Variational and numerical analysis of a dynamic frictionless contact problem with adhesion

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Received 11 March 2002; received in revised form 25 October 2002

Abstract

We study a dynamic frictionless contact problem between a viscoelastic body and an obstacle, the so-called foundation. The contact is subjected to an adhesion effect, whose evolution is described by an ordinary differential equation. For the variational formulation of the contact problem, we present and prove an existence and uniqueness result. A fully discrete scheme is introduced to solve the problem. Under certain solution regularity assumptions, we derive an optimal order error estimate. Some numerical examples are included to show the performance of the method.

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1. Introduction

Processes of adhesion are very important in industry, especially when composite materials are involved. There exists extensive engineering literature on various aspects of the subject. A novel approach to the modeling of contact with adhesion, based on thermodynamical consideration, can be found in Frémond [7,8]. The main new idea in these papers is the introduction of a surface internal variable, the bonding or adhesion field, which has values between zero and one and describes the fractional density of the active bonds on the contact surface.

Recent modeling, analysis and numerical simulations of adhesive contact with or without friction can be found in [2,3,9,16] and references therein. In [16] a consistent model for adhesion, friction and unilateral contact has been constructed, and numerical analysis and numerical simulations are presented. In [3] the dynamic frictionless adhesive contact problem has been modeled and analyzed,

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doi:10.1016/S0377-0427(02)00909-3
and the quasistatic version has been considered in [2] where numerical simulations have been provided. Existence and uniqueness results in the study of dynamic and quasistatic process of adhesive contact for elastic and viscoelastic beams can be found in [9].

In this paper we describe a model for the dynamic frictionless adhesive contact between a viscoelastic body and an obstacle, the so-called foundation. Our purpose is to describe the delamination process when the tangential tractions on the adhesive contact surface are negligible. We use a nonlinear Kelvin–Voigt viscoelastic constitutive law to model the material behavior and we describe the contact with a modified normal compliance contact condition, involving a truncation operator. As in [2,3] we use the adhesion field as an additional dependent variable, its evolution being described by an ordinary differential equation. We prove that the model has a unique weak solution. Then we consider numerical approximations of the problem, and derive error estimates. Finally, we present representative numerical simulations, depicting the evolution of the state of the system and, in particular, the evolution of the adhesion field.

This work is a companion and an extension of the results in [3]. There, the dynamic frictionless adhesive contact problem was investigated when the material is linearly viscoelastic and the contact is described by a normal compliance condition involving a truncation operator; neither numerical analysis, nor numerical simulations were included in [3]. The trait of novelty of the present paper consists of the use of a nonlinear constitutive law, a different regularization operator on the normal compliance contact condition and results on numerical analysis and numerical simulations of the model.

The paper is organized as follows. In the next section, we introduce some notation and function spaces that are needed later. In Section 3, we describe the mechanical problem, derive its variational formulation, and present a well-posedness result. Proof of the well-posedness result is the content of Section 4. It is based on arguments of evolutionary equations with monotone operators and fixed point. To solve the contact problem, we introduce a fully discrete approximation scheme in Section 5. We derive an optimal order error estimate under certain solution regularity assumptions. In the last section, we present some numerical examples showing the performance of the numerical scheme.

2. Notation and preliminaries

Here we introduce the notation and some preliminary materials to be used later. For further details we refer the reader to [6,12,15].

We denote by $S_d$ the space of second order symmetric tensors on $\mathbb{R}^d$ ($d = 1, 2, 3$), while \(\cdot\) and \(\|\cdot\|\) represent the inner product and the Euclidean norm on $S_d$ and $\mathbb{R}^d$, respectively. Let $\Omega \subset \mathbb{R}^d$ be a bounded domain with a Lipschitz boundary $\Gamma$ and let $\nu$ denote the unit outer normal on $\Gamma$. The indices $i$ and $j$ run between 1 and $d$. The summation convention over repeated indices is adopted and the index that follows a comma indicates a partial derivative with respect to the corresponding component of the independent variable. We will use the spaces

$$H = [L^2(\Omega)]^d = \{u = (u_i); u_i \in L^2(\Omega)\}, \quad Q = \{\sigma = (\sigma_{ij}); \sigma_{ij} = \sigma_{ji} \in L^2(\Omega)\},$$

$$H_1 = \{u \in H; \varepsilon(u) \in Q\}, \quad Q_1 = \{\sigma \in Q; \text{Div } \sigma \in H\}.$$
Here \( \varepsilon : H_1 \to Q \) and \( \text{Div} : Q_1 \to H \) are the deformation and divergence operators, respectively, defined by
\[
\varepsilon(u) = (\varepsilon_{ij}(u)), \quad \varepsilon_{ij}(u) = \frac{1}{2}(u_{i,j} + u_{j,i}), \quad \text{Div} \, \sigma = (\sigma_{ij,j}).
\]
The spaces \( H, Q, H_1 \) and \( Q_1 \) are real Hilbert spaces endowed with their canonical inner products
\[
(u,v)_H = \int_{\Omega} u_i v_i \, dx, \quad (\sigma, \tau)_Q = \int_{\Omega} \sigma_{ij} \tau_{ij} \, dx,
\]
\[
(u,v)_{H_1} = (u,v)_H + (\varepsilon(u), \varepsilon(v))_Q, \quad (\sigma, \tau)_Q = (\sigma, \tau)_Q + (\text{Div} \, \sigma, \text{Div} \, \tau)_H.
\]
The associated norms on the spaces \( H, Q, H_1 \) and \( Q_1 \) are denoted by \( \| \cdot \|_H, \| \cdot \|_Q, \| \cdot \|_{H_1} \) and \( \| \cdot \|_{Q_1} \), respectively.

Let \( H_\Gamma = [H^{1/2}(\Gamma)]^d \) and let \( \gamma : H_1 \to H_\Gamma \) be the trace map. For every element \( v \in H_1 \) we also use the notation \( v \) to denote the trace \( \gamma v \) of \( v \) on \( \Gamma \) and we denote by \( v_v \) and \( v_t \) the normal and tangential components of \( v \) on the boundary \( \Gamma \),
\[
v_v = v \cdot n, \quad v_t = v - v_v n.
\]
Similarly, for a regular (say \( C^1 \)) tensor field \( \sigma : \Omega \to S_d \) we define its normal and tangential components by
\[
\sigma_v = (\sigma v) \cdot n, \quad \sigma_t = \sigma v - \sigma_v n
\]
and we recall that the following Green’s formula holds:
\[
(\sigma, \varepsilon(v))_Q + (\text{Div} \, \sigma, v)_H = \int_{\Gamma} \sigma v \cdot n \, ds \quad \forall v \in H_1.
\]
Finally, for any real Hilbert space \( X \) we use the classical notation for the spaces \( L^p(0,T;X) \) and \( W^{k,p}(0,T;X) \), \( 1 \leq p \leq + \infty \), \( k \geq 1 \), and we denote by \( \mathcal{C}([0,T];X) \) and \( \mathcal{C}^1([0,T];X) \) the spaces of continuous and continuously differentiable functions from \([0,T]\) to \( X \).

3. Problem statement and variational formulation

In this section we describe the model for the process, present its variational formulation and state our main existence and uniqueness result, Theorem 3.1.

The physical setting is as follows. A viscoelastic body occupies the domain \( \Omega \) with a Lipschitz continuous boundary \( \Gamma \) that is divided into three disjoint measurable parts \( \Gamma_1, \Gamma_2 \) and \( \Gamma_3 \) such that \( \text{meas}(\Gamma_1) > 0 \). Let \([0,T]\) be the time interval of interest, \( T > 0 \). The body is clamped on \( \Gamma_1 \times (0,T) \) and therefore the displacement field vanishes there. A volume force of density \( f_0 \) acts in \( \Omega \times (0,T) \) and surface tractions of density \( f_2 \) act on \( \Gamma_2 \times (0,T) \). The body may come in contact with an obstacle, the so-called foundation. A gap \( g \) exists between the potential contact surface \( \Gamma_3 \) and the foundation, and is measured along the outward normal vector \( n \).

We denote by \( u \) the displacement field, \( \sigma \) the stress field and \( \varepsilon(u) \) the small strain tensor. We assume that the material is viscoelastic and its deformation follows a constitutive law of the form
\[
\sigma = A \varepsilon(u) + G \varepsilon(u),
\]
where \( A \) and \( G \) are given nonlinear constitutive functions which will be described below. To simplify the notation, we usually do not indicate explicitly the dependence of various functions on the variables.
\( x \in \Omega \cup \Gamma \) and \( t \in [0, T] \). Moreover, dots above a quantity represent derivatives of the quantity with respect to the time variable, i.e.
\[
\dot{u} = \frac{du}{dt}, \quad \ddot{u} = \frac{d^2u}{dt^2}.
\]

Next we describe the conditions on the potential contact surface \( \Gamma_3 \). Following Frémond [8], we introduce an internal state variable \( \beta \), which represents the intensity of adhesion, \( 0 \leq \beta \leq 1 \), where \( \beta = 1 \) means the total adhesion, \( \beta = 0 \) means the lack of adhesion and \( 0 < \beta < 1 \) is the case of partial adhesion. Then we assume that the normal stress satisfies the normal compliance contact condition with adhesion
\[
\sigma_v = -p_v(u_v) - \gamma_v \beta^2 (-u_v)_+ \quad \text{on } \Gamma_3 \times (0, T),
\]
where \( u_v \) represents the normal displacement, \( \gamma_v \) is the adhesion coefficient, \( p_v \) is a prescribed function such that \( p_v(r) = 0 \) for \( r \leq 0 \). When it is positive, \( u_v \) represents the penetration of the body into the foundation.

General expressions of this normal compliance condition were used in [11] in the study of static contact problems for elastic materials, and in [17] in the study of quasistatic contact problems for viscoelastic materials. As an example of normal compliance function \( p_v \), we may consider
\[
p_v(r) = c_v r_+,
\]
where \( c_v \) is a positive constant and \( r_+ = \max\{0, r\} \). Formally, Signorini’s nonpenetration condition is obtained in the limit \( c_v \to +\infty \). We can also consider the normal compliance function
\[
p_v(r) = \begin{cases} 
  c_v r_+ & \text{if } r \leq \alpha, \\
  c_v \alpha & \text{if } r > \alpha,
\end{cases}
\]
where \( \alpha \) is a positive coefficient related to the wear and hardness of the surface. In this case the contact condition means that when the penetration is too large, i.e. it exceeds \( \alpha \), the obstacle disintegrates and offers no additional resistance to the penetration.

