2} c_1^2 s_2^2 + \frac{\alpha^2 EI^2}{\rho_1^2} c_2^2 s_2^2
\]

\[+ O(\omega^7).
\]

If (4.10) is not true, then it follows from (4.12) that there is a sequence \( \{\omega_n\} \subset \mathbb{R} \) such that \( \omega_n \to \infty \) as \( n \to \infty \), and

\[
\sin^2(-i\alpha_{1n}\ell) \sin^2(-i\alpha_{2n}\ell) \to 0,
\]

\[
\cos^2(-i\alpha_{1n}\ell) \cos^2(-i\alpha_{2n}\ell) \to 0,
\]

\[
\cos^2(-i\alpha_{1n}\ell) \sin^2(-i\alpha_{2n}\ell) \to 0,
\]

\[
\cos^2(-i\alpha_{2n}\ell) \sin^2(-i\alpha_{1n}\ell) \sin^2(-i\alpha_{2n}\ell) \to 0
\]

as \( n \to +\infty \), where
\( \alpha_{1n} = i \sqrt{\frac{a_n + b_n + \sqrt{(a_n - b_n)^2 + 4a_n c}}{2}}, \)
\( \alpha_{2n} = i \sqrt{\frac{a_n + b_n - \sqrt{(a_n - b_n)^2 + 4a_n c}}{2}}, \)
\( a_n = \rho_1^2 \omega_n^2, \quad b_n = \rho_2^2 \omega_n^2. \)

Therefore
\[
\sin^2(-i\alpha_{1n} \ell) + \cos^2(-i\alpha_{1n} \ell) \to 0,
\]
\[
\sin^2(-i\alpha_{2n} \ell) + \cos^2(-i\alpha_{2n} \ell) \to 0
\]
as \( n \to +\infty \), a contradiction.

The proof for the case of \( \rho_1 < \rho_2 \) is similar.

In the case of \( \rho_1 = \rho_2 \), again from (4.9) it follows that

\[
\frac{|G_1|^2}{4\omega^2} = \left( |F| c - \rho_1^2 K^2 \right)^2 - \left( |F| c + \rho_1^2 K^2 \right) \cos(-i(\alpha_1 + \alpha_2) \ell) + O(\omega^{-1}).
\]

If
\[
|\sin(-i(\alpha_1 + \alpha_2) \ell)| \geq \frac{1}{2} \sqrt{1 - \left( \frac{|F| c - \rho_1^2 K^2}{|F| c + \rho_1^2 K^2} \right)^2} \geq \delta_1,
\]
then (4.10) follows immediately from (4.13). Otherwise, we have

\[
\frac{|G_1|^2}{4\omega^2} \geq \left( -|F| c - \rho_1^2 K^2 \right)^2 + \left( |F| c + \rho_1^2 K^2 \right) \cos(-i(\alpha_1 + \alpha_2) \ell) + O(\omega^{-1})
\]
\[
\geq \left( -|F| c - \rho_1^2 K^2 \right)^2 + \left( |F| c + \rho_1^2 K^2 \right) \sqrt{1 - \delta_1^2}
\]
\[
+ O(\omega^{-1}),
\]
from which (4.10) follows.

From the definition of \( \alpha_1, \alpha_2, \beta_1 \) and \( \beta_2 \), for large \( \omega \), it follows that

\( \alpha_1 = \max\{\rho_1, \rho_2\} i \omega + O(\omega^{-1}), \)
\( \alpha_2 = \min\{\rho_1, \rho_2\} i \omega + O(\omega^{-1}), \)
\( \beta_1 = \begin{cases} -c \omega^{-2} |\rho_2^2 - \rho_1^2|^{-1} + O(\omega^{-4}), & \text{if } \rho_1 \neq \rho_2, \\ -\sqrt{c} \rho_1^{-1} \omega^{-1}, & \text{if } \rho_1 = \rho_2, \end{cases} \)
\( \beta_2 = \begin{cases} \rho_1^{-2} |\rho_2^2 - \rho_1^2| + O(\omega^{-2}), & \text{if } \rho_1 \neq \rho_2, \\ -\sqrt{c} \rho_1^{-1} \omega^{-1}, & \text{if } \rho_1 = \rho_2. \end{cases} \)
Thus we obtain

