A bivariate rational interpolation with a bi-quadratic denominator

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Abstract

In this paper a new rational interpolation with a bi-quadratic denominator is developed to create a space surface using only values of the function being interpolated. The interpolation function has a simple and explicit rational mathematical representation. When the knots are equally spaced, the interpolating function can be expressed in matrix form, and this form has a symmetric property. The concept of integral weights coefficients of the interpolation is given, which describes the “weight” of the interpolation points in the local interpolating region.

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1. Introduction

The construction method of the curve and surface and their mathematical description is a key issue in computer-aided geometric design (CAGD). There are many ways to tackle this problem [1–6,9–18], for example, the polynomial spline method, the NURBS method and the Bézier method. These methods are effective and are applied widely in shape design of industrial products. Generally speaking, most of the polynomial spline methods are interpolating methods, which means that the curves or surfaces constructed by these methods pass through interpolating points. To construct the polynomial spline, usually derivative values are needed as the interpolating data besides the function values. Unfortunately, in many practical problems, such as the description of rainfall in some rainy region and some industrial geometric shapes, derivative values are difficult to obtain. On the other hand, one of the disadvantages of the polynomial
spline method is its global property; it is impossible for the local modification under the condition that the given data are not changed. The NURBS and Bèzier methods are the so-called “no-interpolating-type” methods; this means that the constructed curve and surface do not pass through the given data, and the given points play the role of control points. Thus, construction of an interpolating function which satisfies the following conditions will be necessary in CAGD: only function values be available as the interpolating data; the interpolating functions should have simple and explicit representations, so that they may be convenient for use both in practical applications and in theoretical studies; and the constructed curves and surfaces should be modifiable under the condition that the given data are not changed.

A univariate rational cubic spline interpolation with parameters which are only based on the function values has been constructed [5,8]. These kinds of interpolation splines not only have simple mathematical representation, but can be used for the modification of local curves by selecting suitable parameters under the condition that the interpolating data are not changed. In this case, the uniqueness of the interpolating curves for the given interpolating data becomes the uniqueness of the interpolating curves for the given interpolating data and the parameters. Based on univariate rational cubic spline interpolation [5], a bivariate rational spline was established in [7]. By a similar method, another kind of bivariate rational spline with a bi-quadratic denominator is constructed in this paper. It is interesting that all the results on these two kinds of bivariate splines are very similar although they are different splines.

The paper is organized as follows: in Section 2, a new bivariate rational spline based on function values with parameters is constructed, and this spline would have a bicubic numerator and a bi-quadratic denominator. Section 3 deals with the smoothness of the interpolating surfaces, when some of the parameters satisfy a simple condition, and the interpolating function is $C^1$ in the interpolating region. More interesting is the fact that when the parameter is selected suitably, the interpolating function can be expressed in a matrix form, and this form has a symmetric property. They are discussed in Section 4. Section 5 presents some properties of interpolation, and the concept of integral weights coefficients of the interpolation is given, which describes the “weight” of the interpolation points in the local interpolating region. The constrained control of the interpolating surfaces and numerical examples will be presented in a subsequent paper.

2. Interpolation

Let $\Omega : [a, b; c, d]$ be the plane region, and $\{(x_i, y_j, f_{i,j}), i = 1, 2, \ldots, n, n + 1; j = 1, 2, \ldots, m, m + 1\}$ be a given set of data points, where $a = x_1 < x_2 < \cdots < x_n < x_{n+1} = b$ and $c = y_1 < y_2 < \cdots < y_m < y_{m+1} = d$ are the knot spacings. Let $h_i = x_{i+1} - x_i$ and $l_j = y_{j+1} - y_j$, and for any point $(x, y) \in [x_i, x_{i+1}; y_j, y_{j+1}]$ in the $xy$-plane, let $\theta = (x - x_i)/h_i$ and $\eta = (y - y_j)/l_j$. First, for each $y = y_j$, $j = 1, 2, \ldots, m + 1$, construct the $x$-direct interpolating curve $P_{i,j}^*(x)$ in $[x_i, x_{i+1}]$ [8]; this is given by