Moreover we assume that there is no tangential friction during the process, that is
\[
\sigma_t = 0 \quad \text{on } \Gamma_3 \times (0, T).
\]

Finally, the evolution of the adhesion field is governed by the following differential equation:
\[
\dot{\beta} = -(\gamma_v \beta R(u_v)^2 - \epsilon_a)_+ \quad \text{on } \Gamma_3 \times (0, T).
\]
Here \( \epsilon_a \) represents a limit bound and, for any \( u \in [H^1(\Omega)]^d \), \( R(u_v) \) is a regularization of \( u_v \) defined by
\[
R(u_v)(x) = \frac{1}{m(\Gamma_3 \cap B(x, h))} \int_{\Gamma_3 \cap B(x, h)} u_v \, da, \quad x \in \Gamma_3,
\]
where \( m \) denotes the \((d-1)\)-dimensional Lebesgue measure, \( h > 0 \) is a fixed small parameter and \( B(x, h) \) is the ball centered at \( x \) with radius \( h \). We suppose that
\[
I(h) = \inf_{x \in \Gamma_3} m(\Gamma_3 \cap B(x, h)) > 0 \quad \forall h > 0.
\]
The choice of \( R \) is justified by the theory of Lebesgue points; indeed we know that
\[
R(u_v) \to u_v \quad \text{a.e. on } \Gamma_3 \quad \text{as } h \to 0.
\]
On the other hand, it is easy to check that for any \( u \in [H^1(\Omega)]^d \), \( R(u_\nu) \in L^\infty(\Gamma_3) \) and
\[
\|R(u_\nu)\|_{L^\infty(\Gamma_3)} \leq \frac{c_0}{\sqrt{I(h)}} |u|_{[H^1(\Omega)]^d},
\]
where \( c_0 > 0 \) is a constant depending only on \( \Omega \) and \( \Gamma \). Finally, we remark that for the one-dimensional case \( (d = 1) \), we have \( R(u_\nu) = u_\nu \).

Let us denote by \( \rho \) the mass density, \( u_0 \) the initial displacement and \( v_0 \) the initial velocity field. Under the above assumptions, the classical form of the mechanical problem of frictionless contact with normal compliance of the viscoelastic body may be stated as follows.

**Problem P.** Find a displacement field \( u: \Omega \times [0, T] \to \mathbb{R}^d \), a stress field \( \sigma: \Omega \times [0, T] \to S_d \) and an adhesion field \( \beta: \Gamma_3 \times [0, T] \to \mathbb{R} \) such that

\[
\rho \ddot{u} = \text{Div} \sigma + f_0 \quad \text{in} \ \Omega \times (0, T),
\]
\[
\sigma = \mathcal{A} e(\dot{u}) + \mathcal{G} e(u) \quad \text{in} \ \Omega \times (0, T),
\]
\[
u_0 = 0 \quad \text{on} \ \Gamma_1 \times (0, T),
\]
\[
\sigma v = f_2 \quad \text{on} \ \Gamma_2 \times (0, T),
\]
\[
- \sigma_z = p_\nu(u_\nu) + \gamma_\nu \beta^2 (-u_\nu)_+, \quad \sigma_z = 0 \quad \text{on} \ \Gamma_3 \times (0, T),
\]
\[
\dot{\beta} = - (\gamma_\nu \beta R(u_\nu)^2 - e_\nu)_{+} \quad \text{on} \ \Gamma_3 \times (0, T),
\]
\[
u(0) = u_0, \quad \dot{\nu}(0) = v_0 \quad \text{in} \ \Omega,
\]
\[
\beta(0) = \beta_0 \quad \text{on} \ \Gamma_3.
\]

To obtain a variational formulation of problem (3.1)–(3.8), we need additional notation. Let \( V \) denote the closed subspace of \( H_1 \) defined by
\[
V = \{ v \in H_1; v = 0 \ \text{on} \ \Gamma_1 \}. \quad (3.10)
\]

Since \( \text{meas}(\Gamma_1) > 0 \), Korn’s inequality holds: There exists \( C_K > 0 \) depending only on \( \Omega \) and \( \Gamma_1 \) such that
\[
\|\varepsilon(v)\|_Q \geq C_K \|v\|_{H_1} \quad \forall v \in V. \quad (3.9)
\]

A proof of Korn’s inequality may be found in [14, p. 79]. On \( V \) we consider the inner product given by
\[
(u, v)_V = (\varepsilon(u), \varepsilon(v))_Q \quad \forall u, v \in V \quad (3.10)
\]
and let \( \| \cdot \|_V \) be the associated norm, i.e.
\[
\|v\|_V = \|\varepsilon(v)\|_Q \quad \forall v \in V. \quad (3.11)
\]
It follows that \( \| \cdot \|_{H^1} \) and \( \| \cdot \|_V \) are equivalent norms on \( V \) and therefore \( (V, \| \cdot \|_V) \) is a real Hilbert space. Moreover, by the Sobolev trace theorem and (3.9) we have a constant \( C_0 \) depending only on the domain \( \Omega, \Gamma_1 \) and \( \Gamma_3 \) such that
\[
\| \varphi \|_{L^2(\Gamma_1)} \leq C_0 \| \varphi \|_V \quad \forall \varphi \in V.
\] (3.12)

In the study of the mechanical problem (3.1)–(3.8), we assume that the viscosity operator satisfies
\[
(a) \quad \mathcal{A} : \Omega \times S_d \rightarrow S_d.
\]
(b) There exists \( C_{1}^\mathcal{A}, C_{2}^\mathcal{A} > 0 \) such that
\[
\| \mathcal{A}(x, \xi) \| \leq C_{1}^\mathcal{A} \| \xi \| + C_{2}^\mathcal{A} \quad \forall \xi \in S_d, \text{ a.e. } x \in \Omega.
\]
(c) There exists \( m_{\mathcal{A}} > 0 \) such that
\[
\begin{align*}
(\mathcal{A}(x, \xi_1) - \mathcal{A}(x, \xi_2)) \cdot (\xi_1 - \xi_2) \\
\geq m_{\mathcal{A}} \| \xi_1 - \xi_2 \|^2 \quad \forall \xi_1, \xi_2 \in S_d, \text{ a.e. } x \in \Omega.
\end{align*}
\]
(d) The mapping \( x \mapsto \mathcal{A}(x, \xi) \) is Lebesgue measurable on \( \Omega \) for any \( \xi \in S_d \).
(e) The mapping \( \xi \mapsto \mathcal{A}(x, \xi) \) is continuous on \( S_d \), a.e. \( x \in \Omega \). \hfill (3.13)

The elasticity operator satisfies the usual properties of ellipticity and symmetry, i.e.
\[
(a) \quad \mathcal{G} = (g_{ijkh}) : \Omega \times S_d \rightarrow S_d.
\]
(b) \( g_{ijkh} \in L^\infty(\Omega) \).
(c) \( \mathcal{G} \sigma \cdot \tau = \sigma \cdot \mathcal{G} \tau \quad \forall \sigma, \tau \in S_d, \text{ a.e. in } \Omega. \)
(d) There exists \( m_{\mathcal{G}} > 0 \) such that
\[
\mathcal{G} \tau \cdot \tau \geq m_{\mathcal{G}} \| \tau \|^2 \quad \forall \tau \in S_d, \text{ a.e. in } \Omega.
\] (3.14)

The normal compliance function satisfies
\[
(a) \quad p_r : \Gamma_3 \times \mathbb{R} \rightarrow \mathbb{R}^+.
\]
(b) There exists an \( L_v > 0 \) such that
\[
|p_v(x, r_1) - p_v(x, r_2)| \leq L_v |r_1 - r_2| \quad \forall r_1, r_2 \in \mathbb{R}, \text{ a.e. } x \in \Gamma_3.
\]
(c) The mapping \( x \mapsto p_v(x, r) \) is Lebesgue measurable on \( \Gamma_3 \), \( \forall r \in \mathbb{R}. \)
(d) The mapping \( x \mapsto p_v(x, r) = 0 \) for all \( r \leq 0. \) \hfill (3.15)

The adhesion coefficient and the limit bound satisfy
\[
\gamma_v \in L^\infty(\Gamma_3), \quad \gamma_v \geq 0,
\]
\[
\varepsilon_a \in L^\infty(\Gamma_3), \quad \varepsilon_a \geq 0.
\] (3.16) (3.17)

We suppose that the mass density satisfies
\[
\rho \in L^\infty(\Omega) \quad \text{and there exists } \rho^* > 0 \text{ such that } \rho(x) \geq \rho^* \text{ a.e. } x \in \Omega
\] (3.18)
and the body forces and surface tractions have the regularity
\[ f_0 \in L^2(0, T; H), \quad f_2 \in L^2(0, T; [L^2(I_2)]^d). \] (3.19)

Finally, the initial data satisfy
\[ u_0 \in V, \quad v_0 \in H, \quad \beta_0 \in L^\infty(I_3), \quad 0 \leq \beta_0 \leq 1. \] (3.20)

On the space \( H \) we will use a new inner product
\[ ((u, v))_H = (\rho u, v)_H \quad \forall u, v \in H. \] (3.21)

Let \( |||\cdot|||_H \) be the associated norm, i.e.
\[ |||v|||_H = ((\rho v, v))^{1/2}_H \quad \forall v \in H. \] (3.22)

Using assumption (3.18), it follows that \( |||\cdot|||_H \) and \( \| \cdot \|_H \) are equivalent norms on \( H \). Moreover, the inclusion mapping of \((V, \| \cdot \|_V)\) into \((H, |||\cdot|||_H)\) is continuous and dense. We denote by \( V' \) the dual space of \( V \). Identifying \( H \) with its own dual, we can write
\[ V \subset H \subset V'. \]

We use the notation \( \langle \cdot, \cdot \rangle_{V' \times V} \) to represent the duality pairing between \( V' \) and \( V \). We have
\[ \langle u, v \rangle_{V' \times V} = ((u, v))_H \quad \forall u \in H, \ v \in V. \] (3.23)

Finally, we denote by \( \| \cdot \|_{V'} \) the norm on the dual space \( V' \).