\[
g_{11} = \begin{cases} 
-|\rho_2^2 - \rho_1^2|(EIC_2 + \gamma \rho_2^{-1} S_2)\omega^2 + d_2(C_1 - C_2)i\omega + O(1), & \text{if } \rho_1 \neq \rho_2, \\
(-\sqrt{KE}\rho_1(C_1 + C_2) + d_2(C_1 - C_2)i - \gamma \sqrt{c}(S_1 + S_2))\omega + O(1), & \text{if } \rho_1 = \rho_2,
\end{cases}
\]

\[
g_{12} = \begin{cases} 
O(\omega^2), & \text{if } \rho_1 \neq \rho_2, \\
O(\omega), & \text{if } \rho_1 = \rho_2,
\end{cases} \tag{4.15}
\]

\[
g_{21} = \begin{cases} 
\frac{d_2(\rho_1^2 - \rho_2^2)\rho_1^{-1} S_1\omega^2 + [EIC(\rho_1 S_1 - \rho_2 S_2) + c\gamma(C_1 - C_2)]i\omega + O(1),}{} & \text{if } \rho_1 > \rho_2, \\
\frac{d_2(\rho_2^2 - \rho_2^2)\rho_2^{-1} S_1\omega^2 + [EIC(\rho_2 S_1 - \rho_1 S_2) + c\gamma(C_1 - C_2)]i\omega + O(1),}{} & \text{if } \rho_1 < \rho_2, \\
[\rho_1 K(S_1 - S_2)i - d_1\sqrt{c}(S_1 + S_2) + \alpha(C_1 - C_2)i]\omega + O(1), & \text{if } \rho_1 = \rho_2,
\end{cases}
\]

\[
g_{22} = \begin{cases} 
\frac{(KC_1 + \alpha \rho_1^{-1} S_1)(\rho_2^2 - \rho_1^2)\omega^2 + d_1c(C_1 - C_2)i\omega + O(1),}{} & \text{if } \rho_1 > \rho_2, \\
\frac{(K\rho_2^2 \rho_1^{-2} C_1 + \alpha \rho_2 \rho_1^{-2} S_1)(\rho_2^2 - \rho_1^2)\omega^2 + d_1c(C_1 - C_2)i\omega + O(1),}{} & \text{if } \rho_1 < \rho_2, \\
[K\rho_1 \sqrt{c}(C_1 + C_2) + d_1c(C_1 - C_2)i + \alpha \sqrt{c}(S_1 + S_2)]\omega + O(1), & \text{if } \rho_1 = \rho_2,
\end{cases} \tag{4.17}
\]

\[
\begin{aligned}
\alpha_1^2 - \alpha_2^2 &= \begin{cases} 
-|\rho_2^2 - \rho_1^2|\omega^2 + O(1), & \text{if } \rho_1 \neq \rho_2, \\
-2\rho_1 \sqrt{c}\omega, & \text{if } \rho_1 = \rho_2,
\end{cases} \\
&= \begin{cases} 
O(\omega^2), & \text{if } \rho_1 \neq \rho_2, \\
O(\omega), & \text{if } \rho_1 = \rho_2.
\end{cases} \tag{4.19}
\end{aligned}
\]

By the same argument as above, we derive, when \(\omega\) is large, that
\omega w_2(x), w'_2(x), \omega \varphi_4(x), \varphi'_4(x) = O(1),
\omega w_4(x), w'_4(x), \omega \varphi_2(x), \varphi'_2(x) = \begin{cases} O(\omega^{-1}), & \text{if } \rho_1 \neq \rho_2, \\
O(1), & \text{if } \rho_1 = \rho_2. \end{cases} \tag{4.20}

