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Conservation Laws of Evolution Equations: Generic Non-existence

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Evolution-type partial differential equations in one space variable are formulated in terms of exterior differential systems. The space of conservation laws is discussed in this geometric context, and a familiar classical condition for conservation laws is derived. It is shown that the generic even-order evolution equation with one space variable possesses no conservation laws of order greater than the order of the equation. © 1999 Academic Press

1. INTRODUCTION

It is well known that some partial differential equations, such as the Korteweg–de Vries equation [5, 8] and the vortex filament equation [6], are completely integrable systems; in particular, they possess infinite sequences of local conservation laws. Some other equations, such as the lubrication equation and modifications, possess finite numbers of local conservation laws which can be exploited in the analysis of the equation (see, for example, [2]). A generic equation, on the other hand, possesses no local conservation laws, and the task of computing the space of conservation laws of a given equation remains difficult in general. Some progress in this direction is given by the following theorem (see Theorem 5-1 and Theorem 5-2 for a more precise statement and the precise meaning of generic):

THEOREM 1-1. *The generic even-order system of evolution-type partial differential equations in one space variable possesses no local conservation laws of order greater than the order of the equation.*

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In this context, the term "generic" encompasses every scalar equation, including the heat equation, Burger's equation, and the lubrication equation, as well as many of the vector-valued generalizations of these.

The organization of this paper is as follows. In Section 2, precise definitions and explicit assumptions are given. Sections 3 and 4 contain a geometric description of evolution equations, the main result of which is a well-known necessary condition for existence of conservation laws, Proposition 4-3. Section 3 also contains a technical result concerning local existence of solutions, Lemma 2-3. In Section 5 (which can be read independent of Sections 3 and 4), the generic non-existence theorem for high-order conservation laws is proved.

2. DEFINITIONS

Let $F(t, x, u_0, \ldots, u_r)$ be an $\mathbf{R}^n$-valued function of $t \in \mathbf{R}$, $x \in \mathbf{R}$ or S^1, and $u_0, \ldots, u_r \in \mathbf{R}^n$. An *evolution equation* with one space variable, for the $\mathbf{R}^n$-valued function $u(t, x)$, is a partial differential equation of the form

$$u_t = F\big(t, x, u(t, x), u_x(t, x), \ldots, u_{x^{(r)}}(t, x)\big). \tag{2-1}$$

The *order* of this evolution equation is r. (Assume that the $n \times n$ matrix F_{u_r} is non-zero; that is, F depends explicitly on u_r.) The u_i can be thought of as coordinates representing $\partial^i u / \partial x^i$; this will be made precise in Section 3.

Associated with this differential equation, define differential operators D_x and D_t as follows. Let $H(t, x, u_0, \ldots, u_p)$ be a function (scalar or vector valued) of a finite number of variables; define

$$D_x H = \frac{\partial H}{\partial x} + \frac{\partial H}{\partial u_0} u_1 + \cdots + \frac{\partial H}{\partial u_p} u_{p+1}, \tag{2-2}$$

where the notation $(\partial H / \partial u_i) u_{i+1}$ represents matrix multiplication,

$$\frac{\partial H}{\partial u_i} u_{i+1} = \left[\frac{\partial H}{\partial u_i} u_{i+1} \right]^{\alpha} = \sum_{\beta} \frac{\partial H^{\alpha}}{\partial u_i^{\beta}} u_{i+1}^{\beta}$$

(similar matrix notation is used throughout). Define functions

$$F_i = (D_x)^i F \tag{2-3}$$

for $i = 0, 1, 2, \ldots$ $(F_0 = F)$; and finally, define

$$D_t H = \frac{\partial H}{\partial t} + \frac{\partial H}{\partial u_0} F_0 + \cdots + \frac{\partial H}{\partial u_p} F_p . \tag{2-4}$$

It is worth noting that $D_x u_i = u_{i+1}$ and that $D_t u_i = F_i$. Moreover, if $u(t,x)$ is any (smooth, local) solution to (2-1), and if $u_i(t,x) = \partial^i u(t,x)/\partial x^i$, then

$$\begin{aligned} &(D_x H)\big(t, x, u_0(t,x), \ldots, u_{p+1}(t,x)\big) \\ &\qquad = \frac{\partial}{\partial x}\Big(H\big(t, x, u_0(t,x), \ldots, u_p(t,x)\big)\Big) \\ &(D_t H)\big(t, x, u_0(t,x), \ldots, u_{p+r}(t,x)\big) \\ &\qquad = \frac{\partial}{\partial t}\Big(H\big(t, x, u_0(t,x), \ldots, u_p(t,x)\big)\Big). \end{aligned}$$

Thus, D_x and D_t represent space and time differentiation, respectively, on solutions of (2-1). (Also see Section 3, where the operators D_x and D_t are seen as dual to the forms dx and dt.)

A *local conservation law density* for the equation (2-1) is a pair of functions

$$H\big(t, x, u_0, \ldots, u_q\big),\ K\big(t, x, u_0, \ldots, u_k\big)$$

such that

$$D_t H = D_x K. \tag{2-5}$$

If such a pair exists, the function K is determined up to addition of an arbitrary function $k(t)$ by the function H; for this reason, the local conservation law density will be referred to by the function H without reference to K (but, for technical reasons, it is better to think of the pair (H, K)).

If $H(t, x, u_0, \ldots, u_q)$ is a local conservation law density, then, for any (local, sufficiently smooth) solution $u(t,x)$ of the evolution equation (2-1),

$$\begin{aligned} &\frac{\partial}{\partial t}\big(H\big(t, x, u(t,x), \ldots, u_{x^{(q)}}(t,x)\big)\big) \\ &\qquad = \frac{\partial}{\partial x}\big(K\big(t, x, u(t,x), \ldots, u_{x^{(k)}}(t,x)\big)\big); \end{aligned}$$

moreover,

$$\frac{d}{dt}\int H(t, x, u(t, x), \dots, u_{x^{(q)}}(t, x))\, dx$$
$$= \int \frac{\partial}{\partial x} K(t, x, u(t, x), \dots, u_{x^{(k)}}(t, x))\, dx$$

will vanish modulo boundary terms. (This is the motivation for the terminology: $\int H\,dx$ is a quantity which is conserved whenever $u(t, x)$ evolves by (2-1), assuming appropriate behaviour at the boundary.)

The local conservation law density H is *trivial* if

$$H = D_x G$$

for some function $G(t, x, u_0, \dots, u_p)$. If (H, K) is a trivial local conservation law density, then $K = D_t G$ (up to a function $k(t)$), and $D_t H = D_t D_x G = D_x D_t G = D_x K$.

Recall (see, for example, [1] or [9]) that the Euler–Lagrange operator (often referred to as the variational derivative $\frac{\delta}{\delta u}$) in x,

$$EL_x = \sum_l (-D_x)^l \circ \frac{\partial}{\partial u_l}, \tag{2-6}$$

has the property that $EL_x \circ D_x = 0$. Moreover, if $EL_x(H) = 0$, then locally, $H = D_x(G)$ for some function G. Thus, the non-trivial local conservation law densities are (locally) described by their images under the Euler–Lagrange operator. Define the space of *conservation laws* for the evolution equation (2-1) to be the set of all $\mathbf{R^n}$-valued functions

$$A(t, x, u_0, \dots, u_m) = {}^{t}EL_x(H), \tag{2-7}$$

where H is a local conservation law density as above. (This formalism requiring the Euler–Lagrange operator can be bypassed; see the remark near the end of Section 4. However, this formalism motivates the definition, below, of the order of the conservation law.)

