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Characterization of *BMO* in terms of rearrangement-invariant Banach function spaces

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Abstract

The atomic decomposition of Hardy spaces by atoms defined by rearrangement-invariant Banach function spaces is proved in this paper. Using this decomposition, we obtain the characterizations of *BMO* and Lipschitz spaces by rearrangement-invariant Banach function spaces. We also provide the sharp function characterization of the rearrangement-invariant Banach function spaces.

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1. Introduction and preliminarily results

In this paper, we extend some of the main results from harmonic analysis to the setting of rearrangement-invariant (r.-i.) Banach function space. More precisely, we are interested in the atomic decomposition of Hardy space, $H^p(\mathbb{R}^n)$, $0 < p \leqslant 1$, the characterization of *BMO* and the characterization of r.-i. Banach function spaces by the sharp function. A study on using the r.-i. quasi-Banach function space together with the notions of other function spaces such as Triebel-Lizorkin spaces and Morrey spaces is given in Ho [5].

In Section 2, we show that the atoms for the non-smooth atomic decomposition of Hardy spaces can be defined via the r.-i. Banach function space instead only of the Lebesgue space. Using this decomposition, we provide a new characterization of *BMO* by r.-i. Banach

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function spaces in Section 3. Finally, we present the sharp function characterization of r.-i. Banach function spaces in Section 4.

Our result is a combination of results from harmonic analysis and the theory of r.-i. Banach function space. Therefore, we first introduce some definitions and properties for the r.-i. Banach function spaces on $\mathbb{R}^n$.

For any Banach function space on $\mathbb{R}^n$, Y, let Y' be its associate space. We have the following Hölder inequality on Y.

Theorem 1. *Let Y be a Banach function space on $\mathbb{R}^n$ with associate space, Y'. If $f \in Y$ and $g \in Y'$, then fg is integrable and*

$$\int_{\mathbb{R}^n} |f(x)g(x)|\,dx \leqslant \|f\|_Y \|g\|_{Y'}.$$

Theorem 2.9 in Chapter 1 of [1] gives the following result. Indeed, the following result is a consequence of the fact that the identification, $Y=(Y')'$, is valid for any Banach function space.

Theorem 2. *For any Banach function space on $\mathbb{R}^n$, Y, we have*

$$\|f\|_Y = \sup_{\substack{h\in Y' \\ \|h\|_{Y'}\leqslant 1}} \left| \int_{\mathbb{R}^n} f(x)h(x)\,dx \right|.$$

We use the definition of Boyd indices from [1], Chapter 3, Definitions 5.10 and 5.12.

Definition 1.1. For each $t>0$ any Lebesgue measurable function f, let E_t denote the dilation operator defined by

$$(E_t f)(x) = f(tx), \quad x \in \mathbb{R}^n.$$

The Boyd indices of a r.-i. Banach function space Y are the numbers defined by

$$\underline{\alpha}_Y = \sup_{0<t<1} \frac{\log(\|E_{1/t}\|_{Y\to Y})}{n \log t}, \quad \overline{\alpha}_Y = \inf_{1<t<\infty} \frac{\log(\|E_{1/t}\|_{Y\to Y})}{n \log t},$$

where $\|E_{1/t}\|_{Y\to Y}$ is the operator norm of the linear operator, $E_t : Y \to Y$.

Lemma 3. *Let Y be a r.-i. Banach function space on $\mathbb{R}^n$ and let Y' be its associated space. Then,*

$$\|\chi_E\|_Y \|\chi_E\|_{Y'} = |E| \tag{1.1}$$

for any Lebesgue measurable set, E, with $|E|<\infty$.

The proof of the above lemma is given in [1], Chapter 2, Theorem 5.2.

For any Lebesgue measurable function on $\mathbb{R}^n$, f, let f^* be its decreasing-rearrangement. We recall the definition of joint weak type from [1], Chapter 3, Definitions 5.1 and 5.4.