Similarly, by virtue of Cauchy integral inequality, it follows that
\begin{align*}
h_1(\ell), h_2(\ell) &= O\left(\|f'\| + \|f'_1\| + \|g\| + \|g_1\|\right). \\
\text{Moreover, since } Y_1(x) &= \Phi_2(x)[a_2, a_4]^\top, \text{ it follows from (4.10), (4.15)--(4.21) that}
\omega \tilde{w}_1(x), \omega \tilde{w}'_1(x), \omega \tilde{\varphi}_1(x), \omega \tilde{\varphi}'_1(x) &= O\left(\|f'\| + \|f'_1\| + \|g\| + \|g_1\|\right). \tag{4.22}
\end{align*}

Therefore, from (4.1) and (4.5), it follows that
\begin{align*}
z_1(x), \tilde{w}'_1(x), \psi_1(x), \tilde{\varphi}'_1(x) &= O\left(\|f'\| + \|f'_1\| + \|g\| + \|g_1\|\right). \tag{4.23}
\end{align*}

By using Sobolev embedding theorem, for any \( Y(x) = [w(x), z(x), \varphi(x), \psi(x)]^\top \in \mathcal{H} \) the norm
\[ \|Y\|_1 \overset{\Delta}{=} \left(\|w'\|^2 + \|z\|^2 + \|\varphi'\|^2 + \|\psi\|^2\right)^{1/2} \]
and the original norm \( \|Y\| \) in \( \mathcal{H} \) are equivalent. Thus Lemma 4.1 follows from (4.23). \(\blacksquare\)

From Lemma 4.1 and [11], we can deduce the following result.

**Theorem 4.2.** Assume that \( B \geq 0, \text{ rank}(B) > 0 \) and \( d_1 \neq -d_2 \). Then, the energy of the closed loop system (2.1) decays asymptotically if and only if it decays exponentially.

Combining the results in Section 3 with Theorem 4.2, we obtain the following main result of this section.

**Theorem 4.3.** Assume that \( B \geq 0 \). Then:

(i) In the case of \( d_1 = d_2 \), for all \( \rho, I_\rho, K, EI > 0 \), the energy of the closed loop system (2.1) decays exponentially if and only if \( B > 0 \).

(ii) In the case of \( d_1 \neq \pm d_2 \) and \( \text{rank}(B) = 1 \), the energy of the closed loop system (2.1) decays exponentially if and only if the system on \( \omega \)
\begin{align*}
\begin{aligned}
\alpha_2 \beta_1 (\eta^2 - \alpha_1^2 \beta_1^2)S_1 &= \alpha_1 \beta_2 (\eta^2 - \alpha_2^2 \beta_2^2)S_2, \\
\alpha_2 \beta_1 (\eta^2 - \alpha_1^2 \beta_1^2)C_1 &= (\eta^3 + \beta_1 (a - b + c)\eta)S_2, \\
\alpha_2 \beta_1 (\eta^2 - \alpha_2^2 \beta_2^2)C_2 &= (\eta^3 + \beta_1 (a - b + c)\eta)S_2, \\
(\alpha_2 + a \beta_2 \alpha_2^{-1})S_2C_1 + (\alpha_1 + a \beta_1 \alpha_1^{-1})S_1C_2 &= 0. 
\end{aligned}
\end{align*} \tag{4.24}

does not have nonzero solution.
Remark 4.4. If $B = 0$, then the closed loop system (2.1) is a conservative one; i.e., the energy of the closed system (2.1) is constant.

Remark 4.5. Here we point out that under the condition of $B \geq 0$, $\text{rank}(B) = 1$ and $d_1 = -d_2$, to study the relation between the asymptotic stability and the exponential stability of the closed loop system (2.1), we still need a further analysis for $|G_1|$. Hence, in this case, the question of the equivalence of these two types of the stability is not clear just for the time being.

References