$$P_{i,j}^*(x) = \frac{p_{i,j}^*(x)}{q_{i,j}^*(x)}, \quad i = 1, 2, \ldots, n,$$

where

$$p_{i,j}^*(x) = (1 - \theta)^3 z_{i,j}^* f_{i,j} + \theta (1 - \theta)^2 V_{i,j}^* + \theta^2 (1 - \theta) W_{i,j}^* + \theta^3 f_{i+1,j} \beta_{i,j}^*,$$

$$q_{i,j}^*(x) = (1 - \theta)^2 z_{i,j}^* + 2\theta (1 - \theta) + \theta^2 \beta_{i,j}^*,$$
and
\[ V_{i,j} = 2 f_{i,j} + x_{i,j}^* f_{i+1,j}, \]
\[ W_{i,j} = (2 + \beta_{i,j}^*) f_{i+1,j} - h_i \beta_{i,j}^* A_{i+1,j}, \]
with \( x_{i,j}^* > 0, \beta_{i,j}^* > 0 \) and \( A_{i,j}^* = (f_{i+1,j} - f_{i,j})/h_i \). This interpolation is called the rational cubic interpolation based on function values which satisfy

\[
P_{i,j}(x_i) = f_{i,j}, \quad P_{i,j}(x_{i+1}) = f_{i+1,j}, \quad P_{i,j}'(x_i) = A_{i,j}^*, \quad P_{i,j}'(x_{i+1}) = A_{i+1,j}^*.\]

Obviously, the interpolating function \( P_{i,j}(x) \) on \([x_i, x_{i+1}]\) is unique for the given data \((x_r, f(x_r, y_j))\), \( r = i, i+1, i+2 \) and parameter \( x_{i,j}^*, \beta_{i,j}^* \).

Using an \( x \)-direction interpolation function \( P_{i,j}^*(x) \) to define the bivariate rational bicubic interpolating function in \([x_1, x_n; y_1, y_m]\), for each pair of \((i,j)\), \( i = 1, 2, \ldots, n-1 \) and \( j = 1, 2, \ldots, m-1 \), let the bivariate interpolating function \( P_{i,j}(x, y) \) on \([x_i, x_{i+1}; y_j, y_{j+1}]\) be as follows:

\[
P_{i,j}(x, y) = \frac{p_{i,j}(x, y)}{q_{i,j}(y)}, \quad i = 1, 2, \ldots, n-1; \quad j = 1, 2, \ldots, m-1, \tag{2}\]

where
\[
p_{i,j}(x, y) = (1 - \eta)^3 x_{i,j} P_{i,j}^*(x) + \eta (1 - \eta)^2 V_{i,j} + \eta^2 (1 - \eta) W_{i,j} + \eta^3 \beta_{i,j}^* P_{i,j+1}^*(x),
\]
\[
q_{i,j}(y) = (1 - \eta)^2 x_{i,j} + 2 \eta (1 - \eta) + \eta^2 \beta_{i,j},
\]
and
\[
V_{i,j} = 2 P_{i,j}^*(x) + x_{i,j} P_{i,j+1}^*(x),
\]
\[
W_{i,j} = (2 + \beta_{i,j}^*) P_{i,j+1}^*(x) - l_j \beta_{i,j}^* A_{i,j+1}(x),
\]
with \( x_{i,j} > 0, \beta_{i,j} > 0 \), and \( A_{i,j} = (P_{i,j+1}(x) - P_{i,j}(x))/l_j \). \( P_{i,j}(x, y) \) is called a bivariate rational interpolation with a bi-quadratic denominator based on function values which satisfy

\[
P_{i,j}(x_r, y_s) = f(x_r, y_s), \quad r = i, i+1, s = j, j+1.
\]

It is easy to understand that the interpolating function \( P_{i,j}(x, y) \) on \([x_i, x_{i+1}; y_j, y_{j+1}]\) is unique for the given data \((x_r, y_s, f(x_r, y_s))\), \( r = i, i+1, i+2, s = j, j+1, j+2 \) and parameters \( x_{i,j}, \beta_{i,j} \).