Definition 1.2. Let $1 \leqslant p < q \leqslant \infty$. A quasilinear operator is said to be of joint weak type (p, p, q, q) if there exists a constant $C > 0$ such that

$$(Tf)^*(t) \leqslant C \left(t^{-1/p} \int_0^t s^{1/p} f^*(s) \frac{ds}{s} + t^{-1/q} \int_t^\infty s^{1/q} f^*(s) \frac{ds}{s} \right), \quad 0 < t < \infty.$$

We state the Lorentz and Shimogaki theorem on the boundedness of the maximal operator on the r.-i. Banach function space (see [1], Chapter 3, Theorem 5.17).

Theorem 4. *Let Y be a r.-i. Banach function space on $\mathbb{R}^n$. Then, the Hardy–Littlewood maximal operator is bounded on Y if and only if the upper Boyd index of Y satisfies $\overline{\alpha}_Y < 1$.*

Finally, we state the Boyd interpolation theorem. The proof of the following result can be found in [1], Chapter 3, Theorem 5.16.

Theorem 5. *Let $1 \leqslant p < q \leqslant \infty$ and Y be a r.-i. Banach function space on $\mathbb{R}^n$. Let T be a quasilinear operator of joint weak type $(p, p; q, q)$. Then T is bounded on Y if and only if the Boyd indices of Y satisfy $1/q < \underline{\alpha}_Y \leqslant \overline{\alpha}_Y < 1/p$.*

At the end of this section, we present the notations used in this paper. For any $x_0 \in \mathbb{R}^n$ and $r > 0$, let $B(x_0, r) = \{x \in \mathbb{R}^n : |x - x_0| < r\}$ be a ball with center, x_0, and radius, r. Define $\mathbb{B} = \{B(x_0, r) : x_0 \in \mathbb{R}^n, r > 0\}$. Let $\mathscr{S}(\mathbb{R}^n)$ be the Schwartz function space and $\mathscr{S}_0(\mathbb{R}^n) = \{f \in \mathscr{S}(\mathbb{R}^n) : \int_{\mathbb{R}^n} x^\gamma f(x)\,dx = 0, \forall \gamma \in \mathbb{N}^n\}$. Let $\mathscr{P}_k$ denote the set of polynomials on $\mathbb{R}^n$ with degree less than or equal to k, $k \in \mathbb{N}$ and $\mathscr{P} = \cup_{k \in \mathbb{N}} \mathscr{P}_k$.

2. Atomic decomposition of the Hardy space

The atoms in the "standard" non-smooth atomic decomposition of the Hardy space are defined to be a compactly supported function satisfying some vanishing moment condition and L^r-condition for some $1 < r \leqslant \infty$. In this section, we extend the atomic decomposition of the Hardy space by replacing the L^r-condition with a condition on the norm, $\| \cdot \|_Y$, where Y is a r.-i. Banach function space fulfilling a mild condition on the Boyd indices. The precise condition is given in Theorem 6.

Definition 2.1. Let Y be a r.-i. Banach function space on $\mathbb{R}^n$. We call that function, $A(x)$, a non-smooth (p, Y)-atom if there exists a $B \in \mathbb{B}$ such that

$$\operatorname{supp} A \subset 3B, \tag{2.1}$$

$$\int_{\mathbb{R}^n} x^\gamma A(x)\,dx = 0, \quad |\gamma| \leqslant \left[\frac{n}{p} - n \right], \quad \gamma \in \mathbb{N}^n, \tag{2.2}$$

$$\|A\|_Y \leqslant \|\chi_B\|_Y |B|^{-1/p}. \tag{2.3}$$

We call B the ball associated with the non-smooth (p, Y)-atom, $A(x)$. We denote the set of non-smooth (p, Y)-atoms as $\mathscr{A}_{p,Y}$.

Theorem 6. *Let $0 < p \leqslant 1$ and Y be a r.-i. Banach function space on $\mathbb{R}^n$ with its Boyd indices satisfying $\overline{\alpha}_Y \neq 1/p$. Then,*

$$\|f\|_{H^p(\mathbb{R}^n)} \approx \inf \left\{ \|\{r_i\}_{i\in\mathbb{N}}\|_{l^p} : f = \sum_{i\in\mathbb{N}} r_i A_i, \text{ and } A_i \in \mathscr{A}_{p.Y} \right\}, \tag{2.4}$$

where $f = \sum_{i\in\mathbb{N}} r_i A_i$ converges in $H^p(\mathbb{R}^n)$.