Using assumption (3.19) we can define \( f(t) \in V' \) by
\[ \langle f(t), v \rangle_{V' \times V} = (f_0(t), v)_H + (f_2(t), v)_{([L^2(I_2)]^d)} \quad \forall v \in V, \quad \text{a.e.} \ t \in (0, T). \] (3.24)

Note that conditions (3.19) imply
\[ f \in L^2(0, T; V'). \] (3.25)

Let \( j : L^\infty(I_3) \times V \times V \to \mathbb{R} \) be the functional
\[ j(\beta, u, v) = \int_{I_3} p_a(u_r)v_r \, d\alpha + \int_{I_3} \gamma \beta^2(-u_r)_+v_r \, d\alpha \quad \forall \beta \in L^\infty(I_3), \quad \forall u, v \in V. \] (3.26)

Keeping in mind (3.15) and (3.16), we observe that the integrals in (3.26) are well defined.

We turn now to derive a variational formulation for the mechanical problem P. To this end, assume \( \{u, \sigma, \beta\} \) are regular functions satisfying (3.1)–(3.8) and let \( v \in V, \ t \in [0, T] \). Using (2.3) and (3.1) we have
\[ (\rho \dot{u}(t), v)_H + (\sigma(t), \dot{v}(t))_Q = \int_{\Omega} f_0(t) \cdot v \, dx + \int_{R} \sigma(t)v \cdot v \, da \]
and by (3.3), (3.4), (3.19) and (3.22)–(3.24) we find
\[ \langle \ddot{u}(t), v \rangle_{V' \times V} + (\sigma(t), \dot{v}(t))_Q = \langle f(t), v \rangle_{V' \times V} + \int_{I_3} \sigma(t)v \cdot v \, da. \] (3.27)

Using now (2.1), (2.2), (3.5), (3.26) and (3.27), we find
\[ \langle \ddot{u}(t), v \rangle_{V' \times V} + (\sigma(t), \dot{v}(t))_Q = \langle f(t), v \rangle_{V' \times V}. \] (3.28)
In conclusion, from (3.2), (3.6) and (3.28), we obtain the following variational formulation of the mechanical problem $P$.

**Problem $P_V$.** Find a displacement field $u : [0, T] \to V$, a stress field $\sigma : [0, T] \to Q$ and an adhesion field $\beta : [0, T] \to L^\infty(\Gamma_3)$ such that for a.e. $t \in (0, T)$,

$$\sigma(t) = \mathcal{A}\varepsilon(\dot{u}(t)) + \mathcal{B}\varepsilon(u(t)), \quad \text{a.e. in } \Omega,$$

$$\dot{\beta}(t) = -(\gamma, \beta(t)R(u_c(t)))^2 - \varepsilon_a +, \quad 0 \leq \beta \leq 1, \quad \text{a.e. on } \Gamma_3,$$

$$\langle \ddot{u}(t), w \rangle_{V' \times V} + \langle \sigma(t), \varepsilon(w) \rangle_Q + j(\beta(t), u(t), w) = \langle f(t), w \rangle_{V' \times V} \quad \forall w \in V,$$

$$u(0) = u_0, \quad \dot{u}(0) = v_0, \quad \beta(0) = \beta_0. \quad (3.32)$$

The well-posedness of Problem $P_V$ is stated below and is proved in the next section.

**Theorem 3.1.** Assume that (3.13)–(3.20) hold. Then there exists a unique solution $\{u, \sigma, \beta\}$ to Problem $P_V$. Moreover, the solution satisfies

$$u \in W^{1,2}(0, T; V) \cap \mathcal{C}^1([0, T]; H), \quad \frac{d^2u}{dt^2} \in L^2(0, T; V'),$$

$$\sigma \in L^2(0, T; Q), \quad \text{Div } \sigma \in L^2(0, T; V'),$$

$$\beta \in \mathcal{C}^1([0, T]; L^\infty(\Gamma_3)). \quad (3.33)$$

**4. Proof of the existence and uniqueness result**

The proof of Theorem 3.1 will be carried out in several steps. We first prove an existence and uniqueness result for the viscoelastic problem in terms of the displacement field $u$ and the stress field $\sigma$, defined by Eqs. (3.29), (3.31) with (3.32), where the adhesion field $\beta$ is given. Then we study the continuous dependence of the displacement solution with respect to $\beta$. We assume (3.13)–(3.20) hold. Define the set of admissible adhesion fields

$$\Theta = \{\beta \in \mathcal{C}([0, T]; L^\infty(\Gamma_3)); 0 \leq \beta \leq 1 \text{ a.e. on } \Gamma_3\}.$$

**Proposition 4.1.** For any $\beta \in \Theta$ there exists a unique displacement field $u_\beta : [0, T] \to V$ and a unique stress field $\sigma_\beta : [0, T] \to Q$ satisfying Eqs. (3.29) and (3.31), with initial conditions (3.32) and with regularities (3.33) and (3.34).
The proof of Proposition 4.1 is carried out in several steps, and is based on results of evolution equations with monotone operators and a fixed point argument. It is similar to [17,18], however, with a different choice of the operators. In the following we fix an element $\beta \in \Theta$.

In the first step, let $\eta \in L^2(0; T'; V')$ be given and consider the following variational problem.

**Problem $Q^\eta_V$.** Find a displacement field $u_\eta : [0, T] \to V$ such that

$$
\langle \ddot{u}_\eta(t), v \rangle_{V' \times V} + \langle A\dot{u}_\eta(t), \dot{v} \rangle_Q + \langle \eta(t), v \rangle_{V' \times V} = \langle f(t), v \rangle_{V' \times V}, \quad \forall v \in V, \text{ a.e. } t \in (0, T),
$$

$$
u_\eta(0) = u_0, \quad \dot{u}_\eta(0) = v_0.
$$

To solve Problem $Q^\eta_V$, we use an abstract existence and uniqueness result (see, e.g., [1]) which we now recall, for the convenience of the reader. Let $V$ and $H$ be real Hilbert spaces such that $V$ is dense in $H$ and the injection map is continuous: $H$ is identified with its own dual and with a subspace of the dual $V'$ of $V$, i.e. $V \subset H \subset V'$ is a Gelfand triplet. The notations $\| \cdot \|_V$, $\| \cdot \|_{V'}$, and $\langle \cdot, \cdot \rangle_{V' \times V}$ represent the norms on $V$ and $V'$ and the duality pairing between them, respectively. An operator $A : V \to V'$ is said to be hemicontinuous if the real function $t \mapsto \langle A(u + tv), w \rangle$ is continuous on $[0, 1]$ for all $u, v, w \in V$. The following abstract result may be found in [1, p. 140].

**Theorem 4.2.** Let $V, H$ be as above, and let $A : V \to V'$ be a hemicontinuous monotone operator which satisfies

$$
\langle Au, u \rangle_{V' \times V} \geq \omega \| u \|_V^2 + \alpha \quad \forall u \in V,
$$

where $\omega > 0$ and $\alpha \in \mathbb{R}$, and

$$
\| Au \|_{V'} \leq C(\| u \|_V + 1) \quad \forall u \in V
$$

for $C > 0$. Then, given $u_0 \in H$ and $f \in L^2(0, T; V')$, there exists a unique function $u$ which satisfies:

$$
u \in L^2(0, T; V) \cap \mathcal{C}([0, T]; H), \quad \frac{\partial u}{\partial t} \in L^2(0, T; V'),
$$

$$
\frac{\partial u}{\partial t} + Au(t) = f(t) \quad \text{a.e. } t \in (0, T),
$$

$$
u(0) = u_0.
$$

We show that this theorem implies the following result concerning Problem $Q^\eta_V$.

**Lemma 4.3.** There exists a unique solution for Problem $Q^\eta_V$ which satisfies (3.33).

**Proof.** We define the operator $A : V \to V'$ by

$$
\langle Au, v \rangle_{V' \times V} = \langle A\varepsilon(u), \varepsilon(v) \rangle_Q \quad \forall u, v \in V.
$$

Using (4.8), (3.12) and (3.13) it follows that

$$
\| Au - Av \|_{V'} \leq \| A\varepsilon(u) - A\varepsilon(v) \|_Q \quad \forall u, v \in V.
$$
By the Krasnosel’ski Theorem (see for instance in [13, p. 60]) we deduce that \( A : V \to V' \) is continuous, and is hence hemicontinuous. Now, by (4.8), (3.13)(b) and (3.11) we find
\[
\langle Au - Av, u - v \rangle_{V' \times V} \geq m_\omega \| u - v \|^2_V \quad \forall u, v \in V,
\] (4.9)
i.e. \( A : V \to V' \) is a monotone operator. Choosing \( v = 0_V \) in (4.9) we obtain
\[
\langle Au, u \rangle_{V' \times V} \geq m_\omega \| u \|^2_V - \| A0_V \|_V \| u \|_V
\geq \frac{1}{2} m_\omega \| u \|^2_V - \frac{1}{2} m_\omega \| A0_V \|^2_{V'} \quad \forall u \in V.
\]
Thus \( A \) satisfies condition (4.3) with \( \omega = 3m_\omega / 2 \) and \( z = -\| A0_V \|^2_{V'}/(2m_\omega) \). Moreover, by (4.8), (3.13)(a) we deduce that
\[
\| Au \|_{V'} \leq \| \mathcal{A} \mathcal{E}(u) \|_\Omega \leq C_1 \| \mathcal{E}(u) \|_\Omega + C_2 \quad \forall u \in V.
\]
This inequality and (3.11) imply that the operator \( A \) satisfies condition (4.4). Finally, we recall that \( f - \eta \in L^2(0, T; V') \) and \( v_0 \in H' \) (see (3.25)).

It follows now from Theorem 4.2 that there exists a unique function \( v_\eta \) which satisfies
\[
v_\eta \in L^2(0, T; V') \cap C([0, T]; H), \quad \frac{\partial v_\eta}{\partial t} \in L^2(0, T; V'), \quad (4.10)
\]
\[
\frac{\partial v_\eta}{\partial t} + Av_\eta(t) + \eta(t) = f(t) \quad \text{a.e.} \; t \in (0, T),
\]
\[
v_\eta(0) = v_0.
\] (4.12)
Let \( u_\eta : [0, T] \to V \) be the function defined by
\[
u(t) = \int_0^t \nu(s) \, ds + u_0, \quad t \in [0, T].
\] (4.13)

It follows from (4.8), (4.10)–(4.13) that \( u_\eta \) is a solution of the variational problem \( P_\eta_V \) such that (3.32) holds. This concludes the existence part of Lemma 4.3. The uniqueness part follows from the uniqueness of the solution of problem (4.10)–(4.12), guaranteed by Theorem 4.2. \( \square \)

For \( \eta \in L^2(0, T; V') \) we denote by \( u_\eta \) the solution of Problem \( Q_\eta^V \) obtained in Lemma 4.3 and we let \( A\eta(t) \) denote the element of \( V' \) defined by
\[
\langle A\eta(t), w \rangle_{V' \times V} = \langle \mathcal{G} \mathcal{E}(u_\eta(t)), w \rangle_{V} + j(\beta(t), u_\eta(t), w),
\] (4.14)
for \( w \in V \) and \( t \in [0, T] \). We have the following

**Lemma 4.4.** For \( \eta \in L^2(0, T; V') \), the function \( A\eta : [0, T] \to V' \) is continuous. Moreover, there exists a unique element \( \eta^* \in L^2(0, T; V') \) such that \( A\eta^* = \eta^* \).