If $A(t, x, u_0, \dots, u_m)$ is a conservation law, and $A_{u_m} \neq 0$, the *order* of the conservation law A is the number m. (If $A = A(t, x)$, then the order m is -1.) Let C_m be the set of all conservation laws of order at most m. C_m is a vector space over $\mathbf{R}$, for every m; and

$$C_{-1} \subseteq C_0 \subseteq C_1 \subseteq C_2 \subseteq \cdots . \tag{2-8}$$

Note. This definition of the order of a conservation law differs from the definitions given by some other authors. The definition used here is useful mainly for the purposes of Section 5. In the scalar ($n = 1$) case, a different definition would be the minimum of the order of the function H, over all local conservation law densities H which differ from each other by a trivial conservation law density; this definition corresponds to one-half the order as defined here. However, in the vector ($n \geq 2$) case, the correspondence between definitions is more subtle.

The main result which will be proved below is that, when r is even, and F is a generic (in a precise sense) function of order r, this sequence (2-8) of vector spaces stabilizes at C_r. (The scalar case ($n = 1$) was proved in [10], and generalizes one of the result in [4] for scalar parabolic second-order equations of evolution type.)

This provides a useful first step in the search for conservation laws: in this generic even-order case, all of the conservation laws depend on at most r space derivatives of u, and therefore the PDE system which describes conservation laws involves a bounded number of independent variables, a fact which greatly simplifies the problem.

3. PROLONGATION

A more geometric description of the evolution equation (2-1) will now be constructed. This will give (among other things) a more precise meaning to the interpretation of the u_i as space derivatives of u (see Lemma 3-1).

For each $k = 0, 1, \ldots,$ define

$$X_k = \{(t, x, u_0, \ldots, u_{k+r})\} \cong \mathbf{R}^{2+n(k+r+1)} \tag{3-1}$$

and, on X_k, define the differential forms

$$\begin{aligned} \theta_i^{(k)} &= du_i - u_{i+1}\,dx - F_i\,dt \qquad i = 0, \ldots, k \\ \psi_\alpha^{(k)} &= (du_{k+\alpha} - u_{k+\alpha+1}\,dx) \wedge dt \qquad \alpha = 1, \ldots, r-1. \end{aligned} \tag{3-2}$$

(In the future, the superscript (k) will be suppressed, except where it will be needed to distinguish between different k.) Let $\mathscr{I}_k$ be the differential ideal in $\Omega^*(T^*X_k)$ defined by

$$\mathscr{I}_k = \{\theta_i, \psi_\alpha\}, \tag{3-3}$$

that is, the ideal which is algebraically generated by $\{\theta_i, d\theta_i, \psi_\alpha, d\psi_\alpha\}$. (See [3] for an exposition of the theory of differential ideals.)

LEMMA 3-1. *Let*

$$\tilde{u}\colon \mathbf{R}^2 \to X_k\colon (t, x) \mapsto \big(t, x, u_0(t, x), u_1(t, x), \ldots, u_{k+r}(t, x)\big)$$

be a (sufficiently smooth) function. Then

$$\tilde{u}^*\mathscr{I}_k = \{0\} \tag{3-4}$$

if and only if $u(t, x) = u_0(t, x)$ *is a solution of* (2-1) *and* $u_j(t, x) = \partial^j u(t, x)/\partial x^j$ *for* $j = 0, \ldots, k + r$.

Proof. Let $\tilde{u}(t, x)$ be as above. Suppose (3-4) holds; that is, given any $\varphi \in \mathscr{I}_k$, $\tilde{u}^*\varphi = 0$. In particular, for every $i = 0, 1, 2, \ldots, k$,

$$\begin{aligned} 0 &= \tilde{u}^*\theta_i \\ &= (u_i)_x\, dx + (u_i)_t\, dt - u_{i+1}\, dx - F_i\, dt \\ &= \big((u_i)_x(t, x) - u_{i+1}(t, x)\big)\, dx \\ &\quad + \big((u_i)_t(t, x) - F_i\big(t, x, u_0(t, x), \ldots, u_{i+r}(t, x)\big)\big)\, dt. \end{aligned}$$

Thus

$$u_{i+1}(t, x) = (u_i)_x(t, x),$$

$$u_i(t, x) = \frac{\partial^i u}{\partial x^i}(t, x) \qquad i = 0, \ldots, k,$$

and

$$(u_i)_t(t, x) = F_i\big(t, x, u_0(t, x), \ldots, u_{i+r}(t, x)\big).$$

In particular (when $i = 0$), $(u_0)_t(t, x) = F(t, x, u_0(t, x), \ldots, u_r(t, x))$, so $u_0(t, x)$ satisfies (2-1). Moreover,

$$\begin{aligned} 0 &= \tilde{u}^*\psi_\alpha \\ &= \big((u_{k+\alpha})_x\, dx - u_{k+\alpha+1}\, dx\big) \wedge dt \\ &= \big((u_{k+\alpha})_x - u_{k+\alpha+1}\big)\, dx \wedge dt \end{aligned}$$

so

$$(u_{k+\alpha})_x(t, x) = \frac{\partial^{k+\alpha} u}{\partial x^{k+\alpha}} \qquad \alpha = 1, \ldots, r.$$

Conversely, suppose $u(t,x) = u_0(t,x)$ is a solution of (2-1), and $u_j(t,x) = u_{x^{(j)}}(t,x)$. Then

$$\begin{aligned} 0 &= \big((u_0)_x(t,x) - u_1(t,x)\big)\,dx \\ &\quad + \big((u_0)_t(t,x) - F\big(t,x,u_0(t,x),\dots,u_r(t,x)\big)\big)\,dt \\ &= \tilde{u}^*\theta_0. \end{aligned}$$

Moreover,

$$\begin{aligned} F_1\big(t,x,u_0(t,x),\dots,u_{r+1}(t,x)\big) &= D_x(F)\big(t,x,u_0(t,x),\dots,u_{r+1}(t,x)\big) \\ &= \frac{\partial}{\partial x}\big(F\big(t,x,u_0(t,x),\dots,u_r(t,x)\big)\big) \end{aligned}$$

so

$$\begin{aligned} 0 &= \big((u_1)_x(t,x) - u_2(t,x)\big)\,dx \\ &\quad + \big((u_0)_t(t,x) - F\big(t,x,u_0(t,x),\dots,u_r(t,x)\big)\big)_x\,dt \\ &= \big((u_1)_x(t,x) - u_2(t,x)\big)\,dx \\ &\quad + \big((u_1)_t(t,x) - F_1\big(t,x,u_0(t,x),\dots,u_{r+1}(t,x)\big)\big)\,dt \\ &= \tilde{u}^*\theta_1. \end{aligned}$$

Continuing in this manner, it is easy to see that

$$0 = \tilde{u}^*\theta_i$$

for all $i = 0,\dots,k$; and

$$\begin{aligned} 0 &= \big((u_{k+1})_x(t,x) - u_{k+2}(t,x)\big)\,dx \wedge dt \\ &= \big((u_{k+1})_x(t,x)\,dx - u_{k+2}(t,x)\,dx\big) \wedge dt \\ &= \big((u_{k+1})_x(t,x)\,dx + (u_{k+1})_t(t,x)\,dt - u_{k+2}(t,x)\,dx\big) \wedge dt \\ &= \big(du_{k+1}(t,x) - u_{k+2}(t,x)\,dx\big) \wedge dt \\ &= \tilde{u}^*\psi_1 \end{aligned}$$

and similarly for the other ψ_α.

Since $\mathcal{I}_k$ is generated by $\{\theta_i, \psi_\alpha\}$, it follows that $\tilde{u}^*\mathcal{I}_k = \{0\}$, as desired. ∎

A function $\tilde{u}$ as above is an *integral manifold* of $\mathcal{I}_k$.