Proof. From the standard non-smooth atomic decomposition for $H^p(\mathbb{R}^n)$, there exists a family of non-smooth (p, L^∞)-atoms, $\{A_i\}_{i\in\mathbb{N}}$ and a sequence of scalars, $\{r_i\}_{i\in\mathbb{N}}$, such that $f = \sum_{i\in\mathbb{N}} r_i A_i$ and $\|\{r_i\}_{i\in\mathbb{N}}\|_{l^p} \leqslant C\|f\|_{H^p(\mathbb{R}^n)}$ where the constant $C > 0$ is independent of f. By applying $\|\cdot\|_Y$ on both sides of the inequality, $|A_i(x)| \leqslant \|A_i\|_{L^\infty}\chi_{3B_i}$ where B_i is the ball associated with A_i in Definition 2.1, we find that

$$\|A_i\|_Y \leqslant \|A_i\|_{L^\infty}\|\chi_{3B_i}\|_Y \leqslant 3\|\chi_{B_i}\|_Y |B_i|^{-1/p}.$$

Therefore, A_i is a constant multiple of a non-smooth (p, Y)-atom. The convergence of the expansion $f = \sum_{i\in\mathbb{N}} r_i A_i$ in $H^p(\mathbb{R}^n)$ is given by the standard non-smooth atomic decomposition (see [6], Chapter III, Section 2.3.2).

Hence, it is sufficient to prove that there exists a constant, $C > 0$, such that, for any non-smooth (p, Y)-atom, A, we have

$$\|A\|_{H^p(\mathbb{R}^n)} \leqslant C. \tag{2.5}$$

We first consider the case when $0 < p < 1$. Notice that the condition, $\overline{\alpha}_Y < 1/p$, is satisfied by all r.-i. Banach function spaces (see [1], Chapter 3, Proposition 5.13). Given a (p, Y)-atom, A, using Theorem 1, we have a constant $C > 0$ independent of A such that

$$\|A\|_{L^1(\mathbb{R}^n)} \leqslant \|A\|_Y \|\chi_{3B}\|_{Y'} \leqslant C\|\chi_B\|_Y \|\chi_B\|_{Y'} |B|^{-1/p} = C|B|^{1-1/p}.$$

For the last equality, we use identity (1.1). Hence, A is a constant multiple of a (p, L^1)-atom for $H^p(\mathbb{R}^n)$, $0 < p < 1$. Therefore, the (p, Y)-atom, A, satisfies (2.5).

For the case $p = 1$, we do not have atomic decomposition with $(1, L^1)$-atom (in this connection, see [3], Chapter III, Definition 4.2 and Theorem 4.10), so, we use the maximal function characterization of Hardy space to prove (2.5).

As Y is rearrangement-invariant, to prove (2.5), we can assume that the center of the ball associated with A is the origin.

Let $\Phi \in \mathscr{S}(\mathbb{R}^n)$ satisfy $\operatorname{supp}\Phi \in B(0, 1)$ and $\int_{\mathbb{R}^n} \Phi(x)\,dx \neq 0$. For any locally integrable function, f, we consider the mapping

$$\mathrm{M}_\Phi(f) = \sup_{t>0} |f * \Phi_t|,$$

where $\Phi_t(x) = t^{-n}\Phi(x/t)$, $t > 0$. As the Boyd indices of Y satisfy $\overline{\alpha}_Y < 1$ and $\mathrm{M}_\Phi(f) < C\mathrm{M}(f)$ for some constant $C > 0$, where M is the Hardy–Littlewood maximal operator (see [7], Chapter III, Section 1.2.1), by applying Theorem 4, we find that there exists a constant, $C > 0$, such that

$$\|\mathrm{M}_\Phi(f)\|_Y \leqslant C\|f\|_Y, \quad \forall f \in Y. \tag{2.6}$$

We split the estimate of $\|A\|_{H^1(\mathbb{R}^n)}$ into two components as follows:

$$\begin{aligned}\|A\|_{H^1(\mathbb{R}^n)} &= \|\mathrm{M}_\Phi(A)\|_{L^1} \leqslant 2(\|\chi_{2B}\mathrm{M}_\Phi(A)\|_{L^1} + \|(1-\chi_{2B})\mathrm{M}_\Phi(A)\|_{L^1})\\ &= I + II.\end{aligned}$$