**Proof.** Let \( \eta \in L^2(0, T; V') \) and let \( t_1, t_2 \in [0, T] \). Using (4.14), (3.24), (3.11) and (3.12) we obtain
\[
\| A\eta(t_1) - A\eta(t_2) \|_{V'} \leq \| \mathcal{G} \mathcal{E}(u_\eta(t_1)) - \mathcal{G} \mathcal{E}(u_\eta(t_2)) \|_{V'}.
\]
\[
+ C_0 \| p_{\nu}(u_{\eta_1}(t_1)) - p_{\nu}(u_{\eta_1}(t_2)) \|_{L^2_{\nu}(T_3)}
\]
\[
+ C_0 \| \gamma_{\nu} \|_{L^\infty(T_3)} \| \beta^2(t_1)(-u_{\eta_1}(t_1))_+ - \beta^2(t_2)(-u_{\eta_1}(t_2))_+ \|_{L^2_{\nu}(T_3)}.
\]
We write
\[
\beta^2(t_1)(-u_{\eta_1}(t_1))_+ - \beta^2(t_2)(-u_{\eta_1}(t_2))
= (\beta(t_1) - \beta(t_2))(\beta(t_1) + \beta(t_2))(-u_{\eta_1}(t_1))
+ \beta^2(t_2)[(-u_{\eta_1}(t_1))_+ - (-u_{\eta_1}(t_2))_+].
\]
Keeping in mind (3.12) and (3.14)–(3.15), since \(u_{\eta} \in C([0, T]; V)\), we find
\[
\|A\eta(t_1) - A\eta(t_2)\|_V \leq c\|u_{\eta}(t_1) - u_{\eta}(t_2)\|_V + c(u_{\eta})\|\beta(t_1) - \beta(t_2)\|_{L^2(T_1)}.
\] (4.15)
Then we deduce from inequality (4.15) that \(A\eta \in C([0, T]; V')\).

Let now \(\eta_1, \eta_2 \in L^2(0, T; V')\). We use the notation \(u_{\eta_i} = u_i, \dot{u}_{\eta_i} = v_i\) for \(i = 1, 2\). Arguments similar to those in the proof of (4.15) yield
\[
\|A\eta_1(t) - A\eta_2(t)\|_{V'} \leq c\|u_1(t) - u_2(t)\|_{V'}^2
\]
for \(t \in [0, T]\). Then by (4.13) we have
\[
\|A\eta_1(t) - A\eta_2(t)\|_{V'} \leq \int_0^t \|v_1(s) - v_2(s)\|_{V'}^2 \, ds
\] (4.16)
for \(t \in [0, T]\). Moreover, from (4.1) we obtain
\[
\langle \dot{v}_1 - v_2, v_1 - v_2 \rangle_{V' \times V} + \langle \mathcal{A} \varepsilon(v_1) - \mathcal{A} \varepsilon(v_1), \varepsilon(v_1) - \varepsilon(v_2) \rangle_{V^*} + \langle \eta_1 - \eta_2, v_1 - v_2 \rangle_{V' \times V} = 0
\]
for a.e. \(t \in (0, T)\). We integrate this inequality with respect to time and use \(v_1(0) = v_2(0) = v_0\), (3.13)(b) and (3.11) to find
\[
m_{\mathcal{A}} \int_0^{t} \|v_1(s) - v_2(s)\|_{V'}^2 \, ds \leq -\int_0^{t} \langle \eta_1(s) - \eta_2(s), v_1(s) - v_2(s) \rangle_{V' \times V} \, ds
\]
for all \(t \in [0, T]\). Then
\[
\int_0^{t} \|v_1(s) - v_2(s)\|_{V'}^2 \, ds \leq C \int_0^{t} \|\eta_1(s) - \eta_2(s)\|_{V'}^2 \, ds
\] (4.17)
for all \(t \in [0, T]\). Now from (4.16) and (4.17) we have
\[
\|A\eta_1(t) - A\eta_2(t)\|_{V'} \leq C \int_0^{t} \|\eta_1(s) - \eta_2(s)\|_{V'}^2 \, ds
\]
for all \(t \in [0, T]\). Reiterating this inequality \(n\) times leads to
\[
\|A^n \eta_1 - A^n \eta_2\|_{L^2(0, T; V')}^2 \leq \frac{(CT)^n}{n!} \|\eta_1 - \eta_2\|_{L^2(0, T; V')}^2,
\]
which implies that for \(n\) sufficiently large a power \(A^n\) of \(A\) is a contraction in the Hilbert space \(L^2(0, T; V')\). Then, there exists a unique \(\eta^* \in L^2(0, T; V')\) such that \(A^n \eta^* = \eta^*\) and, moreover, \(\eta^*\) is the unique fixed point of \(A\). Certainly, \(\eta^*\) depends on \(\beta\). \(\square\)

We have now all the ingredients needed to prove Proposition 4.1.

**Existence.** Let \(\eta^* \in L^2(0, T; V')\) be the fixed point of \(A\) and let \(u = u_{\beta}\) be the solution of the variational problem \(Q^V_{\eta^*}\) for \(\eta = \eta^*\), i.e. \(u = u_{\eta^*}\). We denote by \(\sigma = \sigma_{\beta}\) the function given by (3.29).
Using (4.1), (4.2), (4.14) and keeping in mind that $A\eta^* = \eta^*$, we find that the couple $\{u, \sigma\}$ is a solution of (3.29), (3.31) and (3.32). The regularity (3.33) follows from Lemma 4.3. Moreover, since $u \in W^{1,2}(0, T; V)$, it follows from (3.29), (3.13), and (3.14) that $\sigma \in L^2(0, T; Q)$. Choosing now $w = \phi$, where $\phi \in [D(\Omega)]^d$, in (3.31) and using (3.24), (3.26), we have

$$\rho \ddot{u}(t) = \text{Div} \sigma(t) + f_0(t) \quad \text{a.e.} \ t \in (0, T).$$

(4.18)

Now, the assumptions (3.18) and (3.19), the fact that $\ddot{u} \in L^2(0, T; V')$ and (4.18) imply that $\text{Div} \sigma \in L^2(0, T; V')$.

**Uniqueness.** The uniqueness part of Proposition 4.1 is a consequence of the uniqueness of the fixed point of the operator $A$ given by (4.14). $\Box$

Next we derive some a priori estimates on the function $u_\beta$ and study its dependence on $\beta$.

**Lemma 4.5.** There exists $c > 0$ depending only on the data such that

1. $\forall \beta \in \Theta \ \forall t \in [0, T], \ ||u_\beta(t)||_V \leq c$.
2. $\forall \beta_1, \beta_2 \in \Theta \ \forall t \in [0, T], \ ||u_{\beta_1}(t) - u_{\beta_2}(t)||_V^2 \leq c \int_0^t ||\beta_1(s) - \beta_2(s)||_{L^\infty(\Gamma_3)} ds$.

**Proof.** (1) Taking $w = \ddot{u}_\beta(t)$ in (3.31), we have

$$\langle \ddot{u}_\beta(t), \ddot{u}_\beta(t) \rangle_{V' \times V} + (\mathcal{A}(u_\beta(t)), e(\ddot{u}_\beta(t)))_Q + (\mathcal{E}(u_\beta(t)), e(\ddot{u}_\beta(t)))_Q$$

$$+ j(\beta(t), u_\beta(t), \ddot{u}_\beta(t)) = (f(t), \ddot{u}_\beta(t))_{V' \times V}.$$

Notice that

$$\langle \ddot{u}_\beta(t), \ddot{u}_\beta(t) \rangle_{V' \times V} = \frac{d}{dt} \frac{1}{2} ||\ddot{u}_\beta(t)||_{H}^2,$$

$$(\mathcal{E}(u_\beta(t)), e(\ddot{u}_\beta(t)))_Q = \frac{d}{dt} \frac{1}{2} (\mathcal{E}(u_\beta(t)), e(u_\beta(t)))_Q,$$

$$(\mathcal{A}(u_\beta(t)), e(\ddot{u}_\beta(t)))_Q \geq m_{\mathcal{A}} ||\ddot{u}_\beta(t)||_V^2 + (\mathcal{A}(0), e(\ddot{u}_\beta(t)))_Q.$$

Integrating over $[0, t]$ and denoting $(\mathcal{E}(w), e(w))_Q = ||w||_V^2$, we get

$$\frac{1}{2} ||\ddot{u}_\beta(t)||_H^2 - \frac{1}{2} ||v_0||_H^2 + \int_0^t ||\ddot{u}_\beta(s)||_V^2 ds + \frac{1}{2} ||u_\beta(t)||_V^2 - \frac{1}{2} ||u_0||_V^2$$

$$\leq \int_0^t |(\mathcal{A}(e(0), e(\ddot{u}_\beta(s)))_Q| ds + \int_0^t |j(\beta(s), u_\beta(s), \ddot{u}_\beta(s))| ds$$

$$+ \int_0^t |(f(s), \ddot{u}_\beta(s))_{V' \times V}| ds.$$

We use then the following inequalities:

$$|\langle \mathcal{A}(0), e(\ddot{u}_\beta(s)) \rangle_Q| \leq \frac{1}{m_{\mathcal{A}}} ||\mathcal{A}(0)||_Q^2 + \frac{m_{\mathcal{A}}}{4} ||\ddot{u}_\beta(s)||_V^2,$$

$$|j(\beta(s), u_\beta(s), \ddot{u}_\beta(s))| \leq c ||u_\beta(s)||_V ||\ddot{u}_\beta(s)||_V$$

$$||u_\beta(t)||_V \leq c.$$
We first bound the terms of \( j \).

\[
|j(\beta_1(t), u_{\beta_1}(t), w) - j(\beta_2(t), u_{\beta_2}(t), w)|
\leq c\|u_{\beta_1}(t) - u_{\beta_2}(t)\|_V \|w\|_V
+ c\|\gamma_v\|_{L^\infty(G_3)}: \beta_1^2(-u_{\beta_1,v})_+ - \beta_2^2(-u_{\beta_2,v})_+ \|L^2(G_3)\|w\|_V.
\]