The following lemma justifies the fact that the u_i can be treated as "independent variables on solutions." This is so that expressions such as

$D_t H - D_x K$ can be differentiated with respect to the u_i, "on solutions," and therefore there must be a family of solutions parameterized by the varying u_i. The method of the proof used below can also be applied (with appropriate modifications) to an arbitrary (analytic, involutive) system of (possibly overdetermined) PDE.

LEMMA 3-2. *Assume the function F is analytic. Then, given arbitrary* $(\hat{t}, \hat{x}, \hat{u}_0, \ldots, \hat{u}_q)$ (*for any q*) *in an open dense subset of* $\mathbf{R}^{2+n(q+1)}$, *there is a local solution $u(t, x)$ of the evolution equation* (2-1) *such that*

$$u(\hat{t}, \hat{x}) = \hat{u}_0$$

$$\vdots$$

$$\frac{\partial^q u}{\partial x^q}(\hat{t}, \hat{x}) = \hat{u}_q.$$

Proof. The open dense subset condition arises from the requirement in the following proof that rk F_{u_r} be constant; henceforth, assume that it is constant.

By Lemma 3-1, it suffices to show that there is an integral manifold passing through the point

$$\left(\hat{t}, \hat{x}, \hat{u}_0, \ldots, \hat{u}_q, \hat{u}_{q+1}, \ldots, \hat{u}_{q+r}\right) \in X_q$$

for some $\hat{u}_{q+1}, \ldots, \hat{u}_{q+r}$. In other words, it suffices to show that, for any k, there is a local integral manifold through any point in X_k.

Note that (for any fixed k)

$$d\theta_i = -\theta_{i+1} \wedge dx - \sum_{j=0}^{i+r} (F_i)_{u_j} \theta_j \wedge dt$$

for $i = 0, \ldots, k - r$,

$$d\theta_i = -\theta_{i+1} \wedge dx - \sum_{j=0}^{k} (F_i)_{u_j} \theta_j \wedge dt - \sum_{j=k+1}^{i+r} (F_i)_{u_j} \psi_{j-k}$$

for $i = k - r + 1, \ldots, k - 1$, and

$$d\theta_k = -du_{k+1} \wedge dx - \sum_{j=0}^{k} (F_k)_{u_j} \theta_j \wedge dt - \sum_{j=k+1}^{k+r-1} (F_k)_{u_j} \psi_{j-k}$$

$$- \left((F_k)_x + \sum_{j=0}^{k+r-1} (F_k)_{u_j} u_{j+1} \right) dt \wedge dx - (F_k)_{u_{k+r}} du_{k+r} \wedge dt;$$

or, briefly,

$$
\begin{aligned}
d\theta_i &\equiv 0 \bmod\{\theta\} \qquad i = 0, \dots, k-r \\
d\theta_i &\equiv 0 \bmod\{\theta, \psi\} \qquad i = k-r+1, \dots, k-1 \\
d\theta_k &\equiv -\left((F_k)_{u_{k+r}} du_{k+r} \right. \\
&\qquad \left. - \left((F_k)_x + \sum_{j=0}^{k+r-1} (F_k)_{u_j} u_{j+1} \right) dx \right) \wedge dt \bmod\{\theta, \psi\}.
\end{aligned} \tag{3-5}
$$

Before continuing, it is useful to introduce some notation which will avoid some possible confusion. Let

$$
\begin{aligned}
e_t &= \partial_t \\
e_x &= \partial_x \\
e_{u_j} &= \left(\partial_{u_j^1}, \dots, \partial_{u_j^n} \right)
\end{aligned}
$$

represent the coordinate basis of tangent vectors on X_k, dual to the basis (dt, dx, du_j) of 1-forms. (Note, in particular, that e_{u_j} is a row vector.)

By the structure equations (3-5), the space of integral 2-planes on which $dt \wedge dx \neq 0$,

$$
V_2(\mathscr{I}_k) = \left\{ E_2 \in G_2(TX_k) : dt \wedge dx|_{E_2} \neq 0,\ \varphi|_{E_2} = 0 \quad \text{for all } \varphi \in \mathscr{I}_k \right\},
$$

which is defined by the (algebraic) equations

$$
\begin{aligned}
0 &= \theta_i \\
0 &= \psi_\alpha \\
0 &= d\theta_i
\end{aligned}
$$

(because the forms of degree three and higher all vanish on every 2-plane), is a smooth sub-bundle of $G_2(TX_k) \to X_k$, given by

$$
\begin{aligned}
V_2(\mathscr{I}_k) &= \left\{ \left(e_t + \sum_{i=0}^{k+1} e_{u_i} F_i + \sum_{\alpha=2}^{r} e_{u_{k+\alpha}} v_\alpha \right) \right. \\
&\qquad \left. \wedge \left(e_x + \sum_{i=0}^{k} e_{u_i} u_{i+1} + \sum_{\alpha=1}^{r} e_{u_{k+\alpha}} u_{k+\alpha+1} \right) \right\} \\
&= X_k \times \left\{ (u_{k+r+1}, v_2, \dots, v_r) \right\} \\
&\cong X_k \times \mathbf{R}^{nr}.
\end{aligned}
$$

(Note that u_{k+r+1} is a fiber coordinate; labelling it as u_{k+r+1} rather than v_1 will prove to be convenient.)

The prolongation $(X_k^{(1)}, \mathscr{I}_k^{(1)})$ of $(X_k, \mathscr{I}_k)$ is defined to be the pullback of the tautological differential ideal on $G_2(TX_k)$ to $V_2(\mathscr{I}_k)$ (see Chap. VI of [3]). In this case, $V_2(\mathscr{I}_k)$ is smooth, and $\mathscr{I}_k^{(1)}$ is generated by

$$\eta_i = du_i - u_{i+1}\,dx - F_i\,dt \qquad i = 0, \ldots, k+1$$
$$\eta_i = du_i - u_{i+1}\,dx - v_{i-k}\,dt \qquad i = k+2, \ldots, k+r.$$

Define

$$\hat{F} = (F_{k+1})_x + \sum_{j=0}^{k+r} (F_{k+1})_{u_j} u_{j+1};$$

then the structure equations of $\mathscr{I}_k^{(1)}$ are

$$d\eta_i \equiv 0 \qquad i = 0, \ldots, k$$
$$d\eta_{k+1} \equiv -\Big((F_{k+1})_{u_{k+r+1}}\,du_{k+r+1} - \big(\hat{F} + v_2\big)\,dx\Big) \wedge dt$$
$$d\eta_{k+\alpha} \equiv -(dv_\alpha - v_{\alpha+1}\,dx) \wedge dt \qquad \alpha = 2, \ldots, r-1$$
$$d\eta_{k+r} = -du_{k+r+1} \wedge dx - dv_r \wedge dt$$

where congruences are mod$\{\eta_0, \ldots, \eta_{k+r}\}$. These structure equations can be rewritten (noting that $(F_{k+1})_{u_{k+r+1}} = F_{u_r}$) as

$$d\begin{bmatrix} \eta_0 \\ \vdots \\ \eta_k \\ \eta_{k+1} \\ \eta_{k+2} \\ \vdots \\ \eta_{k+r-1} \\ \eta_{k+r} \end{bmatrix} \equiv -\begin{bmatrix} 0 & 0 \\ \vdots & \vdots \\ 0 & 0 \\ F_{u_r}\,du_{k+r+1} - \big(\hat{F} + v_2\big)\,dx & 0 \\ dv_2 - v_3\,dx & 0 \\ \vdots & \vdots \\ dv_{r-1} - v_r\,dx & 0 \\ dv_r & du_{k+r+1} \end{bmatrix} \wedge \begin{bmatrix} dt \\ dx \end{bmatrix} \mathrm{mod}\{\eta\}. \tag{3-6}$$