3. The smoothing conditions

The rational interpolating function \( P_{i,j}^*(x) \) defined by (1) has a continuous first-order derivative when \( x \in [x_1, x_n] \), so it is easy to see that the bivariate interpolating function \( P_{i,j}(x, y) \) defined by (2) has a continuous first-order partial derivative \( \partial P_{i,j}(x, y)/\partial y \) and \( \partial P_{i,j}(x, y)/\partial x \) in the interpolating region \([x_1, x_n; y_1, y_m]\) except for every \( y \in [y_j, y_{j+1}] \), \( j = 1, 2, \ldots, m-1 \) at the points \((x_i, y), i = 2, 3, \ldots, n-1 \), so it is sufficient for \( P_{i,j}(x, y) \in C^1 \) in the whole interpolating region \([x_1, x_n; y_1, y_m]\) if \( \partial P_{i,j}(x_i+, y)/\partial x = \partial P_{i,j}(x_i-, y)/\partial x \) holds. This leads to the following theorem.
Theorem 1. The sufficient condition for the interpolating function \( P_{i,j}(x, y), i = 1, 2, \ldots, n - 1; j = 1, 2, \ldots, m - 1 \) to be \( C^1 \) in the whole interpolating region \([x_1, x_n; y_1, y_m]\) is that the parameters \( \alpha_{i,j} = \text{constant} \) and \( \beta_{i,j} = \text{constant} \) for each \( j \in \{1, 2, \ldots, m - 1\} \) and all \( i = 1, 2, \ldots, n - 1 \).

Proof. Based on the above analysis, without loss of generality, for any pair of real numbers \( (i, j), 1 \leq i \leq n - 1, 1 \leq j \leq m - 1 \) and \( y \in [y_j, y_{j+1}] \), it is sufficient to prove that

\[
\frac{\partial P_{i,j}(x_i+, y)}{\partial x} = \frac{\partial P_{i,j}(x_i-, y)}{\partial x}.
\]

Since

\[
\frac{\partial P_{i,j}(x, y)}{\partial x} = \frac{1}{q_{i,j}(y)} \left[ (1 - \eta)^2 \alpha_{i,j} \frac{dP^*_{i,j}(x)}{dx} + \eta(1 - \eta)^2 \frac{dV_{i,j}}{dx} + \eta^2(1 - \eta) \frac{dW_{i,j}}{dx} + \eta^3 \beta_{i,j} \frac{dP^*_{i,j+1}(x)}{dx} \right]
\]

and

\[
P^*_{i,r}(x_i+) = \Delta^*_i, \quad r = j, j + 1, j + 2,
\]

so

\[
V_{i,j}'(x_i+) = 2\Delta^*_i + \alpha_{i,j} \Delta^*_i, j + 1,
\]

\[
W_{i,j}'(x_i+) = (2 + \beta_{i,j}) \Delta^*_i, j + 1 - \frac{l_j}{l_{j+1}} \beta_{i,j} (\Delta^*_i, j + 2 - \Delta^*_i, j + 1),
\]

and thus

\[
\left. \frac{\partial P_{i,j}(x, y)}{\partial x} \right|_{x=x_i+} = \frac{1}{(1 - \eta)^2 \alpha_{i,j} + 2 \eta (1 - \eta) + \eta^2 \beta_{i,j}} \left[ (1 - \eta)^3 \alpha_{i,j} \Delta^*_i, j + \eta(1 - \eta)^2 (2\Delta^*_i, j + \alpha_{i,j} \Delta^*_i, j + 1) + \eta^2(1 - \eta)((2 + \beta_{i,j}) \Delta^*_i, j + 1 - \frac{l_j}{l_{j+1}} (\Delta^*_i, j + 2 - \Delta^*_i, j + 1)) + \eta^3 \beta_{i,j} \Delta^*_i, j + 1 \right].
\]