For the estimate of I, Theorem 1 asserts that

$$I \leqslant \|\mathrm{M}_\Phi(A)\|_Y \|\chi_{2B}\|_{Y'} \leqslant C\|\mathrm{M}_\Phi(A)\|_Y\|\chi_B\|_{Y'}.$$

According to the definition of a non-smooth $(1, Y)$-atom and (2.6), we obtain

$$I \leqslant C\|A\|_Y\|\chi_B\|_{Y'} \leqslant C\|\chi_B\|_Y\|\chi_B\|_{Y'}|B|^{-1} \leqslant C. \tag{2.7}$$

We now consider II. As $x \notin 2B$ and $\operatorname{supp}\Phi \in B(0,1)$, we use the vanishing moment condition for A, and find that, for any $N > 0$,

$$\begin{aligned}|(A * \Phi_t)(x)| &= t^{-n}\left|\int_{3B} A(y)(\Phi_t(x-y) - \Phi_t(x))dy\right|\\ &\leqslant t^{-n}\int_{3B}|A(y)|\frac{C_N|y/t|}{(1+|x/t|)^N}dy\\ &\leqslant \frac{C_N t^{-(1+n)}}{(1+t^{-1}|x|)^N}\int_{3B}|A(y)||y|dy,\end{aligned}$$

where C_N depends on n and N only. Using Theorem 1 and (1.1) again, we obtain

$$|(A*\Phi_t)(x)| \leqslant \frac{C_N t^{-(1+n)}|B|^{1/n}}{(1+t^{-1}|x|)^N}\|A\|_Y\|\chi_B\|_{Y'} \leqslant C_N\frac{t^{-(1+n)}|B|^{1/n}}{(1+t^{-1}|x|)^N}.$$

By choosing $N > 1+n$, we assert that

$$\sup_{t>0}|(A*\Phi_t)(x)| \leqslant C_N\frac{|B|^{1/n}}{|x|^{2(1+n)}}. \tag{2.8}$$

Let $l(B) = 2^a$ where $a \in \mathbb{Z}$. Applying $\|\cdot\|_{L^1}$ on both sides of (2.8), we find that

$$II \leqslant C2^a\left(\sum_{j=a}^{\infty}\frac{2^{jn}}{2^{j(1+n)}}\right) \leqslant C \tag{2.9}$$

for some constant $C > 0$ independent of A. Thus, (2.7) and (2.9) prove (2.5). □

If we consider $Y = L^r(\mathbb{R}^n)$, then, $\overline{\alpha}_Y = 1/r$. Thus, the conditions in the above atomic decomposition of Hardy space reduce to the usual condition imposed on the (p, L^r)-atom. Furthermore, in terms of the Boyd indices, this is the best condition on Y. An obvious example is given by the Hardy space $H^1(\mathbb{R}^n)$ as it does not have non-smooth atomic decomposition with $(1, L^1)$-atom.

Here is a simple application of the above atomic decomposition on the boundedness of linear operator.

Corollary 7. *Let $0 < p \leqslant 1$ and Y be a r.-i. Banach function space on $\mathbb{R}^n$ with its Boyd indices satisfying $\overline{\alpha}_Y \neq 1/p$. Let $p \leqslant r \leqslant 1$ and X be a r-Banach function space and T be a linear operator such that for any (p, Y)-atom, A, $\|T(A)\|_X \leqslant C$ for some constant $C > 0$ independent of A. Then, T can be extended to be a bounded linear operator from $H^p(\mathbb{R}^n)$ to X.*