The integral 2-planes of $\mathscr{I}_k^{(1)}$ are defined by the algebraic equations

$$0 = \eta_j$$
$$0 = d\eta_j$$

which can be solved as follows. Any 2-plane on which $\eta_j = 0$ is defined by

$$\left.\begin{aligned} du_{k+r+1} &\equiv P_1^1\,dt + P_1^2\,dx \\ dv_2 &\equiv P_2^1\,dt + P_2^2\,dx \\ &\vdots \\ dv_r &\equiv P_r^1\,dt + P_r^2\,dx \end{aligned}\right\} \mod\{\eta\} \tag{3-7}$$

(with parameters $P_\alpha^\beta \in \mathbf{R}^n$); and then the equations $d\eta_j = 0$ become (using (3-7)) the following system of equations for the P:

$$\begin{aligned} 0 &= \hat{F} + v_2 - F_{u_r} P_1^2 \\ 0 &= v_3 - P_2^2 \\ &\vdots \\ 0 &= v_r - P_{r-1}^2 \\ 0 &= -P_r^2 + P_1^1. \end{aligned} \tag{3-8}$$

Suppose the matrix F_{u_r} has rank $\operatorname{rk} F_{u_r} = n - p$. (If $p = 0$ then the equations (3-8) for the P always have solutions, in which case much of the following discussion is unnecessary.) The first of the equations (3-8) has a solution only when $\hat{F} + v_2$ is in the image of the linear map F_{u_r}. Recalling the assumption that p is constant (which is true on an open dense subset of X_k), the set of all v_2 satisfying this condition can be parametrized by

$$v_2 = A_2 \hat{v}_2 + B_2,$$

where $\hat{v}_2 \in \mathbf{R}^{n-p}$, $A_2\colon \mathbf{R}^{n-p} \to \mathbf{R}^n$, and $B_2 = -\hat{F}$ is a function of the variables $(t, x, u_0, \dots, u_{k+r}; u_{k+r+1})$. (Note, in particular, that B_2 depends on the fiber variable u_{k+r+1}.) This defines a submanifold $Y_{k,2}$ of $X_k^{(1)}$, given by

$$\begin{aligned} Y_{k,2} &= \{(t, x, u_0, \dots, u_{k+r}; u_{k+r+1}, A_2\hat{v}_2 + B_2, v_3, \dots, v_r)\} \\ &\subseteq \{(t, x, u_0, \dots, u_{k+r}; u_{k+r+1}, v_2, v_3, \dots, v_r)\} = X_k^{(1)}. \end{aligned}$$

Pull back the differential ideal $\mathscr{I}_k^{(1)}$ to $Y_{k,2}$; then repeat the above computations, with the following modifications. The 2-planes over $Y_{k,2}$ on which $\eta_j = 0$ are parametrized as in (3-7) except for

$$d\hat{v}_2 = \hat{P}_2^1\,dt + \hat{P}_2^2\,dx$$

where $\hat{P}_2^1, \hat{P}_2^2 \in \mathbf{R}^{n-p}$; and therefore

$$\begin{aligned} dv_2 &= A_2\left(\hat{P}_2^1\, dt + \hat{P}_2^2\, dx\right) + d(A_2)\hat{v}_2 + d(B_2) \\ &\equiv A_2\left(\hat{P}_2^1\, dt + \hat{P}_2^2\, dx\right) + ((A_2)_x\, dx + (A_2)_t\, dt)\hat{v}_2 \\ &\quad + \left((B_2)_t\, dt + (B_2)_x\, dx + (B_2)_{u_{k+r+1}}\, du_{k+r+1}\right) \quad \operatorname{mod}\{\eta\}. \end{aligned}$$

Upon substituting this into the equations $d\eta_j = 0$ (analogous to (3-8)) defining integral elements on $Y_{k,2}$, another restricting condition arises, namely, that v_3 lie in a certain $(n-p)$-dimensional subspace of $\mathbf{R}^n$; and this defines another submanifold,

$$\begin{aligned} Y_{k,3} &= \left\{\left(t, x, u_0, \ldots, u_{k+r}; u_{k+r+1}, A_2\hat{v}_2 + B_3, A_3\hat{v}_3 + B_2, v_4, \ldots, v_r\right)\right\} \\ &\subseteq \left\{\left(t, x, u_0, \ldots, u_{k+r}; u_{k+r+1}, A_2\hat{v}_2 + B_2, v_3, v_4, \ldots, v_4\right)\right\} = Y_{k,2} \\ &\subseteq \left\{\left(t, x, u_0, \ldots, u_{k+r}; u_{k+r+1}, v_2, v_3, v_4, \ldots, v_r\right)\right\} = X_k^{(1)}, \end{aligned}$$

where $\hat{v}_3 \in \mathbf{R}^{n-p}$, and B_3 is a function of $(t, x, u_0, \ldots, u_{k+r}; u_{k+r+1}, \hat{v}_2)$.

This process continues, resulting in $v_4, \ldots, v_r$ lying in certain $(n-p)$-dimensional subspaces of $\mathbf{R}^n$ and defining the sequence of submanifolds

$$Y_k = Y_{k,r} \subset \cdots \subset Y_{k,2} \subset X_k^{(1)}.$$

(Note, though, that there are no conditions imposed on u_{k+r+1}.) On Y_k, there is also a differential system $\mathcal{J}_k$, the restriction of $\mathcal{I}_k^{(1)}$.

It is now clear (by construction) that, over every point of Y_k, the equations (3-8) (restricted to Y_k) for the P have solutions, and the solution space is parametrized by $P_2^1, P_3^1, \ldots, P_r^1, P_r^2$, and p of the P_1^2 (because $\dim\ker F_{u_r} = p$); in other words, the space $V_2(\mathcal{J}_k)$ of integral 2-planes of the differential system $\mathcal{J}_k$ has dimension $r(n-p)+p$:

$$\dim V_2(\mathcal{J}_k) = nr - pr + p.$$

Finally, the Cartan characters s_i (see Chap. IV of [3]) can be easily computed. Briefly, s_1 is the maximal (allowing invertible row and column operations) number of independent 1-forms in the first column of the matrix Π, where

$$d\eta \equiv -\Pi \wedge \begin{bmatrix} dt \\ dx \end{bmatrix} \operatorname{mod}\{\eta\};$$

s_2 is the maximal of additional independent 1-forms in the second column; and so on. In the case considered here, by adding a generic multiple of the second column to the first, all of the 1-forms can be put into the first row, and therefore $s_1 = n + (n-p)(r-1)$ and $s_2 = 0$. Note that $s_1 + 2s_2 = nr - pr + p = \dim V_2(\mathcal{J}_k)$, and therefore, by Cartan's test, the differential ideal $\mathcal{I}_k^{(1)}$ pulled back to Y_k is involutive, and therefore by the Cartan–Kähler theorem, there are local integral manifolds through every point of Y_k (and hence over every point of X_k). (Recall the assumption of constant rank $n-p$. To be precise, Y_k should be defined not as above, but restricted to those points at which p is minimal.) Since the integral manifolds of $\mathcal{J}_k$ on Y_k are in one-to-one correspondence with the integral manifolds of $\mathcal{I}_k$ on X_k, there are integral manifolds through every point of X_k, as desired. ∎

Note that, although Matsushima's theorem (see Chap. VI of [3]) guarantees that all prolongations of an involutive ideal are involutive, it is, in this case, rather inconvenient to work with the prolongations of the involutive ideal on Y_k as constructed in the preceeding proof. Instead, it is simpler to work with the ideals $\mathcal{I}_k$ defined above.