Similarly, since \( P^*_{i-1,r}(x_i-) = \Delta^*_i, \quad r = j, j + 1, j + 2 \), it can be shown that

\[
\left. \frac{\partial P_{i-1,j}(x, y)}{\partial x} \right|_{x=x_i-} = \frac{1}{(1 - \eta)^2 \alpha_{i-1,j} + 2 \eta (1 - \eta) + \eta^2 \beta_{i-1,j}} \left[ (1 - \eta)^3 \alpha_{i-1,j} \Delta^*_i, j + \eta(1 - \eta)^2 (2\Delta^*_i, j + \alpha_{i-1,j} \Delta^*_i, j + 1) + \eta^2(1 - \eta)((2 + \beta_{i-1,j}) \Delta^*_i, j + 1 - \frac{l_j}{l_{j+1}} \beta_{i-1,j} (\Delta^*_i, j + 2 - \Delta^*_i, j + 1)) + \eta^3 \beta_{i-1,j} \Delta^*_i, j + 1 \right].
\]
Comparing (4) and (5), we see that when \(x_{i-1,j} = x_{i,j} \) and \(\beta_{i-1,j} = \beta_{i,j} \), (3) holds. This completes the proof. □

4. Matrix expression and the symmetry of the interpolation

In the rest of this paper, consider the equally spaced knots case, namely, for all \(i = 1, 2, \ldots, n \) and \(j = 1, 2, \ldots, m \), \(h_i = h_j \) and \(l_i = l_j \). Let \(z_{i,j}^* = \text{constant} \) and \(\beta_{i,j}^* = \text{constant} \) for each \(i \in \{1, 2, \ldots, n-1\} \) and all \(j = 1, 2, \ldots, m+1 \). Then the interpolating function \(P_{i,j}(x)\) defined by (2) can be expressed in a simple matrix form in each region \([x_i, x_{i+1}; y_j, y_{j+1}]\), where, \(i = 1, 2, \ldots, n-1; j = 1, 2, \ldots, m-1\).

\[ P_{i,j}^*(x) = \omega_0(\theta, z_{i,j}^*, \beta_{i,j}^*) f_{i,j} + \omega_1(\theta, z_{i,j}^*, \beta_{i,j}^*) f_{i+1,j} + \omega_2(\theta, z_{i,j}^*, \beta_{i,j}^*) f_{i+2,j}, \]

where

\[
\begin{align*}
\omega_0(\theta, z_{i,j}^*, \beta_{i,j}^*) &= \frac{(1-\theta)^2((1-\theta)z_{i,j}^* + 2\theta)}{(1-\theta)^2z_{i,j}^* + 2\theta(1-\theta) + \theta^2\beta_{i,j}^*}, \\
\omega_1(\theta, z_{i,j}^*, \beta_{i,j}^*) &= \frac{\theta(1-\theta)^2z_{i,j}^* + \theta^2(2-\theta)\beta_{i,j}^* + 2\theta^2(1-\theta)}{(1-\theta)^2z_{i,j}^* + 2\theta(1-\theta) + \theta^2\beta_{i,j}^*}, \\
\omega_2(\theta, z_{i,j}^*, \beta_{i,j}^*) &= \frac{-\theta^2(1-\theta)\beta_{i,j}^*}{(1-\theta)^2z_{i,j}^* + 2\theta(1-\theta) + \theta^2\beta_{i,j}^*}.
\end{align*}
\]

Thus, \(P_{i,j}^*(x)\) could be expressed in the following matrix form:

\[ P_{i,j}^*(x) = A_1 B_1, \]

where

\[ A_1 = (\omega_0(\theta, z_{i,j}^*, \beta_{i,j}^*) \ \omega_1(\theta, z_{i,j}^*, \beta_{i,j}^*) \ \omega_2(\theta, z_{i,j}^*, \beta_{i,j}^*)) \]

\[ B_1 = (f_{i,j} \ f_{i+1,j} \ f_{i+2,j})^T. \]