Proof. For any $f \in H^p(\mathbb{R}^n)$, we have $f = \sum_{i\in\mathbb{N}} r_i A_i$ for a family of non-smooth (p, Y)-atoms, $\{A_i\}_{i\in\mathbb{N}}$ and a sequence of scalars, $\{r_i\}_{i\in\mathbb{N}}$, such that $\|\{r_i\}_{i\in\mathbb{N}}\|_{l^p} \leqslant C\|f\|_{H^p(\mathbb{R}^n)}$. Thus, $T(f)$ can be defined as $T(f) = \sum_{i\in\mathbb{N}} r_i T(A_i)$. It is well-defined and bounded from $H^p(\mathbb{R}^n)$ to X because

$$\begin{aligned}\|T(f)\|_X^r &\leqslant \sum_{i\in\mathbb{N}} |r_i|^r \|T(A_i)\|_X^r \leqslant C \sum_{i\in\mathbb{N}} |r_i|^r \\ &\leqslant C\left(\sum_{i\in\mathbb{N}} |r_i|^p\right)^{r/p} \leqslant C\|f\|_{H^p(\mathbb{R}^n)}^r. \quad \square\end{aligned}$$

3. Characterization of *BMO*

Let f_B denote the mean value of f over $B \in \mathbb{B}$; that is, $f_B = (1/|B|)\int_B f(x)\,dx$.

Definition 3.1. Let Y be a r.-i. Banach function space. The function space, BMO_Y, consists of those locally integrable function, f, satisfying

$$\|f\|_{BMO_Y} = \sup_{B\in\mathbb{B}} \frac{\|(f - f_B)\chi_B\|_Y}{\|\chi_B\|_Y} < \infty. \tag{3.1}$$

Similar to *BMO*, BMO_Y endowed with the norm, $\|\cdot\|_{BMO_Y}$, is a Banach space.

We say that two Banach spaces, A_1 and A_2, are equal if $A_1 = A_2$ as sets and we have the continuous embedding $A_1 \hookrightarrow A_2$ and $A_2 \hookrightarrow A_1$. The following theorem is our main result.

Theorem 8. *Let Y be a r.-i. Banach function space on $\mathbb{R}^n$ having Boyd indices satisfying $0 < \underline{\alpha}_Y$. Then, BMO is equal to BMO_Y.*

Proof. Let $f \in BMO_Y$. For any $B \in \mathbb{B}$, according to the Hölder inequality on Y and using Lemma 3, we have

$$\frac{1}{|B|}\int_B |f(x) - f_B|\,dx \leqslant \frac{\|(f - f_B)\chi_B\|_Y \|\|\chi_B\|_{Y'}}{|B|} = \frac{\|(f - f_B)\chi_B\|_Y}{\|\chi_B\|_Y}.$$

Thus, we establish the continuous embedding, $BMO_Y \hookrightarrow BMO$.

We use Theorem 2 and the fact that the dual space of $H^1(\mathbb{R}^n)$ is equal to *BMO* to prove the reserve direction.

For any $f \in BMO$ and $B \in \mathbb{B}$, by Theorem 2 we have a $h \in Y'$ satisfying $\|h\|_{Y'} \leqslant 1$, $\operatorname{supp} h \subseteq B$ and

$$\|(f - f_B)\chi_B\|_Y \leqslant 2 \left| \int_B h(x)(f(x) - f_B)\,dx \right|.$$

It is obvious that there exists a $\tilde{B} \in \mathbb{B}$ such that $|B| = |\tilde{B}|$, $B \cap \tilde{B} = \emptyset$ and $\operatorname{dist}(B, \tilde{B}) = 0$. Define A by

$$A(x) = \begin{cases} h(x), & x \in B, \\ -\dfrac{1}{|B|}\displaystyle\int_B h(y)\,dy, & x \in \tilde{B}, \\ 0, & x \in \mathbb{R}^n \backslash (B \cup \tilde{B}). \end{cases}$$

Thus, A fulfills conditions (2.1) and (2.2) with $\gamma = 0$. Moreover, by Lemma 3, we obtain

$$\begin{aligned} \|A\|_{Y'} &\leqslant \|h\|_{Y'} + \left| \frac{1}{|B|} \int_B h(y)\,dy \right| \|\chi_{\tilde{B}}\|_{Y'} \\ &\leqslant \|h\|_{Y'} + \frac{1}{|B|} \|h\|_{Y'} \|\chi_B\|_Y \|\chi_{\tilde{B}}\|_{Y'} \leqslant 2\|h\|_{Y'} \leqslant 2 \end{aligned}$$

as Y' is rearrangement-invariant. Hence, A is a constant multiple of a $(1, Y')$-atom. Since $\overline{\alpha}_{Y'} = 1 - \underline{\alpha}_Y < 1$, using Lemma 3 again, we conclude that A belongs to $H^1(\mathbb{R}^n)$ with

$$\|A\|_{H^1} \leqslant C \frac{|B|}{\|\chi_B\|_{Y'}} = C\|\chi_B\|_Y$$

for some constant $C > 0$ independent of h.