We write

\[
\beta_1^2(-u_{\beta_1,v})_+ - \beta_2^2(-u_{\beta_2,v})_+ = (\beta_1 - \beta_2)(\beta_1 + \beta_2)(-u_{\beta_1,v})_+ + \beta_2^2(-u_{\beta_1,v})_+ - (-u_{\beta_2,v})_+.
\]

Using then Lemma 4.5 and the continuous imbedding of \( V \) into \( L^2(G_3) \), we have

\[
\|\beta_1^2(-u_{\beta_1,v})_+ - \beta_2^2(-u_{\beta_2,v})_+\|_{L^2(G_3)} \leq c\|\beta_1(t) - \beta_2(t)\|_{L^\infty(G_3)} + c\|u_{\beta_1}(t) - u_{\beta_2}(t)\|_V.
\]

We deduce then that

\[
|j(\beta_1(t), u_{\beta_1}(t), w) - j(\beta_2(t), u_{\beta_2}(t), w)|
\leq c(\|u_{\beta_1}(t) - u_{\beta_2}(t)\|_V + \|\beta_1(t) - \beta_2(t)\|_{L^\infty(G_3)}) \|w\|_V.
\]

On the other hand we have

\[
|\langle \mathcal{E}(\dot{u}_{\beta_1}(t)) - \mathcal{E}(\dot{u}_{\beta_2}(t)), \varepsilon(w) \rangle_Q| \leq c\|u_{\beta_1}(t) - u_{\beta_2}(t)\|_V \|w\|_V.
\]

Using the last two inequalities and the assumption (3.13)(c), we integrate (4.19) over \([0,t]\) to get

\[
m_c \int_0^t \|\dot{u}_{\beta_1}(s) - \dot{u}_{\beta_2}(s)\|^2 \, ds \leq c \int_0^t (\|u_{\beta_1}(s) - u_{\beta_2}(s)\|_V + \|\beta_1(s) - \beta_2(s)\|_{L^\infty(G_3)}) \times \|\dot{u}_{\beta_1}(s) - \dot{u}_{\beta_2}(s)\|_V \, ds.
\]
From this we deduce
\[
\int_0^t \|\hat{u}_{\beta_1}(s) - \hat{u}_{\beta_2}(s)\|^2 \, ds \leq c \int_0^t \|u_{\beta_1}(s) - u_{\beta_2}(s)\|^2 \, ds + c \int_0^t \|\beta_1(s) - \beta_2(s)\|^2_{L^\infty(\Gamma_3)} \, ds.
\]

As
\[
\|u_{\beta_1}(t) - u_{\beta_2}(t)\|^2 \leq c \int_0^t \|\hat{u}_{\beta_1}(s) - \hat{u}_{\beta_2}(s)\|^2 \, ds,
\]
we use Gronwall’s inequality to conclude the second part of Lemma 4.5. □

We then prove an existence and uniqueness result for the adhesion function.

**Proposition 4.6.** For any $\beta_0$ satisfying (3.20), there exists a unique $\beta : [0, T] \rightarrow L^\infty(\Gamma_3)$ satisfying
\[
\dot{\beta}(t) = -(\gamma, \beta(t)R(u_{\beta_0}(t))^2 - \varepsilon_a)_{+}, \quad 0 \leq \beta(t) \leq 1,
\]
for all $t \in [0, T]$, a.e. on $\Gamma_3$, with the initial condition (3.32) and the regularity (3.35).

**Proof.** The problem is rewritten as
\[
\forall t \in [0, T], \quad \beta(t) = \beta_0 - \int_0^t (\gamma, \beta(s)R(u_{\beta_0}(s))^2 - \varepsilon_a)_{+} \, ds.
\]

For any $\beta \in \Theta$, we define $\Delta \beta$ by
\[
\Delta \beta(t) = \beta_0 - \int_0^t (\gamma, \beta(s)R(u_{\beta_0}(s))^2 - \varepsilon_a)_{+} \, ds \quad \forall t \in [0, T].
\]

Let us check that $\Delta \beta \in \Theta$. By the property of $R$ and Lemma 4.5, we see that
\[
\|R(u_{\beta_t}(s))\|_{L^\infty(\Gamma_3)} \leq c \|u_{\beta}(s)\|_V \leq c
\]
for a constant $c$ independent of $s \in [0, T]$ and $\beta$. Hence $\forall t \in [0, T]$, $\Delta \beta(t) \in L^\infty(\Gamma_3)$ and $\Delta \beta \in C^1([0, T]; L^\infty(\Gamma_3))$. Let us show that $0 \leq \Delta \beta \leq 1$. Indeed, $\Delta \beta(0) = \beta_0 \in [0, 1]$ (a.e. on $\Gamma_3$). Suppose for some $t_0 \in [0, 1]$, $\Delta \beta(t_0) = 0$. As $|\Delta \beta| \leq 0$, we deduce that $\Delta \beta(t) \leq 0$ for all $t \in [t_0, 1]$. From the adhesion equation, we get $|\Delta \beta|'(t) = 0$, $\forall t \in [t_0, 1]$. Hence $\Delta \beta(t) = 0$, $\forall t \in [t_0, 1]$.

Now for any $\beta_1, \beta_2 \in \Theta$, and for any $t \in [0, T],$
\[
\|\Delta \beta_1(t) - \Delta \beta_2(t)\|_{L^\infty(\Gamma_3)} \leq c \int_0^t \|\beta_1(s)R(u_{\beta_1}(s))^2 - \beta_2(s)R(u_{\beta_2}(s))^2\|_{L^\infty(\Gamma_3)} \, ds.
\]

Writing $\beta_1(s) = \beta_1(s) - \beta_2(s) + \beta_2(s)$ and using Lemma 4.5, we have
\[
\forall t \in [0, T], \quad |\Delta \beta_1(t) - \Delta \beta_2(t)|^2_{L^\infty(\Gamma_3)} \leq c \int_0^t \|\beta_1(s) - \beta_2(s)\|^2_{L^\infty(\Gamma_3)} \, ds.
\]

Denote the norm $\|\beta\|_* = \sup_{0 \leq t \leq 1} e^{-zt} \|\beta(t)\|_{L^\infty(\Gamma_3)}$ for some $z > 0$ to be chosen. It is easy to see that the norm $\|\cdot\|_*$ is equivalent to the usual norm $\|\cdot\|_{C([0, 1]; L^\infty(\Gamma_3))}$. In the last inequality, writing $\|\beta_1(s) - \beta_2(s)\|^2_{L^\infty(\Gamma_3)} = e^{2zs}e^{-2zs}\|\beta_1(s) - \beta_2(s)\|^2_{L^\infty(\Gamma_3)}$, after some algebraic manipulations, we deduce that
\[
\|\Delta \beta_1 - \Delta \beta_2\|_* \leq \frac{c}{z} \|\beta_1 - \beta_2\|_*^2.
\]
We conclude that for large \( AVT \), the operator \( ASOH : ASTX \rightarrow ASTX \) is a contraction. Therefore, \( ASOH : ASTX \rightarrow ASTX \) has one and only one fixed point, which is the solution of the problem for the adhesion function. \( \square \)

**Proof of Theorem 3.1.** Theorem 3.1 is now a consequence of Propositions 4.1 and 4.6. \( \square \)

5. Analysis of a numerical scheme

In this section, we study a fully discrete scheme for the numerical approximation of the variational problem \( PV \). To this end, we eliminate \( \sigma \) from Problem \( PV \) and introduce the velocity variable

\[
v(t) = \dot{u}(t).
\]

(5.1)

We can express the displacement variable as

\[
u(t) = IV(t) \equiv u_0 + \int_0^t v(s) \, ds.
\]

(5.2)

From Theorem 3.1 we see that

\[
v \in L^2(0, T; V) \cap C([0, T]; H), \quad \dot{v} \in L^2(0, T; V')
\]

(5.3)

and \( v \) satisfies

\[
\langle \dot{v}(t), w \rangle_{V' \times V} + (A_\varepsilon(v(t)), \varepsilon(w))_Q + (B_\varepsilon(IV(t)), \varepsilon(w))_Q + j(\dot{\beta}(t), IV(t), w)
\]

\[
= \langle f(t), w \rangle_{V' \times V} \quad \forall w \in V, \text{ a.e. } t \in (0, T),
\]

(5.4)

\[
v(0) = v_0.
\]

(5.5)

In this section, we make the following additional assumptions on the data and the solution:

\[
f_0 \in C([0, T]; H), \quad f_2 \in C([0, T]; [L^2(\Gamma_2)]^d),
\]

(5.6)

\[
\dot{v} \in C([0, T]; H) \cap L^1(0, T; V),
\]

(5.7)

\[
\ddot{\beta} \in L^1(0, T; L^\infty(\Gamma_3)).
\]

(5.8)

Then we have

\[
f \in C([0, T]; V')
\]

and (5.4) can be replaced by

\[
\langle \dot{v}(t), w \rangle_H + (A_\varepsilon(v(t)), \varepsilon(w))_Q + (B_\varepsilon(IV(t)), \varepsilon(w))_Q + j(\dot{\beta}(t), IV(t), w)
\]

\[
= \langle f(t), w \rangle_{V' \times V} \quad \forall w \in V, \ t \in (0, T).
\]

(5.9)

Also we need an assumption stronger than (3.13)(b):

\[
\| A(x, \xi) - A(x, \eta) \| \leq C \| \xi - \eta \| \quad \forall \xi, \eta \in S_d, \text{ a.e. } x \in \Omega.
\]

(5.10)
We will consider a general setting of arbitrary finite-dimensional spaces \( V^h \subset V \) and \( B^h \subset L^\infty(\Gamma_3) \), used to approximate spaces \( V \) and \( L^\infty(\Gamma_3) \), respectively. Here, \( h > 0 \) is a discretization parameter. Let \( \mathcal{P}_b^h : L^2(\Gamma_3) \rightarrow B^h \) be a projection operator satisfying
\[
\| \mathcal{P}_b^h \gamma \|_{L^\infty(\Gamma_3)} \leq \| \gamma \|_{L^\infty(\Gamma_3)} \quad \forall \gamma \in L^\infty(\Gamma_3).
\] (5.11)
We divide the time interval \([0, T]\) into \( N \) equal parts: \( t_n = nk, \ n = 0, 1, \ldots, N \), with the time step \( k = T/N \). Then we introduce the following fully discrete scheme.