Similarly, the bivariate rational interpolating function \(P_{i,j}\) defined by (2) can be rewritten in the matrix form as

\[ P_{i,j}(x, y) = A_2 B_2, \]

where

\[ A_2 = (\omega_0(\eta, z_{i,j}, \beta_{i,j}) \ \omega_1(\eta, z_{i,j}, \beta_{i,j}) \ \omega_2(\eta, z_{i,j}, \beta_{i,j})) \]

\[ B_2 = (P_{i,j}^* P_{i,j+1}^* P_{i,j+2}^*)^T. \]

Based on the assumption at the beginning of this section, \(h_{i-1} = h_i = h_{i+1}, x_{i,j}^* = x_{i,j+1} = x_{i,j+2}^* \) and \(\beta_{i,j}^* = \beta_{i,j+1}^* = \beta_{i,j+2}^* \), (8) can be written as

\[ P_{i,j}(x, y) = A_1 C A_2^T, \]
where
\[
C = \begin{pmatrix}
  f_{i,j} & f_{i,j+1} & f_{i,j+2} \\
  f_{i+1,j} & f_{i+1,j+1} & f_{i+1,j+2} \\
  f_{i+2,j} & f_{i+2,j+1} & f_{i+2,j+2}
\end{pmatrix}.
\]

It is obvious that (9) can be rewritten as
\[
P_{i,j}(x, y) = A_2 C^T A_1^T = (A_1 C A_2^T)^T.
\]
This is called the symmetry of the interpolating function \( P_{i,j}(x, y) \).

5. Some properties of the interpolation

In the interpolating expression (6), consider \( \omega_r(\theta, \beta^*_i, \beta^*_j), r = 0, 1, 2 \) as the interpolating bases of the interpolating function \( P^*_i(x) \), there is a property of the bases.

**Property 1.** In any subinterval \([x_i, x_{i+1}]\), no matter what positive number the parameters \( \beta^*_i, \beta^*_j \) are, the sum of values of the interpolating base functions is equal to unity, that is
\[
\sum_{r=0}^{2} \omega_r(\theta, \beta^*_i, \beta^*_j) = 1.
\]

Denote
\[
\int_{x_i}^{x_{i+1}} P^*_i(x) \, dx = h_i(a_0 f_{i,j} + a_1 f_{i+1,j} + a_2 f_{i+2,j}),
\]
where \( a_r = \int_{0}^{1} \omega_r(\theta, \beta^*_i, \beta^*_j) \, d\theta, r = 0, 1, 2 \). \( a_r \) are called the integral weights coefficients of the interpolation defined by (1). With respect to coefficients, it can be easily seen that
\[
\sum_{r=0}^{2} a_r = 1.
\]

From Property 1, the following property can be obtained.

**Property 2.** In any subinterval \([x_i, x_{i+1}]\), no matter what positive number the parameters \( \beta^*_i, \beta^*_j \) are, if \( f_{i,j} = f_{i+1,j} = f_{i+2,j} = 1 \) in (6), then
\[
\int_{x_i}^{x_{i+1}} P^*_i(x) \, dx = h_i.
\]
Table 1

<table>
<thead>
<tr>
<th>$x^*_i, j$</th>
<th>$\beta^*_i, j$</th>
<th>$a_0$</th>
<th>$a_1$</th>
<th>$a_2$</th>
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</tr>
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</table>

If $x_i = 0$ and $x_{i+1} = 1$, then (12) becomes

$$\int_{x_i}^{x_{i+1}} P^*_i, j(x) \, dx = 1. \quad (13)$$

So, Property 2 can be called the unit area property of univariate interpolation (1).

With respect to the integral weight coefficients $a_r$, some other relations such as $a_0 - a_2 = 1/2$ and $a_1 + 2a_2 = 1/2$ can be derived from the definition; thus, there exists the following estimation property of the integral weight coefficients.