Using the fact that *BMO* is the dual space of $H^1(\mathbb{R}^n)$, we assert that

$$\begin{aligned} \frac{\|(f - f_B)\chi_B\|_Y}{\|\chi_B\|_Y} &\leqslant \frac{2}{\|\chi_B\|_Y} \left| \int_B h(x)(f(x) - f_B)\,dx \right| \\ &= \frac{2}{\|\chi_B\|_Y} \left| \int_{\mathbb{R}^n} A(x)(f(x) - f_B)\chi_B(x)\,dx \right| \\ &\leqslant \frac{2\|A\|_{H^1}\|f\|_{BMO}}{\|\chi_B\|_Y} \leqslant C\|f\|_{BMO} \end{aligned}$$

because *BMO* is a lattice. □

The above theorem generalizes the following well-known result of *BMO*: the norm,

$$\|f\|_{*,p} = \sup_{B \in \mathbb{B}} \left(\frac{1}{|B|} \int_B |f - f_B|^p\,dx \right)^{1/p}, \quad 1 \leqslant p < \infty$$

is an equivalent norm on *BMO*.

In Theorem 8, the condition, $0 < \underline{\alpha}_Y$, cannot be removed. For instance, if we consider BMO_Y with $Y = L^\infty(\mathbb{R}^n)$, we find that for any $f \in BMO_{L^\infty}$, there exists a constant $C > 0$

such that for any $Q \in \mathscr{Q}$, $|f(x) - f(y)| \leqslant C$, $x, y \in Q$. This is only possible when f is essentially bounded. That is, $BMO_{L^\infty} = L^\infty(\mathbb{R}^n)$.

The proof of Theorem 8 relies on the atomic decomposition, Theorem 6, and the duality, $(H^1(\mathbb{R}^n))^* = BMO$. When $0 < p < 1$, the dual space of $H^p(\mathbb{R}^n)$ is the homogeneous Lipschitz space, $\dot{\Lambda}^\alpha$, $\alpha = n(1/p - 1)$. It is also a special case of the Campanato space. In fact, for any $\alpha > 0$, $1 \leqslant r \leqslant \infty$, $\dot{\Lambda}^\alpha$ is equal to $\mathscr{L}_{r,\alpha}$ where $\mathscr{L}_{r,\alpha}$ consists of those locally integrable function, f, satisfying

$$\|f\|_{\mathscr{L}_{r,\alpha}} = \sup_{B \in \mathbb{B}} \inf_{P_B \in \mathscr{P}_{[\alpha]}} \frac{1}{|B|^{\alpha/n}} \left(\frac{1}{|B|} \int_B |f(y) - P_B(y)|^r \, dy \right)^{1/r} < \infty. \tag{3.2}$$

We now give a characterization of the Lipschitz space by r.-i. Banach function spaces.

Definition 3.2. Let $\alpha > 0$ and Y be a r.-i. Banach function space. The function space, $\mathscr{L}_{Y,\alpha}$, consists of those locally integrable function, f, satisfying

$$\|f\|_{\mathscr{L}_{Y,\alpha}} = \sup_{B \in \mathbb{B}} \inf_{P_B \in \mathscr{P}_{[\alpha]}} \frac{1}{|B|^{\alpha/n}} \frac{\|(f - P_B)\chi_B\|_Y}{\|\chi_B\|_Y} < \infty. \tag{3.3}$$

Theorem 9. *Let $\alpha > 0$ and Y be a r.-i. Banach function space on $\mathbb{R}^n$. Then, $\dot{\Lambda}_\alpha$ is equal to $\mathscr{L}_{Y,\alpha}$.*

Proof. With some simple modifications, the proof for Theorem 8 carries over to the proof for the above theorem. For simplicity, we just demonstrate the construction of the (p, Y')-atom A from h.