**Problem \( P^h \).** Find a velocity field \( \mathbf{v}^h = \{ \mathbf{v}^h_n \}_{n=0}^N \subset V^h \) and an adhesion field \( \beta^h = \{ \beta^h_n \}_{n=0}^N \subset B^h \) such that
\[
\mathbf{v}^h_0 = \mathbf{v}_0, \quad \beta^h_0 = \beta_0
\] (5.12)
and for \( n = 1, 2, \ldots, N \),
\[
\delta \beta^h_n = -\mathcal{P}_b^h (\gamma \iota \beta^h_{n-1}) [R((\mathbf{u}^h_{n-1})_v)^2 - e_a)_+ \quad \text{on} \quad \Gamma_3,
\] (5.13)
\[
((\delta \mathbf{v}^h_n, \mathbf{w}^h)_H + (\mathcal{A} \mathbf{v}^h_n + \mathcal{B} \mathbf{v}^h_n, \mathbf{w}^h)_Q + j(\beta^h_{n-1}, \mathbf{u}^h_{n-1})_v = (f_n, \mathbf{w}^h)_v \quad \forall \mathbf{w}^h \in V^h.
\] (5.14)
Here, \( \delta \beta^h_n = (\beta^h_n - \beta^h_{n-1})/k \), \( \delta \mathbf{v}^h_n = (\mathbf{v}^h_n - \mathbf{v}^h_{n-1})/k \) and
\[
\mathbf{u}^h_n = \mathbf{u}^h_0 + k \sum_{j=0}^{n} \mathbf{v}^h_j, \quad 0 \leq n \leq N.
\] (5.15)
Moreover, \( \mathbf{u}^h_0, \mathbf{v}^h_0 \in V^h \) and \( \beta^h_0 \in B^h \) are suitable approximations of the initial values \( \mathbf{u}_0, \mathbf{v}_0 \) and \( \beta_0 \).

It is easy to verify that Problem \( P^h \) has a unique solution. Moreover, using a discrete version of Gronwall’s lemma, we can prove that \( \| \beta^h_n \|_{L^\infty(\Gamma_3)}, \| \mathbf{v}^h_n \|_V, 0 \leq n \leq N \) are bounded by a constant depending only on the initial data.

We now turn to an error analysis of the numerical solution. For a continuous function \( \mathbf{w} \in C([0, T]; X) \) with values in a space \( X \), we use the notation \( \mathbf{w}_n = \mathbf{w}(t_n) \in X \). We are interested in estimating the numerical errors \( \beta_n - \beta^h_n \) and \( \mathbf{e}_n = \mathbf{v}_n - \mathbf{v}^h_n, 0 \leq n \leq N \).

First, using (3.30), (5.13) and writing
\[
\delta (\beta_n - \beta^h_n) = \delta \beta_n + \hat{\beta}_n + (I - \mathcal{P}_b^h) \hat{\beta}_n + \mathcal{P}_b^h \hat{\beta}_n - \delta \beta^h_n
\] with
\[
\mathcal{P}_b^h \hat{\beta}_n - \delta \beta^h_n = -\mathcal{P}_b^h [(\gamma \iota \beta_n R((\mathbf{u}^h_n)_v)^2 - e_a)_+ - (\gamma \iota \beta^h_{n-1} R((\mathbf{u}^h_{n-1})_v)^2 - e_a)_+],
\] we get
\[
\| \delta (\beta_n - \beta^h_n) \|_{L^\infty(\Gamma_3)} \leq \| \delta \beta_n - \hat{\beta}_n \|_{L^\infty(\Gamma_3)} + \| (I - \mathcal{P}_b^h) \hat{\beta}_n \|_{L^\infty(\Gamma_3)} + c(\beta) \| \mathbf{u}_n - \mathbf{u}^h_n \|_V
+ c \| \beta_n - \beta_{n-1} \|_{L^\infty(\Gamma_3)} + c \| \beta_{n-1} - \beta^h_{n-1} \|_{L^\infty(\Gamma_3)}.
\] (5.16)
Denote
\[ r_n(\beta) = k \sum_{j=1}^{n} \| (I - \mathcal{P}_{B_h}) \beta_j \|_{L^\infty(T_3)} + k \sum_{j=1}^{n} \| \delta \beta_j - \hat{\beta}_j \|_{L^\infty(T_3)} \]
\[ + \left( k \sum_{j=1}^{n} \| \beta_j - \beta_{j-1} \|_{L^\infty(T_3)}^2 \right)^{1/2} \]

Using
\[ \beta_n - \beta_n^{hk} = (\beta_0 - \beta_0^h) + k \sum_{j=1}^{n} \delta(\beta_j - \beta_j^{hk}), \]
we obtain
\[ \| \beta_n - \beta_n^{hk} \|_{L^\infty(T_3)} \leq \| \beta_0 - \beta_0^h \|_{L^\infty(T_3)} + r_n(\beta) + c(\beta)k \sum_{j=1}^{n} \| u_j - u_j^{hk} \|_V \]
\[ + ck \sum_{j=1}^{n} \| \beta_{j-1} - \beta_{j-1}^{hk} \|_{L^\infty(T_3)}. \]  \hspace{1cm} (5.17)

On the other hand, from (5.2) and (5.15), we have
\[ \| u_n - u_n^{hk} \|_V \leq \| u - u_0^h \|_V + k \sum_{j=0}^{n-1} \| v_j^{hk} - v_j \|_V + I_n, \]  \hspace{1cm} (5.18)
where
\[ I_n = \left\| \int_0^t v(s) \, ds - k \sum_{j=1}^{n-1} v_j \right\|_V. \]

We deduce then from (5.17) and (5.18) that
\[ \| \beta_n - \beta_n^{hk} \|_{L^\infty(T_3)}^2 \leq c \| \beta_0 - \beta_0^h \|_{L^\infty(T_3)}^2 + c(\beta)\| u_0 - u_0^h \|_V^2 + c(\beta)k \sum_{j=1}^{n} I_j^2 + r_n(\beta)^2 \]
\[ + c(\beta)k \sum_{j=1}^{n} \left( k \sum_{i=0}^{j-1} \| e_i \|_V^2 \right) + ck \sum_{j=0}^{n-1} \| \beta_j - \beta_j^{hk} \|_{L^\infty(T_3)}^2. \]  \hspace{1cm} (5.19)

To continue, we consider the quantity
\[ E_n = \left( \begin{array}{c} e_n - e_{n-1} \\
\frac{k}{k} \\
e_n \end{array} \right) \right)_H \left. \mp \mathcal{A}(\mathcal{E}(v_n)) - \mathcal{A}(\mathcal{E}(v_n^{hk})) \right)_Q. \]  \hspace{1cm} (5.20)
First we have a lower bound using the property (3.13)(c):
\[ E_n \geq \frac{1}{2k} (\| e_n \|_H^2 - \| e_{n-1} \|_H^2) + m_{\mathcal{A}} \| e_n \|_V^2. \]  \hspace{1cm} (5.21)
Using (5.9) and (5.14) with \( w \) and \( w^h \) replaced by \( w^h_n - v^h_n \) (\( w^h_n \in V^h \) is arbitrary), we obtain

\[ E_n = I_{1,n} + I_{2,n} + I_{3,n} + I_{4,n} + I_{5,n}, \]

where

\[ I_{1,n} = \left( \left( \frac{v_n - v_{n-1}}{k} - \dot{v}_n, e_n \right) \right)_H - \left( \left( \frac{v_n - v_{n-1}}{k} - \dot{v}_n, v_n - w^h_n \right) \right)_H, \]

\[ I_{2,n} = \left( \left( \frac{e_n - e_{n-1}}{k}, v_n - w^h_n \right) \right)_H, \]

\[ I_{3,n} = (\mathcal{A}e(v_n) - \mathcal{A}e(v^h_n), e(v_n - w^h_n))_Q, \]

\[ I_{4,n} = (\mathcal{B}e(u_n) - \mathcal{B}e(u^h_{n-1}), e(v^h_n - w^h_n))_Q, \]

\[ I_{5,n} = j(\beta_n, u_n, v^h_n - w^h_n) - j(\beta^h_{n-1}, u^h_{n-1}, v^h_n - w^h_n). \]

Using (5.21), we have

\[ \frac{1}{2k} (||e_n||_H^2 - ||e_{n-1}||_H^2) + m_\mathcal{A} ||e_n||_V^2 \leq I_{1,n} + I_{2,n} + I_{3,n} + I_{4,n} + I_{5,n}. \] (5.22)

Let us estimate the terms on the right-hand side of (5.22).

\[ |I_{1,n}| \leq \left\| \left( \frac{v_n - v_{n-1}}{k} - \dot{v}_n \right) \right\|_H (||e_n||_H + ||v_n - w^h_n||_H). \] (5.23)

Using (5.10) and (3.14), we obtain:

\[ |I_{3,n}| \leq c ||e_n||_V ||v_n - w^h_n||_V, \] (5.24)

\[ |I_{4,n}| \leq c ||u_n - u^h_{n-1}||_V (||e_n||_V + ||v_n - w^h_n||_V). \] (5.25)

From (3.26) we get

\[ |I_{5,n}| \leq c(\|\beta_n - \beta^h_{n-1}\|_{L^\infty(\Gamma_3)} + \|u_n - u^h_{n-1}\|_V)(||e_n||_V + ||v_n - w^h_n||_V). \] (5.26)

Finally, taking into account (6.17) we have

\[ ||e_n||^2_H - ||e_0||^2_H + 2m_\mathcal{A} k \sum_{j=1}^n ||e_j||_V^2 \leq 2k \sum_{j=1}^n [I_{1,j} + I_{2,j} + I_{3,j} + I_{4,j} + I_{5,j}]. \] (5.27)

Now

\[ k \sum_{j=1}^n I_{2,j} = \sum_{j=1}^n \left( (e_j - e_{j-1}, v_j - w^h_j) \right)_H \]

\[ = ((e_n, v_n - w^h_n))_H + \sum_{j=1}^{n-1} ((e_j, (v_j - w^h_j) - (v_{j+1} - w^h_{j+1})))_H \]

\[ - ((e_0, v_1 - w^h_1))_H, \]
which implies
\[ k \sum_{j=1}^{n} I_{2,j} \leq |||e_n|||_{H}|||v_n - w^h_n|||_H + \sum_{j=1}^{n-1} |||e_j|||_H |||(v_j - w^h_j) - (v_{j+1} - w^h_{j+1})|||_H \\
+ |||e_0|||_H|||v_1 - w^h_1|||_H. \tag{5.28} \]