**Property 3.** In any subinterval $[x_i, x_{i+1}]$, no matter what positive number the parameter $x^*_i, j$ and $\beta^*_i, j$ are, the integral weight coefficients $a_r$, $r = 0, 1, 2$ satisfy

$$1/4 < a_0 < 1/2, \quad 1/2 < a_1 < 1, \quad -1/4 < a_2 < 0.$$ 

**Table 1** gives some integral weight coefficients for the given $x^*_i, j$ and $\beta^*_i, j$.

Furthermore, for the bivariate rational interpolating function $P_{i, j}(x, y)$ defined by (2), there are some parallel properties. From Section 3, it is known that when $h_i = h_j$ and $l_i = l_j$ for all $i = 1, 2, \ldots, n$ and $j = 1, 2, \ldots, m$, and when $x^*_i, j = constant$ and $\beta^*_i, j = constant$ for each $i \in \{1, 2, \ldots, n - 1\}$ and all $j = 1, 2, \ldots, m + 1$, then the interpolating function $P_{i, j}(x)$ defined by (2) can be expressed in a simple matrix form. In this case, since $x^*_i, j = x^*_i, j+1 = x^*_i, j+2$ and $\beta^*_i, j = \beta^*_i, j+1 = \beta^*_i, j+2$, (8) can be rewritten as

$$P_{i, j}(x, y) = \sum_{r=0}^{2} \sum_{s=0}^{2} \omega_{r,s}(\theta, x^*_i, j, \beta^*_i, j; \eta, x_i, j, \beta_i, j) f_{i+r, j+s}, \quad (14)$$

where

$$\omega_{r,s}(\theta, x^*_i, j, \beta^*_i, j; \eta, x_i, j, \beta_i, j) = \omega_r(\theta, x^*_i, j, \beta^*_i, j) \omega_s(\eta, x_i, j, \beta_i, j). \quad (15)$$

Consider $\omega_{r,s}(\theta, x^*_i, j, \beta^*_i, j; \eta, x_i, j, \beta_i, j), r = 0, 1, 2, s = 0, 1, 2$ as the bases of the interpolation expressed by (14); there exists the following property of the bases.

**Property 4.** In any subinterval $[x_i, x_{i+1}; y_j, y_{j+1}]$, if $x^*_i, j = x^*_i, j+1 = x^*_i, j+2$ and $\beta^*_i, j = \beta^*_i, j+1 = \beta^*_i, j+2$, no matter what positive number the parameters $x_i, j, \beta_i, j$ are, the sum of the values of the interpolating
base function is equal to unity, namely
\[
\sum_{r=0}^{2} \sum_{s=0}^{2} \omega_{rs}(\theta, \xi, \beta; \eta, \xi, \beta) = 1.
\]

Let \( D \) be the rectangular region in the \( xy \)-plane enclosed by \( x = x_i, x = x_{i+1}, y = y_j, \) and \( y = y_{j+1} \), and denote
\[
\int \int_{D} P_{i,j}(x, y) \, dx \, dy = h_{i} l_{j} \sum_{r=0}^{2} \sum_{s=0}^{2} a_{rs} f_{i+r, j+s}, \tag{16}
\]
where
\[
a_{rs} = \int \int_{\tilde{D}} \omega_{rs}(\theta, \xi, \beta; \eta, \xi, \beta) \, d\theta \, d\eta, \quad r = 0, 1, 2, \quad s = 0, 1, 2,
\]
and \( \tilde{D} \) is the corresponding rectangular region of \( D \) in the \( \theta \eta \)-plane enclosed by \( \theta = 0, \theta = 1, \eta = 0, \) and \( \eta = 1 \), calling \( a_{rs} \) the integral weights coefficients of the interpolation defined by (8). From Property 4, it is easy to obtain the following property.

**Property 5.** In any subinterval \([x_i, x_{i+1}; y_j, y_{j+1}]\), no matter what positive number the parameter \( \xi, \beta \) are, if \( f_{i+r, j+s} = 1 \), for all \( r = 0, 1, 2, s = 0, 1, 2 \) in (16), then
\[
\int \int_{D} P_{i,j}(x, y) \, dx \, dy = h_{i} l_{j}. \tag{18}
\]

Eq. (18) gives \( \sum_{r=0}^{2} \sum_{s=0}^{2} a_{rs} = 1 \). As for Property 2, Property 5 can be called the unit volume property of the bivariate interpolation defined by (2). From the definition of the integral weight coefficients \( a_{rs} \) defined by (17), using the results given by Property 3, for any positive parameters \( \xi, \beta \) is the following is obtained.