As Y' and $\dot{\Lambda}_\alpha$ are translation invariant, we can assume that $\operatorname{supp} h \subseteq B(c_r, r)$ for some $r > 0$ where $c_r = (0, 0, \ldots, 0, 2r) \in \mathbb{R}^n$. Let $\{\varphi_\gamma\}_{|\gamma| \leqslant [\alpha], \gamma \in \mathbb{N}^n} \subset \mathscr{S}(\mathbb{R}^n)$ satisfy

$$\int_{B(0,1)} x^\lambda \varphi_\gamma(x) \, dx = \delta_{\lambda\gamma}, \quad \lambda \in \mathbb{N}^n, \quad |\lambda| \leqslant [\alpha].$$

Define A by

$$A(x) = \begin{cases} h(x), & x \in B(c_r, r), \\ -\displaystyle\sum_{|\gamma| \leqslant [\alpha]} \left(\int_{\mathbb{R}^n} x^\gamma h(x) \, dx \right) r^{-|\gamma| - n} \varphi_\gamma(x/r), & x \in B(0, r), \\ 0, & \text{otherwise.} \end{cases}$$

It is obvious that $\operatorname{supp} A \subset B(0, 3r)$ and A satisfies the vanishing moment condition (2.2). It remains to show that A fulfills the size condition (2.3). Applying the Hölder inequality, we have

$$\int_{\mathbb{R}^n} |x^\gamma h(x)| \, dx \leqslant (3r)^{|\gamma|} \|h\|_{Y'} \|\chi_{B(0,r)}\|_Y.$$

So, Lemma 3 gives

$$\begin{aligned} \|A\chi_{B(0,r)}\|_{Y'} &\leqslant C r^{|\gamma|} \|h\|_{Y'} \|\chi_{B(0,r)}\|_Y r^{-|\gamma| - n} \|\varphi_\gamma\|_{L^\infty} \|\chi_{B(0,r)}\|_{Y'} \\ &\leqslant C \|h\|_{Y'} \leqslant C. \end{aligned}$$

That is, A belongs to $H^p(\mathbb{R}^n)$ with $\|A\|_{H^p} \leqslant C|B(0, 3r)|^{1/p-1}\|\chi_{B(0,3r)}\|_Y$ for some constant $C > 0$ independent of h. The rest of the proof follows from Theorem 8. □

4. The sharp function

In this section, we show that any separable r.-i. Banach function space on $\mathbb{R}^n$ can be characterized by the sharp function that we had established for the Lebesgue spaces, $L^p(\mathbb{R}^n)$, $1 < p < \infty$. We present a preliminary result for r.-i. Banach function spaces.

Theorem 10. *Let Y be a separable r.-i. Banach function space on $\mathbb{R}^n$. If the Boyd indices of Y satisfies $\overline{\alpha}_Y < 1$, then $\mathscr{S}_0(\mathbb{R}^n)$ is dense in Y.*

Proof. According to [1], Chapter 1, Corollaries 4.3 and 5.6, $Y^* = Y'$. Thus, to show the denseness of $\mathscr{S}_0(\mathbb{R}^n)$, we prove $Y' \cap \mathscr{P} = \{0\}$. It suffices to show that the constant function, $F = 1$, does not belong to Y'. According to Definition 1.1, there exists a $t_0 < 1$ such that for any $0 < t < t_0$, we have

$$\|\chi_{(0,1)^n}\|_{Y'} \leqslant t^{n\underline{\alpha}_{Y'}/2}\|\chi_{(0,1/t)^n}\|_{Y'}.$$

Therefore, $t^{-n\underline{\alpha}_{Y'}/2}\|\chi_{(0,1)^n}\|_{Y'} \leqslant \|\chi_{(0,1/t)^n}\|_{Y'} \leqslant \|F\|_{Y'}$ for any sufficiently small t. By [1] Chapter 3, (5.33), we have $\underline{\alpha}_{Y'} = 1 - \overline{\alpha}_Y > 0$, hence, F does not belong to Y'. Thus, $\mathscr{S}_0(\mathbb{R}^n)$ is dense in Y. □

For any locally integrable function, f, recall that the sharp function of f is defined by

$$f^\sharp(x) = \sup_{x \in B} \frac{1}{|B|}\int_B |f(y) - f_B|dy, \tag{4.1}$$

where the supreme is taken over by all $B \in \mathbb{B}$ containing x.