Using the estimates \((5.23)-(5.26)\) and \((5.28)\) in \((5.27)\), we have
\[ |||e_n|||_H^2 + 2m_{\Delta} k \sum_{j=1}^{n} |||e_j|||_V^2 \\
\leq |||e_0|||_H^2 + 2k \sum_{j=1}^{n} \left( \left| \frac{v_j - v_{j-1}}{k} - \dot{v}_j \right|_H (|||e_j|||_H + |||v_j - w^h_j|||_H) \right) \\
+ 2(|||e_n|||_H|||v_n - w^h_n|||_H + 2 \sum_{j=1}^{n-1} |||e_j|||_H|||(v_j - w^h_j) - (v_{j+1} - w^h_{j+1})|||_H \\
+ 2(|||e_0|||_H|||v_1 - w^h_1|||_H + c \sum_{j=1}^{n} |||e_j|||_V|||v_j - w^h_j|||_V \\
+ c k \sum_{j=1}^{n} |||u_j - u^h_{j-1}|||_V (|||e_j|||_V + |||v_j - w^h_j|||_V) \\
+ c k \sum_{j=1}^{n} |||\beta_j - \beta^h_{j-1}|||_{L^\infty(I_\gamma)} (|||e_j|||_V + |||v_j - w^h_j|||_V). \]

Denote
\[ M = \max_n |||e_n|||_H. \]

After some algebraic manipulations, we obtain
\[ |||e_n|||_H^2 + m_{\Delta} k \sum_{j=1}^{n} |||e_j|||_V^2 \\
\leq |||e_0|||_H^2 + c M \left( k \sum_{j=1}^{n} \left| \frac{v_j - v_{j-1}}{k} - \dot{v}_j \right|_H + |||v_n - w^h_n|||_H \\
+ \sum_{j=1}^{n-1} |||(v_j - w^h_j) - (v_{j+1} - w^h_{j+1})|||_H + |||v_1 - w^h_1|||_H \right) \\
+ c k \sum_{j=1}^{n} \left| \frac{v_j - v_{j-1}}{k} - \dot{v}_j \right|_H |||v_j - w^h_j|||_H + c k \sum_{j=1}^{n} |||v_j - w^h_j|||_V^2 \\
+ c k \sum_{j=1}^{n} |||u_j - u^h_{j-1}|||_V^2 + c k \sum_{j=1}^{n} |||\beta_j - \beta^h_{j-1}|||_{L^\infty(I_\gamma)}^2. \]
Using (5.18) and (5.19) again, we obtain
\[
\|\beta_n - \beta_n^{hk}\|^2_{L^\infty(G_1)} + \sum_{j=0}^{n} \|e_j\|^2_{V} + k \sum_{j=0}^{n} \|e_j\|^2_{V} \\
\leq c \|e_0\|^2_H + c \|\beta_0 - \beta_0^{hk}\|^2_{L^\infty(G_1)} + c(\beta) \|u_0 - u_0^{hk}\|^2 + c(\beta) k \sum_{j=0}^{n-1} \|e_j\|^2_{V} \\
+ cM \left( k \sum_{j=0}^{n} \left\| \frac{v_j - v_{j-1}}{k} - \dot{v}_j \right\|_{H} + \|v_n - \dot{w}_n\|_{H} \right) \\
+ \sum_{j=1}^{n-1} \| (v_j - \dot{w}_j) - (v_{j+1} - \dot{w}_{j+1}) \|_{H} + \|v_1 - \dot{w}_1\|_{H} \\
+ c k \sum_{j=1}^{n} \left\| \frac{v_j - v_{j-1}}{k} - \dot{v}_j \right\|_{H} \|v_j - \dot{w}_j\|_{H} + c k \sum_{j=1}^{n} \|v_j - \dot{w}_j\|^2_{V} \\
+ c k \sum_{j=1}^{n} \|\beta_{j-1} - \beta_{j-1}^{hk}\|^2_{L^\infty(G_1)}. \tag{5.29}
\]

It is shown in [4, Lemma 4.1] that for two sequences \(r_n \geq 0, g_n \geq 0\) satisfying
\[
r_n \leq g_n + c k \sum_{j=0}^{n-1} r_j, \quad n = 1, 2, \ldots, N,
\]
we have
\[
\max_{0 \leq n \leq N} r_n \leq c \max_{0 \leq n \leq N} g_n.
\]
Thus from (5.29) we deduce that
\[
\|\beta_n - \beta_n^{hk}\|^2_{L^\infty(G_1)} + \sum_{j=0}^{n} \|e_j\|^2_{V} \\
\leq c \|e_0\|^2_H + c \|\beta_0 - \beta_0^{hk}\|^2_{L^\infty(G_1)} + c(\beta) \|u_0 - u_0^{hk}\|^2 + c(\beta) k \sum_{j=1}^{N} \|e_j\|^2_{V} + \max_{0 \leq j \leq N} r_j (\beta)^2 \\
+ cM \left( k \sum_{j=1}^{N} \left\| \frac{v_j - v_{j-1}}{k} - \dot{v}_j \right\|_{H} + \max_{0 \leq n \leq N} \|v_n - \dot{w}_n\|_{H} \right) \\
+ \sum_{j=1}^{N-1} \| (v_j - \dot{w}_j) - (v_{j+1} - \dot{w}_{j+1}) \|_{H} \\
+ c k \sum_{j=1}^{N} \left\| \frac{v_j - v_{j-1}}{k} - \dot{v}_j \right\|_{H} \|v_j - \dot{w}_j\|_{H} + c k \sum_{j=1}^{N} \|v_j - \dot{w}_j\|^2_{V}.
\]
Using this estimate in (5.29) we get

\[
M^2 + \|\beta_n - \beta^h_n\|_{L^\infty(\Omega)}^2 + k \sum_{j=0}^n \|e_j\|_V^2 \\
\leq c\|e_0\|_H^2 + c\|\beta_0 - \beta_0^h\|_{L^\infty(\Omega)}^2 + c(\beta)\|u_0 - u_0^h\|_V^2 \\
+ c(\beta)k\sum_{j=1}^n I_j^2 + \max_{0 \leq j \leq N} r_j(\beta)^2 \\
+ cM \left( k \sum_{j=1}^N \left\| \frac{v_j - v_{j-1}}{k} - \dot{v}_j \right\|_H + \max_{0 \leq n \leq N} \|v_n - w^h_n\|_H \\
+ \sum_{j=1}^{N-1} \left\| (v_j - w^h_j) - (v_{j+1} - w^h_{j+1}) \right\|_H \right) \\
+ ck\sum_{j=1}^N \left\| \frac{v_j - v_{j-1}}{k} - \dot{v}_j \right\|_H \|v_j - w^h_j\|_H + ck\sum_{j=1}^N \|v_j - w^h_j\|_V^2.
\]

From this we deduce the following estimate:

\[
\max_{0 \leq n \leq N} \|e_n\|_H^2 + \max_{0 \leq n \leq N} \|\beta_n - \beta^h_n\|_{L^\infty(\Omega)}^2 + k \sum_{j=0}^n \|e_j\|_V^2 \\
\leq c\|e_0\|_H^2 + c\|\beta_0 - \beta_0^h\|_{L^\infty(\Omega)}^2 + c(\beta)\|u_0 - u_0^h\|_V^2 \\
+ c(\beta)k\sum_{j=1}^n I_j^2 + \max_{0 \leq j \leq N} r_n(\beta)^2 + c \left( k \sum_{j=1}^N \left\| \frac{v_j - v_{j-1}}{k} - \dot{v}_j \right\|_H + \max_{0 \leq n \leq N} \|v_n - w^h_n\|_H \right) \\
+ \sum_{j=1}^{N-1} \left\| (v_j - w^h_j) - (v_{j+1} - w^h_{j+1}) \right\|_H^2 \\
+ ck\sum_{j=1}^N \left\| \frac{v_j - v_{j-1}}{k} - \dot{v}_j \right\|_H \|v_j - w^h_j\|_H + ck\sum_{j=1}^N \|v_j - w^h_j\|_V^2,
\]

(5.30)

where \(w^h_j \in V^h, j = 1, \ldots, N\), are arbitrary.

Summarizing, we have proved the following result regarding the discrete problem \(P^h_V\).

**Theorem 5.1.** We keep the assumptions of Theorem 3.1 and assume the conditions (5.6)–(5.8). Then, the discrete problem \(P^h_V\) has a unique solution \(v^h \in V^h, \beta^h \in B^h\), and we have the error estimate (5.30).
Inequality (5.30) is a basis for error estimates for particular choices of the finite-dimensional subspaces $V^h$ and $B^h$, under additional solution regularities. As one such example, let us assume $\Omega$ is a polygonal domain and let $\{\mathcal{T}^h\}_{h>0}$ be a regular family of finite element partitions of $\Omega$ in such a way that if a side of an element lies on the boundary, the side belongs entirely to one of the subsets $\tilde{\Gamma}_1$, $\tilde{\Gamma}_2$ and $\tilde{\Gamma}_3$. Let $h$ be the maximal diameter of the elements. Let $V^h \subset V$ be the finite element space consisting of continuous piecewise linear functions and $B^h \subset L^2(\Gamma_3)$ be the finite element space composed by piecewise constant functions corresponding to the partition $\mathcal{T}^h$. Then the piecewise constant averaging operator $\mathcal{P}_B^h : L^2(\Gamma_3) \to B^h$ satisfies relation (5.11). Denote $\Pi^h$ the finite element interpolation operator. For the initial values, assume

$$u_0, v_0 \in [H^2(\Omega)]^d, \quad \beta_0 \in W^{1,\infty}(\Gamma_3)$$  \hspace{1cm} (5.31)

and take

$$u_0^h = \Pi^h u_0, \quad v_0^h = \Pi^h v_0, \quad \beta_0^h = \mathcal{P}_B^h \beta_0$$  \hspace{1cm} (5.32)

for the discrete initial values used in (5.10) and (5.13). Then

$$\|u_0 - u_0^h\|_V + \|v_0 - v_0^h\|_V + \|\beta_0 - \beta_0^h\|_{L^\infty(\Gamma_3)} \leq ch.$$  \hspace{1cm} (5.33)

Assume the following additional solution regularities:

$$\bar{v} \in L^1(0, T; H), \quad v \in C([0, T]; [H^2(\Omega)]^d), \quad v \in BV(0, T; [H^1(\Omega)]^d),$$  \hspace{1cm} (5.33)

$$\bar{\beta} \in L^1(0, T; L^\infty(\Gamma_3)).$$  \hspace{1cm} (5.34)