**Property 6.** In any subinterval \([x_i, x_{i+1}; y_j, y_{j+1}]\), no matter what positive number the parameters \( \xi, \beta \) are, the integral weight coefficients \( a_{rs} \) satisfy
\[
1/16 < a_{00} < 1/4, \quad 1/8 < a_{01} < 1/2, \quad -1/8 < a_{02} < 0,
\]
\[
1/8 < a_{10} < 1/2, \quad 1/4 < a_{11} < 1, \quad -1/4 < a_{12} < 0,
\]
\[
-1/8 < a_{20} < 0, \quad -1/4 < a_{21} < 0, \quad 0 < a_{22} < 1/16.
\]

Tables 2 and 3 give the integral weight coefficients of the interpolation defined by (17) for the different values of parameters \( \xi, \beta \) and \( \xi, \beta \). From Tables 1–3, it can be seen that the integral weight coefficients are different for the different parameters, which explains why the interpolating functions are not unique for the different parameters under the conditions that the interpolating data are not changed.
Table 2

<table>
<thead>
<tr>
<th>(z_{i,j}^*)</th>
<th>(\beta_{i,j}^*)</th>
<th>(z_{i,j})</th>
<th>(\beta_{i,j})</th>
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<th>(a_{01})</th>
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Table 3

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6. Conclusions

Five key issues have been studied in this paper.

(1) This paper gives the simple and explicit expression of the bivariate spline, and the spline based just on the function values of the function being interpolated. There are some parameters in the interpolating functions: \(z_{i,j}^*, \beta_{i,j}^*\) and \(z_{i,j}, \beta_{i,j}\). \(z_{i,j}^*, \beta_{i,j}^*\) are for univariate interpolation on the \(x\)-direct as described by (1); \(z_{i,j}, \beta_{i,j}\) are for a bivariate interpolating surface as described by (2).

(2) The symmetric property of the interpolation depends only on the parameters \(z_{i,j}^*, \beta_{i,j}^*\). For each pair of integer number \((i, j)\), \(i \in \{1, 2, \ldots, n-1\}\) and \(j \in \{1, 2, \ldots, m-1\}\), if \(z_{i,r}^* = \text{constant}\) and \(\beta_{i,r}^* = \text{constant}\) for \(r = j, j+1, j+2\), the interpolating function \(P_{i,j}(x)\) defined by (2) can be expressed in a simple matrix form in each plane region \([x_i, x_{i+1}; y_j, y_{j+1}]\).

(3) The smoothness property condition of the interpolation described in Section 3 depends only on the parameters \(z_{i,j}, \beta_{i,j}\). If \(z_{i,j} = \text{constant}\) and \(\beta_{i,j} = \text{constant}\) for each \(j \in \{1, 2, \ldots, m-1\}\) and all \(i = 1, 2, \ldots, n-1\), then the interpolating functions defined by (2) must be \(C^1\)-continuous in the interpolating region \([x_1, x_n; y_1, y_m]\).

(4) Combining (2) and (3) above, if \(z_{i,j} = \text{constant}\) and \(\beta_{i,j} = \text{constant}\) for each \(j \in \{1, 2, \ldots, m-1\}\) and all \(i = 1, 2, \ldots, n-1\), and if \(z_{i,j}^* = \text{constant}\) and \(\beta_{i,j}^* = \text{constant}\) for each \(i \in \{1, 2, \ldots, n-1\}\) and all \(j = 1, 2, \ldots, m-1\), then the interpolating functions \(P(x, y)\) defined by (2) can be expressed in a simple matrix form in every plane region \([x_i, x_{i+1}; y_j, y_{j+1}]\) and it is \(C^1\)-continuous in the whole interpolating region \([x_1, x_n; y_1, y_m]\).

(5) The integral weights coefficients of the interpolation defined by (1) and (2) describe the “weight” of the interpolating knots in the interpolation spline. It could help us understand the interpolation based only on the function values deeply.
Acknowledgements

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References