Theorem 11. *Let Y be a separable r.-i. Banach function space on $\mathbb{R}^n$ with Boyd indices satisfying $0 < \underline{\alpha}_Y \leqslant \overline{\alpha}_Y < 1$. Then, there exist constants $C_1 > C_2 > 0$ such that*

$$C_2\|f\|_Y \leqslant \|f^\sharp\|_Y \leqslant C_1\|f\|_Y, \quad \forall f \in Y.$$

Proof. The inequality

$$\|f^\sharp\|_Y \leqslant C_1\|f\|_Y, \quad \forall f \in Y \tag{4.2}$$

follows from the pointwise estimate $f^\sharp \leqslant 2\mathrm{M}f$ and the boundedness of M on Y.

For the other direction, we use the following result from [7], Chapter IV, (16): there exists a constant, $C > 0$, such that, for any $g \in H^1(\mathbb{R}^n)$ and $f \in L^\infty(\mathbb{R}^n)$,

$$\left|\int_{\mathbb{R}^n} f(x)g(x)\,dx\right| \leqslant C\int_{\mathbb{R}^n} f^\sharp(x)\mathscr{M}g(x)\,dx, \tag{4.3}$$

where $\mathscr{M} = \mathscr{M}_{\mathscr{F}}$ is the grand maximal function defined in [7], Chapter III, Section 1.2.

Let f be a simple function and $g \in \mathscr{S}_0(\mathbb{R}^n) \subset H^1(\mathbb{R}^n)$. Using (4.3), we obtain

$$\left| \int_{\mathbb{R}^n} f(x)g(x)\,dx \right| \leqslant C \| f^\sharp \|_Y \| \mathscr{M} g \|_{Y'} \leqslant C \| f^\sharp \|_Y \| g \|_{Y'}.$$

We use the boundedness of $\mathscr{M}$ on Y' for the last inequality. The boundedness of $\mathscr{M}$ on Y' follows from Theorem 5, the facts that $\mathscr{M}$ is sublinear and bounded on $L^p(\mathbb{R}^n)$ (see [7], Chapter 3, Sections 1.3–1.4) and that the Boyd indices of Y' satisfy $0 < 1 - \overline{\alpha}_Y = \underline{\alpha}_{Y'}$ and $\overline{\alpha}_{Y'} = 1 - \underline{\alpha}_Y < 1$ (see [1], Chapter 3, Proposition 5.13).

As Y is separable, from [1], Chapter 1, Corollary 5.6, Y has absolutely continuous norms. Furthermore, Theorem 10 guarantees that $\mathscr{S}_0(\mathbb{R}^n)$ is dense in Y. Using [1], Chapter 1, Theorem 2.9, we have a constant, $C > 0$, independent of f such that

$$\| f \|_Y \leqslant C \| f^\sharp \|_Y \quad \text{for any simple function } f. \tag{4.4}$$

Finally, since Y has absolutely continuous norm, according to [1], Chapter 1, Theorem 3.11, the set of simple functions is a dense subset of Y. Thus, for any $f \in Y$, there exists a sequence of simple functions, $\{f_k\}_{k \in \mathbb{N}}$, such that $f_k \to f$ in Y, as $k \to \infty$. As the mapping, $f \to f^\sharp$, is sublinear, by (4.2), we find that $f_k^\sharp \to f^\sharp$ in Y as $k \to \infty$. Thus, the inequality $C_2 \| f \|_Y \leqslant \| f^\sharp \|_Y, \forall f \in Y$, follows from (4.4). □

Theorem 11 is a generalization of the sharp function characterization of $L^p(\mathbb{R}^n)$ (see, for example, [7], Chapter IV, Section 2.2). It also extends the result on Theorem 5.1 of [8] from the Orlicz space when the Orlicz function satisfies the ∇_1^* and ∇_∞ conditions to r.-i. Banach function space satisfying $0 < \underline{\alpha}_Y \leqslant \overline{\alpha}_Y < 1$.

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