Consider now the sequence

$$\cdots \xrightarrow{\pi_{k+1}} X_{k+1} \xrightarrow{\pi_k} X_k \xrightarrow{\pi_{k-1}} X_{k-1} \xrightarrow{\pi_{k-2}} \cdots \xrightarrow{\pi_1} X_1 \xrightarrow{\pi_0} X_0 .$$

Observe that

$$\pi_k^* \theta_i^{(k)} = \theta_i^{(k+1)} \qquad i = 0, \dots, k$$

$$\pi_k^* \psi_1^{(k)} = \theta_{k+1}^{(k+1)} \wedge dt$$

$$\pi_k^* \psi_\alpha^{(k)} = \psi_{\alpha-1}^{(k+1)} \qquad \alpha = 2, \dots, r-1,$$

and therefore $\pi_k^* \mathcal{I}_k \subset \mathcal{I}_{k+1}$; in other words, there is a sequence of inclusions

$$\mathcal{I}_0 \xrightarrow{\pi_0^*} \mathcal{I}_1 \xrightarrow{\pi_1^*} \cdots \xrightarrow{\pi_{k-2}^*} \mathcal{I}_{k-1} \xrightarrow{\pi_{k-1}^*} \mathcal{I}_k \xrightarrow{\pi_k^*} \mathcal{I}_{k+1} \xrightarrow{\pi_{k+1}^*} \cdots .$$

Define X to be the inverse limit

$$X = \varprojlim X_k ;$$

let $\hat{\pi}_k \colon X \to X_k$ be the projection. Define $\mathcal{I}$ on X by $\mathcal{I} = \bigcup_k \hat{\pi}_k^* \mathcal{I}_k$. Formally, $X = \{(t, x, u_0, u_1, \dots)\}$; functions on X are functions of a finite number of variables and hence well-defined on some X_k; and a form in $\mathcal{I}$ is a form in some $\mathcal{I}_k$. (More precisely, the element $\phi \in \mathcal{I}$ is $\hat{\pi}_k^* \phi_k$ for some $\phi_k \in \mathcal{I}_k$.) Clearly $\mathcal{I}$ is a differential ideal in $\Omega^*(T^*X) = \bigcup_k \hat{\pi}_k^* \Omega^*(T^*X_k)$, and $\mathcal{I}$ is generated by

$$\{\theta_i = du_i - u_{i+1}\, dx - F_i\, dt \colon i = 0, 1, 2, \dots\}$$

with structure equations

$$d\theta_i = -\theta_{i+1} \wedge dx - \sum_j (F_i)_{u_j} \theta_j \wedge dt. \tag{3-9}$$

(Note that, even though $d\theta \equiv 0 \bmod \{\theta\}$, the Frobenius theorem cannot be applied, because $\mathscr{I}$ has an infinite number of generators θ_i.)

The horizontal derivatives D_x and D_t can be interpreted as follows. A basis of $T_p^* X$ is $(dt, dx, \theta_0, \theta_1, \ldots)$; the dual basis of $T_p X$ is $(D_t, D_x, \partial/\partial u_0, \partial/\partial u_1, \ldots)$.

Let $\mathscr{I}^j$ be the j-forms in the differential ideal $\mathscr{I}$; that is, $\mathscr{I}^j = \mathscr{I} \cap \Omega^j(X)$. The following normal form decomposition for $\mathscr{I}^2$ will prove useful in computations.

LEMMA 3-3. *Let*

$$\mathscr{E}^2 = \left\{ {}^t A \theta_0 \wedge dx + \sum_i {}^t B^i \theta_i \wedge dt + \sum_{i,j} {}^t C^{ij} \theta_i \wedge \theta_j \right\} \subseteq \mathscr{I}^2,$$

where all sums are finite, and A, B^i, C^{ij} are functions of a finite number of $(t, x, u_0, u_1, \ldots)$. Then

$$\mathscr{I}^2 \cong \mathscr{E}^2 \oplus d(\mathscr{I}^1).$$

Proof. Any 2-form $\varphi \in \mathscr{I}^2$ can be written as

$$\varphi = \sum_{i=0}^{k_1} {}^t A^i \theta_i \wedge dx + \sum_i {}^t B^i \theta_i \wedge dt + \sum_{i,j} C^{ij} \theta_i \wedge \theta_j,$$

where the summation indices i, j have a finite range, and the functions A^i, B^i, and C^{ij} are functions of a finite number of the variables $(t, x, u_0, \ldots)$.

Given any $k_1 > 0$,

$$\begin{aligned} {}^t A^{k_1} \theta_{k_1} \wedge dx = & -d\left({}^t A^{k_1} \theta_{k_1 - 1}\right) - D_x {}^t A^{k_1} \theta_{k_1 - 1} \wedge dx \\ & - \sum_u \left((D_x)^{k_1} F \right)_{u_j} \theta_j \wedge dt + \sum_j ({}^t A^{k_1})_{u_j} \theta_j \wedge \theta_{k_1 - 1} \\ = & -D_x {}^t A^{k_1} \theta_{k_1 - 1} \wedge dx - \sum_j {}^t \tilde{B}^j \theta_j \wedge dt + Q(\theta, \theta) + d\alpha, \end{aligned}$$

where $Q(\theta, \theta)$ is a quadratic term (i.e., a linear combination of $\theta_j \wedge \theta_k$) and $\alpha \in \mathscr{I}^1$ is a 1-form in $\mathscr{I}$.

Thus, by modifying $A^{k_1 - 1}$, B^i, and C^{ij} in the expression for φ, the A^{k_1} coefficient can be absorbed into B^i, C^{ij}, and a term in $d(\mathscr{I}^1)$; iterating this, all but the A^0 coefficients can be similarly absorbed. This yields the desired normal form. ∎

(An easy modification of this proof shows that $\Omega^2(X) \cong \mathscr{E}^2 \oplus d(\mathscr{I}^1) \oplus \operatorname{span}\{dt \wedge dx\}$.)

4. CONSERVATION LAWS

The preceding formulation of the evolution equation can now be applied to the search for conservation laws.

PROPOSITION 4-1. *Let*

$$C_\infty = \bigcup_{m=-1}^{\infty} C_m$$

be the space of all finite-order conservation laws. Then C_∞ is locally isomorphic to

$$H^2(\mathscr{I}) = \frac{\ker d|_{\mathscr{I}^2}}{\{d(\theta)\colon \theta \in \mathscr{I}^1\}}.$$

Proof. Let $U = \{(H, K)\colon D_t H = D_x K\}$ be the vector space of local conservation law densities, and let $U_0 \subseteq U$ be the space of trivial local conservation law densities. Recall that, by the discussion preceding (2-7), $EL_x\colon U \to C_\infty$ has kernel U_0 and therefore induces an isomorphism $U/U_0 \cong C_\infty$. Thus it suffices to show that $U/U_0 \cong H^2(\mathscr{I})$; this will be done by constructing a map $\psi\colon U \to \ker d|_{\mathscr{I}^2}$, which induces an isomorphism $\Psi\colon U/U_0 \to H^2(\mathscr{I})$.

Given $(H, K) \in U$, define

$$\psi(H, K) = \varphi = d(H\,dx + K\,dt).$$

Observe that

$$\begin{aligned}\varphi &= d(H\,dx + K\,dt)\\ &= (D_t H - D_x K)\,dt \wedge dx + \sum_i \left(H_{u_i}\theta_i \wedge dx + K_{u_i}\theta_i \wedge dt\right)\\ &= \sum_i \left(H_{u_i}\theta_i \wedge dx + K_{u_i}\theta_i \wedge dt\right),\end{aligned}$$

so $\varphi \in \mathscr{I}^2$; clearly φ is closed.