Here for a normed space $X$, $BV(0, T; X)$ denotes the space of vector-valued functions of bounded variation:

$$BV(0, T; X) = \{v \in C([0, T]; X) ; V(v) < \infty\},$$

where

$$V(v) = \sup_{\mathcal{P} : \text{partition of } [0, T]} V(v ; \mathcal{P})$$

and for a partition $\mathcal{P}$ of $[0, T]$: $0 = t_0 < t_1 < \cdots < t_N = T$,

$$V(v ; \mathcal{P}) = \sum_{n=1}^{N} \|v_n - v_{n-1}\|_X.$$

Under assumptions (5.31) and (5.33), we have (cf. [5])

$$\|v_j - \Pi^h v_j\|_H + h\|v_j - \Pi^h v_j\|_V \leq ch^2, \quad 0 \leq j \leq N.$$  \hspace{1cm} (5.35)

We also have

$$\sum_{j=1}^{N-1} \| (v_j - \Pi^h v_j) - (v_{j+1} - \Pi^h v_{j+1}) \|_H = \sum_{j=1}^{N-1} \| (v_j - v_{j+1}) - \Pi^h (v_j - v_{j+1}) \|_H \leq ch \sum_{j=1}^{N-1} \|v_j - v_{j+1}\|_{[H^1(\Omega)]^d} \leq ch.$$  \hspace{1cm} (5.36)
In [10, Lemma 11.4], we know that for \( j = 1, \ldots, N \),

\[
\left\| \frac{\mathbf{v}_j - \mathbf{v}_{j-1}}{k} - \mathbf{v}_j \right\|_H \leq \| \mathbf{v} \|_{L^1(I_{j-1}, I_{j}; H)},
\]

\[
\left\| \frac{\beta_j - \beta_{j-1}}{k} - \beta_j \right\|_{L^\infty(I_{j}; L^\infty(\Omega))} \leq \| \tilde{\beta} \|_{L^1(I_{j-1}, I_{j}; L^\infty(\Omega))},
\]

\[I_n \leq ck\| \dot{\mathbf{v}} \|_{H^1(0, T; V)}.
\]

Using these estimates in (5.30) we obtain

\[
\max_{0 \leq n \leq N} \| \beta_n - \beta^h_n \|_{L^\infty(I_{j}; L^\infty(\Omega))} + \max_{0 \leq n \leq N} \| \mathbf{v}_n - \mathbf{v}^h_n \|_H
\]

\[+ \left( k \sum_{n=0}^N \| \mathbf{v}_n - \mathbf{v}^h_n \|_V^2 \right)^{1/2} \leq c(h + k). \tag{5.37}
\]

We remark that (5.37) is only a sample error estimate. If the solution regularity conditions are different or the finite element space is changed, then we will have another different error estimate which can be again derived from (5.30).

6. Numerical examples

We report some numerical results to show the performance of the numerical scheme studied in the previous section.

6.1. A one-dimensional test example

We consider a long thin rod, horizontally positioned which is clamped at its left end \( x = 0 \) and is in adhesive contact with an obstacle at the other end. We assume that the body is subject to the action of a density of body forces \( f_0(x, t) \) in the \( x \) direction (see Fig. 1) and there is an adhesive contact between its right end \( x = L \) and an obstacle with normal compliance. This problem is then written as follows:

**Problem T1D.** Find a displacement field \( u : [0, L] \times [0, T] \rightarrow \mathbb{R} \), a stress field \( \sigma : [0, L] \times [0, T] \rightarrow \mathbb{R} \) and an adhesion field \( \beta : [0, T] \rightarrow \mathbb{R} \) such that

\[
\rho \frac{\partial^2 u}{\partial t^2} = \frac{\partial \sigma}{\partial x} + f_0 \text{ in } [0, L] \times (0, T),
\]

\[
\sigma = \eta \frac{\partial^2 u}{\partial x \partial t} + E \frac{\partial u}{\partial x} \text{ in } [0, L] \times (0, T),
\]

\[
u(0, t) = 0 \text{ for } t \in (0, T),
\]

\[
u(L, t) = p_\nu(u(L, t)) + \gamma \beta^2(t)[-u(L, t)]_+ \text{ for } t \in (0, T),
\]
\[ \frac{\partial \beta}{\partial t}(t) = -\gamma \beta(t) u(L, t)^2 - \epsilon_a^- \quad \text{for} \quad t \in (0, T), \]

\[ u(x, 0) = u_0, \quad \frac{\partial u}{\partial t}(x, 0) = v_0 \quad \text{for} \quad x \in [0, L], \]

\[ \beta(0) = \beta_0, \]

where \( E \) is the Young’s modulus of the material which occupies \((0, L)\) and \( \eta \) is a viscosity constant. In the three simulations considered here, \( V^h \) consists of continuous piecewise linear functions while \( Q^h \) and \( B^h \) piecewise constant functions. We use the discretization parameters \( k = h = 0.01 \).

For computation we have used the following data:

\[ L = 1 \text{ m}, \quad T = 1 \text{ s}, \]
\[ \rho = 1 \text{ N s}^2/\text{m}^3, \quad \eta = 1 \text{ N s/m}, \quad E = \frac{1}{100} \text{ N/m}, \]
\[ p_v(r) = c_v r, \quad c_v = 1 \text{ N/m}^2, \quad \gamma_v = 1 \text{ m}^{-2}, \quad \epsilon_a = 0, \]
\[ u_0 = 0 \text{ m}, \quad v_0 = 0 \text{ m/s}, \quad \beta_0 = 1. \]

We distinguish three cases depending on the direction of the volume forces \( f_0 \).

First, we consider a compression force, i.e. a positive force, \( f_0(x, t) = 10t \text{ N/m} \). In Fig. 2 (left side) we can see the displacement field at several times, and in the right side the evolution of the displacement field for several points is shown. In Fig. 3 the stress field is plotted for several times and the evolution through the time for some finite elements. In Fig. 4 the evolution of the adhesion field through the time is drawn.

The second case corresponds to a debonding force \( f_0(x, t) = -10t \text{ N/m} \). In Fig. 5 we show the displacement field obtained at several times and the evolution of the displacement fields for several points. In Fig. 6 we plot the stress fields calculated at several times and the evolution in time of the stress fields for some finite elements. In Fig. 7 we show the evolution of the adhesion field.
The third case is for a debounding/rebounding force

\[ f_0(x,t) = \sin(4\pi t) \text{ N/m}. \]

In Fig. 8 the displacement field is shown at several times and the evolution of the displacement field for points \( x = 0.25, 0.5, 1 \) is plotted. In Fig. 9 we plot the stress fields at several times and the evolution of the stress field is shown for several finite elements. In Fig. 10 the evolution of the adhesion field is drawn.
6.2. A two-dimensional test example

As a two-dimensional example of problem $P$, we consider the plane stress viscoelastic problem depicted in Fig. 11. The domain $\Omega = (0, 3) \times (0, 1)$ is the cross-section of a three-dimensional body submitted to the action of surface forces on its upper boundary $\Gamma_2$ which we assume to be linearly increasing in time. Moreover, no volume forces are assumed and both ends of the body are clamped (i.e., $\Gamma_1 = \{0, 3\} \times [0, 1]$). Finally, we assume the body is in adhesive contact with an elastic foundation with normal compliance on $\Gamma_3 = [0, 3] \times \{0\}$.

The elasticity tensor $\mathcal{G}$ satisfies:

$$(\mathcal{G} \tau)_{\alpha\beta} = \frac{E \kappa}{1 - \kappa^2} (\tau_{11} + \tau_{22}) \delta_{\alpha\beta} + \frac{E}{1 + \kappa} \tau_{\alpha\beta}, \quad 1 \leq \alpha, \beta \leq 2,$$
Fig. 6. Problem T1D-2: Stress fields at times $t = 0.25, 0.5, 1$ s and evolution for finite elements $[0.24,0.25], [0.49,0.5], [0.99,1]$.

Fig. 7. Problem T1D-2: Evolution of the adhesion field.

where $E$ is the Young’s modulus, $\kappa$ the Poisson’s ratio of the material and $\delta_{\alpha\beta}$ denotes the Kronecker symbol. The viscosity tensor $\mathcal{A}$ has a similar form, i.e.

$$(\mathcal{A}\tau)_{\alpha\beta} = \mu(\tau_{11} + \tau_{22})\delta_{\alpha\beta} + \eta\tau_{\alpha\beta}, \quad 1 \leq \alpha, \beta \leq 2,$$

where $\mu$ and $\eta$ are viscosity constants.

We recall that the Von Mises norm for a plane stress field $\tau = (\tau_{\alpha\beta})$ is given by

$$\|\tau\| = (\tau_{11}^2 + \tau_{22}^2 - \tau_{11}\tau_{22} + 3\tau_{12}^2)^{1/2}.$$
For computation we use the following data:

\[ T = 1 \text{s}, \quad f_0(x_1, x_2, t) = 0 \text{N/m}^3, \quad f_2(x_1, x_2, t) = (0, t) \text{N/m}^2, \]

\[ p_1(r) = c_1 r^+, \quad c_1 = 1 \text{N/m}^3, \quad \gamma = 50 \text{ m}^{-2}, \quad \varepsilon = 0, \]

\[ E = 2 \text{N/m}^2, \quad \kappa = 0.1, \quad \mu = 63.461 \text{ N s/m}^2, \quad \eta = 46.296 \text{ N s/m}^2, \]

\[ u_0 = 0 \text{ m}, \quad v_0 = 0 \text{ m/s}, \quad \beta_0 = 1. \]
Moreover, as it was remarked in Section 2, we use the function $u_A$ as an approximation of $R(u_r)$. Again, continuous piecewise linear functions were considered in order to define spaces $V^h$ and $B^h$ and piecewise constant functions for $Q^h$. Finally, time step $k = 0.01$ was used.

In Figs. 12 and 13 we show the initial boundary and the deformed mesh at final time and the Von Mises norm for stress in the deformed configuration. In Fig. 14 (left side), the adhesion field on $\Gamma_3$ is plotted at several times and, in the right side, the evolution of the adhesion field of the central contact node $x = (1.5, 0)$ is shown.
Fig. 13. Problem T2D: Von Mises stress norm at final time $t = 1$ s.

Fig. 14. Problem T2D: Adhesion field at times $t = 0, 0.4, 0.7, 1$ s and the evolution in time for central contact point $x = (1.5, 0)$.

References