The linear map ψ induces a linear map $[\psi]\colon U \to H^2(\mathscr{I})$ defined by $[\psi](H, K) = [\psi(H, K)]$ (the equivalence class of φ in the quotient $H^2(\mathscr{I})$).

If $[\varphi] \in H^2(\mathscr{I})$, then choose a representative closed 2-form $\varphi \in \mathscr{I}^2$. By the converse of the Poincaré lemma, $\varphi = d\alpha$ for some 1-form α. This 1-form can be written as $\alpha = H\,dx + K\,dt + \theta + dG$, where $\theta \in \mathscr{I}^1$, H and K are some functions, and G is an arbitrary function. Then $d\alpha \equiv (D_t H - D_x K)\,dt \wedge dx \bmod \mathscr{I}$, but $d\alpha \in \mathscr{I}$, so $(H, K) \in U$. Moreover, $d(H\,dx + K\,dt) = d(\alpha - \theta - dG) \equiv d\alpha \bmod d(\mathscr{I}^1)$; thus $[d(H\,dx + K\,dt)] = [\varphi]$. Therefore $[\psi]$ is a surjective map.

The kernel of $[\psi]$ can be computed as follows. Recall that

$$\begin{aligned} U_0 &= \{(H, K): D_t H = D_x K, H = D_x G\} \\ &= \left\{(H, K): D_t H = D_x K, H = D_x \tilde{G}, K = D_t \tilde{G}\right\}, \end{aligned}$$

where G and $\tilde{G}$ differ by some function $g(t)$. Now compute

$$\begin{aligned} \ker[\psi] &= \left\{(H, K): D_t H = D_x K, d(H\,dx + K\,dt) \in d(\mathscr{I}^1)\right\} \\ &= \left\{(H, K): D_t H = D_x K, d(H\,dx + K\,dt) = d\theta,\ \theta \in \mathscr{I}^1\right\} \\ &= \left\{(H, K): D_t H = D_x K, H\,dx + K\,dt = \theta + dG\right\} \\ &= \Big\{(H, K): D_t H = D_x K, H\,dx + K\,dt \\ &\qquad = \theta + D_x G\,dx + D_t G\,dt + \sum_i G_{u_i}\theta_i\Big\} \\ &= \Big\{(H, K): D_t H = D_x K, \theta = -\sum_i G_{u_i}\theta_i, H = D_x G, K = D_t G\Big\} \\ &= U_0. \end{aligned}$$

Since $[\psi]\colon U \to H^2(\mathscr{I})$ is surjective and has kernel U_0, it induces an isomorphism $\Psi\colon U/U_0 \to H^2(\mathscr{I})$ defined by $\Psi([(H, K)]) = [\psi(H, K)]$. ∎

PROPOSITION 4-2. *The space of conservation laws is*

$$C_\infty \cong \ker d|_{\mathscr{E}^2}.$$

Proof.

$$\begin{aligned} C_\infty &\cong \frac{\ker d|_{\mathscr{I}^2}}{d(\mathscr{I}^1)} \\ &\cong \frac{\ker d|_{\mathscr{E}^2 \oplus d(\mathscr{I}^1)}}{d(\mathscr{I}^1)} \end{aligned}$$

by Lemma 3-3; note that $d(\mathscr{I}^1) \subseteq \ker d|_{\mathscr{E}^2 \oplus d(\mathscr{I}^1)}$ because $d^2 = 0$. ∎

Finally, the preceding results can be used to obtain the following

PROPOSITION 4-3. *For every m, C_m injects into the set of solutions to the linear system of partial differential equations*

$$0 = D_t A + \sum_{l=0}^{r} (-D_x)^l \left({}^{\mathrm{t}}(F_{u_l}) A\right) \tag{4-1}$$

for the function $A(t, x, u_0, \dots, u_m)$. That is, the equation (4-1) *is a necessary condition for A to be a conservation law.*

Proof. By Proposition 4-2 and the discussion preceding the definition (2-7), any conservation law $A = {}^{\mathrm{t}}EL_x(H)$ has a unique representative $\varphi \in \mathscr{E}^2$ which is closed, and conversely, any closed $\varphi \in \mathscr{E}^2$ represents a unique conservation law A. Thus, let $\varphi \in \mathscr{E}^2$ and write

$$\varphi = {}^{\mathrm{t}}A\theta_0 \wedge dx + \sum_{i=0}^{k} {}^{\mathrm{t}}B^i\theta_i \wedge dt + \sum_{i,j} {}^{\mathrm{t}}C^{ij}\theta_i \wedge \theta_j.$$

Now examine one of the conditions for φ to be closed. The coefficients of $\theta_i \wedge dt \wedge dx$ in the equation $d\varphi = 0$ are

$$\begin{aligned}
i = 0&: 0 = -D_t{}^{\mathrm{t}}A - {}^{\mathrm{t}}AF_{u_0} + D_x{}^{\mathrm{t}}B^0 \\
i = 1&: 0 = -{}^{\mathrm{t}}AF_{u_1} + D_x{}^{\mathrm{t}}B^1 + {}^{\mathrm{t}}B^0 \\
&\vdots \\
i = r&: 0 = -{}^{\mathrm{t}}AF_{u_r} + D_x{}^{\mathrm{t}}B^r + {}^{\mathrm{t}}B^{r-1} \\
i = r+1&: 0 = D_x{}^{\mathrm{t}}B^{r+1} + {}^{\mathrm{t}}B^r \\
&\vdots \\
i = k&: 0 = D_x{}^{\mathrm{t}}B^k + {}^{\mathrm{t}}B^{k-1} \\
i = k+1&: 0 = {}^{\mathrm{t}}B^k.
\end{aligned}$$

(Without loss of generality, $k > r$; otherwise, set $B^k = 0$, etc., if necessary.) Hence

$$\begin{aligned}
B^i &= 0 \quad \text{for } i \geq r \\
{}^{\mathrm{t}}B^{r-1} &= {}^{\mathrm{t}}AF_{u_r} \\
{}^{\mathrm{t}}B^{r-2} &= {}^{\mathrm{t}}AF_{u_{r-1}} - D_x{}^{\mathrm{t}}B^{r-1} \\
&\vdots \\
{}^{\mathrm{t}}B^0 &= {}^{\mathrm{t}}AF_{u_1} - D_x{}^{\mathrm{t}}B^1 \\
-D_t{}^{\mathrm{t}}A &= {}^{\mathrm{t}}AF_{u_0} - D_x{}^{\mathrm{t}}B^0,
\end{aligned}$$

which, by recursively eliminating the B^i, implies that

$$ {}^{\mathrm{t}}B^{r-1-i} = \sum_{j=0}^{i} (-D_x)^j \left({}^{\mathrm{t}}AF_{u_{r-i+j}} \right) $$

or, relabelling,

$$ {}^{\mathrm{t}}B^{i} = \sum_{j=0}^{r-1-i} (-D_x)^j \left({}^{\mathrm{t}}AF_{u_{i+j+1}} \right), $$

where $i = 0, \ldots, r-1$ and

$$ -D_t {}^{\mathrm{t}}A = \sum_{l=0}^{r} (-D_x)^l \left({}^{\mathrm{t}}AF_{u_l} \right). $$

Taking the transpose of this gives the desired equation. ∎

It is worth noting that, if (H, K) is a local conservation law density, letting $\varphi = d(H\,dx + K\,dt) \in \mathscr{I}^2$ and following the proof of Lemma 3-3, the coefficient of $\theta_0 \wedge dx$ which appears in the normal form is precisely the function A which appears in Proposition 4-3; and $A = {}^{\mathrm{t}}EL_x(H)$, just as in the definition (2-7). In fact, recall the isomorphisms

$$ H^2(\mathscr{I}) \cong U/U_0 $$

$$ H^2(\mathscr{I}) \cong \ker d|_{\mathscr{E}^2}, $$

and consider the mappings defined, respectively, by

$$ (H, K) \mapsto [(H, K)] \in U/U_0 $$

and

$$ (H, K) \mapsto H\,dx + K\,dt \mapsto d(H\,dx + K\,dt) \mapsto \varphi \oplus d\theta, $$

where $\varphi \in \mathscr{E}^2$ and $\theta \in \mathscr{I}^1$; the coefficient of $\theta_0 \wedge dx$ in φ is $A = {}^{\mathrm{t}}EL_x(H)$. This can be used to define the order m of (H, K) without (explicit) reference to (or motivation from) the construction involving EL_x, and can therefore be used to define the spaces C_m and C_∞. A straightforward argument using the above isomorphisms and mappings proves that $C_\infty \cong U/U_0$, thereby providing an interesting alternative to the construction motivated by the variational operator.

Note that the equation (4-1) is well known, in various forms. (It appears, for example, in [7], and as a consequence of a necessary and sufficient condition in [9].) It is perhaps interesting to note that there are some equations, for example, the heat equation $u_t = u_{xx}$, for which (4-1) is sufficient to characterize conservation laws; however, for other equations, such as the linear third-order equation $u_t = u_{xxx}$, the equation (4-1) is not sufficient (consider the function $A = u_3$). The reason (4-1) is, in general, not a sufficient condition for conservation laws is that it came from just the $\theta_i \wedge dt \wedge dx$ coefficients of $d\varphi$; the vanishing of the $\theta_i \wedge \theta_j \wedge dt$, $\theta_i \wedge \theta_j \wedge dx$, and $\theta_i \wedge \theta_j \wedge \theta_k$ coefficients often provide additional conditions independent of (4-1).

5. GENERIC NON-EXISTENCE

The necessary condition for the existence of a conservation law, given by Eq. (4-1), is (when $r \geq 2$) an over-determined system of linear partial differential equations for the (vector-valued) function $A(t, x, u_0, \ldots, u_m)$; being over-determined, it is reasonable to expect there to be (generically) no non-trivial solutions, except for some small class of functions F. The condition(s) on F which allow non-trivial solutions of (4-1) are the integrability conditions of (4-1); these conditions give obstructions to the existence of conservation laws.

The first of these integrability conditions is given in the following

THEOREM 5-1. *If $r \geq 2$ is even and the matrix $\partial F / \partial u_r$ satisfies a generic eigenvalue condition* ((5-3) *and* (5-4)), *then the evolution equation* (2-1) *possesses no conservation laws of order strictly greater than r.*

Proof. A necessary condition for $A(t, x, u_0, \ldots, u_m)$ to be a conservation law (of order m, with $A_{u_m} \neq 0$) of the evolution equation (2-1) is (4-1). When $m \geq r$, this is a polynomial in the (vector) variables $u_{m+1}, \ldots, u_{m+r}$, whose (tensorial) coefficients are differential operators applied to A. In order for (4-1) to hold, each of these coefficients must vanish. In particular, the (matrix) coefficient of u_{m+r} must vanish.

The following observations are useful for computing the coefficient (and indeed, for studying Eq. (4-1) in general):

- Application of the differential operator D_x to a function $B(t, x, u_0, \ldots, u_b)$ of order b yields a function whose order is $b + 1$, and $(D_x(B))_{u_{b+1}} = B_{u_b}$.
- Application of the differential operator D_t to a function B of order b as above yields a function whose order is $b + r$, and $(D_t(B))_{u_{b+r}} = B_{u_b} F_{u_r}$.
- For any $0 \leq l \leq r$, $(D_x)^l (F_{u_l})$ is a function of $(t, x, u_0, \ldots, u_{l+r})$, and $(D_x)^l (F_{u_l})$ is independent of u_{m+r} for any $m > r$ (because $l + r \leq 2r < m + r$).

Using these observations, the coefficient of u_{m+r} in (4-1) is easily seen to be the matrix

$$A_{u_m} F_{u_r} + (-1)^{r}\,{}^{t}(F_{u_r}) A_{u_m}, \tag{5-1}$$

which, as was said before, must vanish in order for A to be a conservation law. This linear algebra problem can be translated into a simpler form, by defining $n \times n$ matrices

$$C = A_{u_m} \neq 0 \quad \text{(by hypothesis)}$$
$$M = F_{u_r}$$

and observing that the vanishing of (5-1) is equivalent to

$$0 \neq C \in \ker \Phi_M, \tag{5-2}$$

where Φ_M is the linear map between matrix spaces

$$\begin{aligned} \Phi_M &: M_{n,n} \to M_{n,n} \\ &: C \mapsto CM + (-1)^{r}\,{}^{t}MC. \end{aligned}$$

To fix notation, write the $n \times n$ matrix $C \in M_{n,n}$ as

$$C = \begin{bmatrix} c_1^1 & \cdots & c_n^1 \\ \vdots & \ddots & \vdots \\ c_1^n & \cdots & c_n^n \end{bmatrix}$$

and similarly for $M \in M_{n,n}$. $M_{n,n}$ can be identified with $\mathbf{R}^{n^2}$ by the map

$$C = (c_j^i) \mapsto v = {}^{t}(c_1^1, \dots, c_n^1, \dots, c_1^n, \dots, c_n^n);$$

then the equation $0 = CM + (-1)^{r}\,{}^{t}MC$ is equivalent to $Nv = 0$, where $N \in M_{n^2,n^2}$ is given in $n \times n$ block form by

$$N = \begin{bmatrix} {}^{t}M & 0 & \cdots & 0 \\ 0 & {}^{t}M & \cdots & 0 \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \cdots & {}^{t}M \end{bmatrix} + (-1)^{r} \begin{bmatrix} m_1^1 I & m_1^2 I & \cdots & m_1^n I \\ m_2^1 I & m_2^2 I & \cdots & m_2^n I \\ \vdots & & \ddots & \vdots \\ m_n^1 I & m_n^2 I & \cdots & m_n^n I \end{bmatrix}.$$

The linear map Φ_M is represented by the matrix N; the condition $\ker \Phi_M \neq \{0\}$ is equivalent to $\det N = 0$.

To study the condition $\det N = 0$, put $F_{u_r} = M$ in Jordan canonical form, $M = PJP^{-1}$, where

$$J = \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 & & & & & \\ 1 & \lambda_1 & & 0 & \cdots & & 0 & & \\ \vdots & & \ddots & \vdots & & & & & \\ 0 & \cdots & 1 & \lambda_1 & & & & & \\ & \vdots & & & \ddots & & \vdots & & \\ & & & & & \lambda_p & 0 & \cdots & 0 \\ & 0 & & & \cdots & 1 & \lambda_p & & 0 \\ & & & & & \vdots & & \ddots & \vdots \\ & & & & & 0 & \cdots & 1 & \lambda_p \end{bmatrix} \tag{5-3}$$

and each Jordan block of M is an $n_i \times n_i$ matrix. Let $K = {}^{\mathrm{t}}PCP$; then $C = {}^{\mathrm{t}}P^{-1}KP^{-1}$, and

$$CM + (-1)^{r}\,{}^{\mathrm{t}}MC = {}^{\mathrm{t}}P^{-1}\big(KJ + (-1)^{r}\,{}^{\mathrm{t}}JK\big)P^{-1},$$

which vanishes if and only if

$$0 = KJ + (-1)^{r}\,{}^{\mathrm{t}}JK.$$

Construct $\tilde{N}$ and $\tilde{v}$ just as N and v were constructed, using J and K instead of M and C; then (because all the transformations are invertible) the condition $Nv = 0$ is equivalent to $\tilde{N}\tilde{v} = 0$. The $n \times n$ block form of $\tilde{N}$ is

$$\tilde{N} = \begin{bmatrix} {}^{\mathrm{t}}J & 0 & \cdots & 0 \\ 0 & {}^{\mathrm{t}}J & \cdots & 0 \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \cdots & {}^{\mathrm{t}}J \end{bmatrix}$$

$$+(-1)^r \begin{bmatrix} \lambda_1 I & I & \cdots & 0 & & & & & \\ 0 & \lambda_1 I & & 0 & \cdots & & 0 & & \\ \vdots & & \ddots & \vdots & & & & & \\ 0 & \cdots & 0 & \lambda_1 I & & & & & \\ & \vdots & & & \ddots & & \vdots & & \\ & & & & & \lambda_p I & I & \cdots & 0 \\ & 0 & & & \cdots & 0 & \lambda_p I & & 0 \\ & & & & & \vdots & & \ddots & \vdots \\ & & & & & 0 & \cdots & 0 & \lambda_p I \end{bmatrix}$$

Note that $\tilde{N}$ is upper triangular, so its determinant can be easily computed:

$$\det \tilde{N} = \prod_{i,j=0}^{p} \left(\lambda_i + (-1)^r \lambda_j\right)^{n_i n_j}.$$

If the evolution equation (2-1) possesses a conservation law of order $m > r$, then $\det \tilde{N} = 0$; this latter condition occurs if and only if there exists a pair (i, j), $1 \le i \le p$, $1 \le j \le p$ (possibly $i = j$), such that

$$\lambda_i + (-1)^r \lambda_j = 0. \tag{5-4}$$

If r is odd, set $i = j$, and then $\lambda_i - \lambda_i = 0$; so the obstruction (5-4) vanishes. However, if r is even, then the obstruction (5-4) is non-vanishing for a generic choice of numbers $(\lambda_1, \ldots, \lambda_p) \in \mathbf{C}^n$, which are eigenvalues of a real matrix, for all pairs (i, j). ∎

In particular, no even-order scalar evolution equation possesses conservation laws beyond order r. On the other hand, if $n > 1$ and (for example) $\det F_{u_r} = 0$, then the obstruction (5-4) vanishes because one of the λ_i vanishes.

The condition (5-4) can be interpreted in a form which is perhaps easier to compute.

THEOREM 5-2. *If the evolution equation* (2-1) *with even order r possesses a conservation law of order m > r, then the characteristic polynomial*

$$c(\lambda) = \det\left(F_{u_r} - \lambda I\right)$$

of F_{u_r} is of one of the following forms:

- $\lambda p(\lambda)$;
- $(\lambda^2 + \alpha^2)p(\lambda)$, *with* $\alpha \in \mathbf{R}$ *non-zero*;
- $(\lambda^2 - \alpha^2)p(\lambda)$, *with* $\alpha \in \mathbf{R}$ *non-zero*;
- $(\lambda^4 + \beta\lambda^2 + \alpha^2)p(\lambda)$, *with* $\alpha, \beta \in \mathbf{R}$, $\alpha > 0$, $-2\alpha < \beta < 2\alpha$,

where p is a polynomial in λ (*of appropriate degree*) *with real coefficients*.

Proof. If the evolution equation possesses a conservation law of order $m > r$, then, by (5-4) and the hypothesis that r is even, there is a pair (i, j) of integers such that the eigenvalues of F_{u_r} satisfy

$$\lambda_i + \lambda_j = 0. \tag{5-5}$$

Recall that the roots of the characteristic polynomial are the eigenvalues $\{\lambda_k\}$.

If F_{u_r} has a zero eigenvalue, say $\lambda_k = 0$, then set $i = k$ and $j = k$. Then $c(0) = 0$, so $c(\lambda) = \lambda p(\lambda)$ for a real polynomial p.

Since F_{u_r} is a real-valued matrix, $c(\lambda)$ has real coefficients, and therefore its roots come in complex-conjugate pairs.

If there is an eigenvalue λ_i with $\operatorname{Re}\lambda_i = 0$, then there is an eigenvalue λ_j with $\lambda_j^* = \lambda_i$, and therefore $\operatorname{Im}\lambda_j = -\operatorname{Im}\lambda_i$; so $\lambda_j = -\lambda_i$, and $\lambda_i + \lambda_j = 0$. If there is such a pair of purely imaginary roots of c, then $c(\lambda) = (\lambda^2 + \alpha^2)p(\lambda)$, where $\alpha = \mathbf{i}\lambda_i \in \mathbf{R}$, $\mathbf{i}^2 = -1$, and p is a real polynomial.

If the condition (5-5) holds for a purely real non-zero eigenvalue, λ_i, then $-\lambda_i$ is also an eigenvalue; therefore, $c(\lambda) = (\lambda^2 - \alpha^2)p(\lambda)$, where $\alpha = \lambda_i \in \mathbf{R}$ and p is a real polynomial.

Finally, if the condition (5-5) holds, and λ_i is neither purely imaginary, purely real, nor zero, then there is another pair of eigenvalues $\lambda_k = \lambda_i^*$ and $\lambda_l = \lambda_j^*$, and $\lambda_k + \lambda_l = 0$. The four distinct complex numbers $(\lambda_i, -\lambda_i, \lambda_i^*, -\lambda_i^*)$ are all roots of c, therefore, setting $\alpha = |\lambda_i|^2$ and $\beta = 2|\lambda_i|^2 - 4(\operatorname{Re}\lambda_i)^2$, it follows that

$$\begin{aligned} c(\lambda) &= (\lambda - \lambda_i)(\lambda - \lambda_i^*)(\lambda + \lambda_i)(\lambda + \lambda_i^*)p(\lambda) \\ &= \left(\lambda^4 + \beta\lambda^2 + \alpha^2\right)p(\lambda) \end{aligned}$$

for some real polynomial p. Moreover, since $\lambda_i \neq 0$, $\alpha > 0$, and, since λ_i is not purely imaginary, $\beta < 2\alpha$, and $\beta = 2\alpha - 4(\operatorname{Re}\lambda_i)^2 > 2\alpha - 4\alpha = -2\alpha$. ∎

It should be pointed out that the obstruction constructed in the proof of Theorem 5-1 is not the only obstruction to existence of conservation laws. (There are numerous equations with no conservation laws at all; for example, the scalar equation $u_t = (u_{xx})^2$.) Moreover, the assumption that r be even is necessary, as illustrated by the KdV equation ($u_t = u_{xxx} + uu_x$), and by the equation $u_t = (u_{xxx})^2$; for this latter equation, C_6 is spanned by the conservation laws $A = \{u_4, u_3u_6 + 3u_4u_5, tu_3u_6 + 3tu_4u_5 + \frac{1}{10}xu_4 + \frac{1}{5}u_3\}$ with corresponding local densities $G = \{\frac{1}{2}(u_2)^2, -\frac{1}{6}(u_3)^3, -\frac{1}{6}t(u_3)^3 + \frac{1}{20}x(u_2)^2\}$. In particular, all of the conservation laws of this equation have order strictly larger than $r = 3$.

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